19. Indefinite Integrals

Exercise 19.2

1. Question

Evaluate the following integrals:

$$\int\left(3x\sqrt{x}+4\sqrt{x}+5\right)dx$$

Answer

Given:

$$\int\left(3x\sqrt{x}+4\sqrt{x}+5\right)dx$$

By Splitting, we get,

$$\Rightarrow \int((3x\sqrt{x})dx+(4\sqrt{x})dx+5dx)$$

$$\Rightarrow \int 3x\sqrt{x}dx+\int 4\sqrt{x}dx+\int 5dx$$

$$\Rightarrow \int 3x^{\frac{3}{2}}\,dx+\int 4x^{\left(\frac{1}{2}\right)}\,d+\int 5dx$$

By using the formula, $\int x^{n}\,dx=\frac{x^{n+1}}{n+1}$

$$\Rightarrow \frac{3x^{\frac{3}{2}+1}}{\frac{3}{2}+1}+\frac{4x^{\frac{1}{2}+1}}{\frac{1}{2}+1}+\int 5dx$$

$$\int kdx=kx+c$$

$$\Rightarrow \frac{3x^{\frac{5}{2}}}{5/2}+\frac{4x^{\frac{3}{2}}}{5/2}+5x+c$$

$$\Rightarrow \frac{6}{5}x^{\frac{5}{2}}+\frac{4}{5}x^{3/2}+5x+c$$

2. Question

Evaluate the following integrals:

$$\int\left(2^{x}+\frac{5}{x}-\frac{1}{x^{1/3}}\right)dx$$

Answer

Given:

$$\int\left(2^{x}+\frac{5}{x}-\frac{1}{x^{1/3}}\right)dx$$

By Splitting them, we get,

$$\Rightarrow \int 2^{x}dx+\int\left(\frac{5}{x}\right)dx-\int\frac{1}{x^{1/3}}dx$$

By using the formula,

$$\int a^x\,dx = \frac{a^x}{\log a}$$

$$\Rightarrow \frac{2^x}{\log 2} + 5\int \left(\frac{1}{x}\right) dx - \int x^{-1/3} dx$$

By using the formula,

$$\int \left(\frac{1}{x}\right) dx = \log x$$

$$\Rightarrow \frac{2^x}{\log 2} + 5\log x - \int x^{-1/3}\,dx$$

By using the formula,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}$$

$$\Rightarrow \frac{2^x}{\log 2} + 5\log x - \frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1}$$

$$\Rightarrow \frac{2^x}{\log 2} + 5\log x - \frac{x^{\frac{2}{3}}}{2/3}$$

$$\Rightarrow \frac{2^x}{\log 2} + 5\log x - \frac{3}{2}x^{2/3} + c$$

3. Question

Evaluate the following integrals:

$$\int \left\{\sqrt{x}\left(ax^2 + bx + c\right)\right\} dx$$

Answer

Given:

$$\int \{\sqrt{x}(ax^2 + bx + c)\} dx$$

$$\Rightarrow \int (\sqrt{x}ax^2 + \sqrt{x}bx + \sqrt{x}c)\,dx$$

By Splitting, we get,

$$\Rightarrow a\int x^2 \times x^{\frac{1}{2}} dx + b\int x^1 \times x^{\frac{1}{2}} dx + c\int x^{1/2} dx$$

$$\Rightarrow a\int x^{\frac{5}{2}} dx + b\int x^{\frac{3}{2}} dx + c\int x^{\frac{1}{2}} dx$$

By using the formula

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}$$

$$\Rightarrow \frac{ax^{\frac{5}{2}+1}}{\frac{5}{2}+1} + \frac{bx^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \frac{cx^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$

$$\Rightarrow \frac{ax^{\frac{7}{2}}}{7/2} + \frac{bx^{\frac{5}{2}}}{5/2} + \frac{cx^{\frac{3}{2}}}{3/2} + c$$

4. Question

Evaluate the following integrals:

$\int (2 - 3x)(3 + 2x)(1 - 2x)dx$

Answer

Given:

$\Rightarrow\int(2 - 3x)(3 + 2x)(1 - 2x)dx$

By multiplying,

$\Rightarrow\int (6 - 4x - 9x - 6x^2)\, dx$

$\Rightarrow\int (6 - 13x - 6x^2)\, dx$

By Splitting, we get,

$\Rightarrow\int 6dx - \int 13\, x\, dx - \int 6x^2\, dx$

By using the formulas,

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} \text{ and}$$

$$\int kdx = kx + c$$

We get,

$$\Rightarrow 6x - \frac{13x^{1+1}}{1+1} - \frac{6x^{2+1}}{2+1} + c$$

$$\Rightarrow 6x - \frac{13x^2}{2} - \frac{6x^3}{3} + c$$

5. Question

Evaluate the following integrals:

$$\int\left(\frac{m}{x} + \frac{x}{m} + m^x + x^m + mx\right)dx$$

Answer

Given:

$$\int \left(\frac{m}{x} + \frac{x}{m} + m^x + x^m + mx\right)dx$$

By Splitting, we get,

$$\Rightarrow \int \frac{m}{x}dx + \int \frac{x}{m}dx + \int x^m dx + \int m^x dx + \int mxdx$$

By using formula,

$$\int \frac{1}{x}dx = \log x + c$$

$$\Rightarrow m\log x + \frac{1}{m}\int xdx + \int x^m dx + \int m^x dx + \int mxdx$$

By using the formula,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}$$

$$\Rightarrow m\log x + \frac{\frac{1}{m}x^{1+1}}{1+1} + \frac{x^{m+1}}{m+1} + \int m^x dx + \frac{mx^{1+1}}{1+1}$$

By using the formula,

$$\int a^x\,dx = \frac{a^x}{\log a}$$

$$\Rightarrow m\log x + \frac{\frac{1}{m}x^2}{2} + \frac{x^{m+1}}{m+1} + \frac{m^x}{\log m} + \frac{mx^2}{2} + c$$

6. Question

Evaluate the following integrals:

$$\int\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$$

Answer

Given:

$$\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$$

By applying $(a - b)^2 = a^2 - 2ab + b^2$

$$\Rightarrow \int\left((\sqrt{x})^2 + \left(\frac{1}{\sqrt{x}}\right)^2 - 2(\sqrt{x})\left(\frac{1}{\sqrt{x}}\right)\right)dx$$

$$\Rightarrow \int\left((\sqrt{x})^2 + \left(\frac{1}{\sqrt{x}}\right)^2 - 2(\sqrt{x})\left(\frac{1}{\sqrt{x}}\right)\right)dx$$

After computing,

$$\Rightarrow \int\left(x + \frac{1}{x} - 2\right)dx$$

By Splitting, we get,

$$\Rightarrow \int x\,dx + \int \frac{1}{x}dx - 2\int dx$$

By applying the formulas:

$$\int \mathbf{x^n\,dx = \frac{x^{n+1}}{n+1}}$$

$$\int \mathbf{\left(\frac{1}{x}\right)dx = \log x}$$

$$\int \mathbf{kdx = kx + c}$$

We get,

$\Rightarrow \frac{x^{1+1}}{1+1} + \log x - 2x + c$**I = 1/2 x^2 + logx - 2x + c**

7. Question

Evaluate the following integrals:

$$\int \frac{(1+x)^3}{\sqrt{x}} dx$$

Answer

Given:

$$\int \frac{(1+x)^3}{\sqrt{x}} dx$$

Applying: $(a + b)^3 = a^3 + b^3 + 3ab^2 + 3a^2b$

$$\Rightarrow \int \frac{1+x^3+3x^2\times1+3\times1^2\times x}{\sqrt{x}} dx$$

$$\Rightarrow \int \frac{1+x^3+3x^2+3x}{\sqrt{x}} dx$$

By Splitting, we get,

$$\Rightarrow \int \frac{1}{\sqrt{x}} dx + \int \frac{x^3}{\sqrt{x}} dx + \int \frac{3x^2}{\sqrt{x}} dx + \int \frac{3x}{\sqrt{x}} dx$$

$$\Rightarrow \int x^{-\frac{1}{2}} dx + \int x^3 \times x^{-\frac{1}{2}} dx + \int 3x^2 \times x^{-\frac{1}{2}} dx + \int 3x \times x^{-\frac{1}{2}} dx$$

$$\Rightarrow \int x^{-\frac{1}{2}} dx + \int x^{\frac{5}{2}} dx + 3\int x^{\frac{3}{2}} dx + 3\int x^{\frac{1}{2}} dx$$

By applying formula,

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\Rightarrow \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + \frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1} + 3\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \frac{3x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$

$$\Rightarrow \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{x^{\frac{7}{2}}}{\frac{7}{2}} + \frac{3x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} + c$$

8. Question

Evaluate the following integrals:

$$\int \left\{x^2 + e^{\log x} + \left(\frac{e}{2}\right)^x\right\} dx$$

Answer

Given:

$$\int \left\{x^2 + e^{\log x} + \left(\frac{e}{2}\right)^x\right\} dx$$

By Splitting, we get,

$$\Rightarrow \int x^2 dx + \int e^{\log x} dx + \int \left(\frac{e}{2}\right)^x dx$$

By applying formula,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}$$

$$\Rightarrow \frac{x^{2+1}}{2+1} + \int e^{\log_e x} dx + \int \left(\frac{e}{2}\right)^x dx$$

$$\Rightarrow \frac{x^3}{3} + \int x\ dx + \frac{1}{\log(\frac{e}{2})} \log\left(\frac{e}{2}\right)^x$$

$$\Rightarrow \frac{x^3}{3} + \int xdx + \frac{1}{\log(\frac{e}{2})} \log\left(\frac{e}{2}\right)^x$$

$$\Rightarrow \frac{x^3}{3} + \frac{x^2}{2} + \frac{1}{\log\left(\frac{e}{2}\right)} \log\left(\frac{e}{2}\right)^x + c$$

9. Question

Evaluate the following integrals:

$\int (x^e + e^x + e^e)\,dx$

Answer

Given:

$$\int (x^e + e^x + e^e) dx$$

By Splitting, we get,

$$\Rightarrow \int x^e dx + \int e^x dx + \int e^e dx$$

By using the formula,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}$$

$$\Rightarrow \frac{x^{e+1}}{e+1} + \int e^x dx + \int e^e dx$$

By applying the formula,

$$\int a^x\,dx = \frac{a^x}{\log a}$$

$$\Rightarrow \frac{x^{e+1}}{e+1} + \frac{e^x}{\log_e e} + \int e^e dx$$

We know that,

$$\int kdx = kx + c$$

$$\Rightarrow \frac{x^{e+1}}{e+1} + \frac{e^x}{\log_e e} + e^e x + c$$

$$\Rightarrow \frac{x^{e+1}}{e+1} + \frac{e^x}{\log_e e} + e^e x + c$$

10. Question

Evaluate the following integrals:

$$\int \sqrt{x}\left(x^3 - \frac{2}{x}\right)dx$$

Answer

Given:

$$\int \sqrt{x}\left(x^3 - \frac{2}{x}\right)dx$$

Opening the bracket, we get,

$$\Rightarrow \int (x^{\frac{1}{2}} \times x^3 - x^{\frac{1}{2}} \times \frac{2}{x})dx$$

$$\Rightarrow \int (x^{\frac{1}{2}+3} - x^{\frac{1}{2}-1} \times 2)dx$$

$$\Rightarrow \int (x^{\frac{7}{2}} - 2x^{-\frac{1}{2}})dx$$

By multiplying,

$$\Rightarrow \int x^{\frac{7}{2}}dx - 2\int x^{-\frac{1}{2}}dx$$

By applying the formula,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}$$

$$\Rightarrow \frac{x^{\frac{7}{2}+1}}{\frac{7}{2}+1} - 2\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$\Rightarrow \frac{x^{\frac{9}{2}}}{\frac{9}{2}} - 2\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$\Rightarrow \frac{2x^{\frac{9}{2}}}{9} - 4x^{\frac{1}{2}} + c$$

11. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{x}}\left(1 + \frac{1}{x}\right)dx$$

Answer

Given:

$$\int \frac{1}{\sqrt{x}}\left\{1 + \frac{1}{x}\right\}dx$$

By multiplying $\frac{1}{\sqrt{x}}$ with inside brackets,

$$\Rightarrow \int \left\{\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{x}} \times \frac{1}{x}\right\}dx$$

$$\Rightarrow \int \left\{\frac{1}{x^{\frac{1}{2}}} + \frac{1}{x^{\frac{1}{2}}} \times \frac{1}{x}\right\}dx$$

$$\Rightarrow \int \left\{\frac{1}{x^{\frac{1}{2}}} + \frac{1}{x^{\frac{1}{2}+1}}\right\} dx$$

$$\Rightarrow \int \left\{\frac{1}{x^{\frac{1}{2}}} + \frac{1}{x^{\frac{3}{2}}}\right\} dx$$

By Splitting them, we get,

$$\Rightarrow \int x^{-\frac{1}{2}} dx + \int x^{-\frac{3}{2}} dx$$

By applying the formula,

$$\int x^n \, dx = \frac{x^{n+1}}{n+1}$$

$$\Rightarrow \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + \frac{x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} + c$$

$$\Rightarrow \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + c$$

$$\Rightarrow 2x^{\frac{1}{2}} - 2x^{-\frac{1}{2}} + c$$

12. Question

Evaluate the following integrals:

$$\int \frac{x^6 + 1}{x^2 + 1} dx$$

Answer

Given:

$$\int \frac{x^6 + 1}{x^2 + 1} dx$$

By applying: $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$

$$\Rightarrow \int \frac{(x^2)^3 + (1)^3}{x^2 + 1} dx$$

$$\Rightarrow \int \frac{(x^2 + 1)((x^2)^2 + (1)^2 - x^2 \times 1)}{(x^2 + 1)} dx$$

$$\Rightarrow \int \frac{(x^2 + 1)(x^4 + 1 - x^2)}{x^2 + 1} dx$$

$$\Rightarrow \int (x^4 + 1 - x^2) \, dx$$

By Splitting

$$\Rightarrow \int x^4 dx + 1 \int dx - \int x^2 dx$$

By using the formula,

$$\int x^n \, dx = \frac{x^{n+1}}{n+1}$$

$$\int k\,dx = kx + c$$

$$\Rightarrow \frac{x^{5+1}}{5+1} + x - \frac{x^{3+1}}{3+1} + c$$

$$\Rightarrow \frac{x^6}{6} + x - \frac{x^4}{4} + c$$

13. Question

Evaluate the following integrals:

$$\int \frac{x^{-1/3} + \sqrt{x} + 2}{\sqrt[3]{x}}\,dx$$

Answer

Given:

$$\int \frac{x^{-\frac{1}{3}} + \sqrt{x} + 2}{\sqrt[3]{x}}\,dx$$

By Splitting them,

$$\Rightarrow \int \frac{x^{-\frac{1}{3}}}{\sqrt[3]{x}}dx + \int \frac{\sqrt{x}}{\sqrt[3]{x}}dx + \int \frac{2}{\sqrt[3]{x}}dx$$

$$\Rightarrow \int x^{-\frac{1}{3}} \times x^{-\frac{1}{3}}dx + \int x^{\frac{1}{2}} \times x^{-\frac{1}{3}}dx + 2\int x^{-\frac{1}{3}}dx$$

$$\Rightarrow \int x^{-\frac{1}{3}-\frac{1}{3}}dx + \int x^{\frac{1}{2}-\frac{1}{3}}dx + 2\int x^{-\frac{1}{3}}dx$$

$$\Rightarrow \int x^{-\frac{2}{3}}dx + \int x^{\frac{5}{6}}dx + 2\int x^{-\frac{1}{3}}dx$$

By applying the formula,

$$\int \mathbf{x^n\,dx} = \frac{\mathbf{x^{n+1}}}{\mathbf{n+1}}$$

We get,

$$\Rightarrow \frac{x^{-\frac{2}{3}+1}}{-\frac{2}{3}+1} + \frac{x^{\frac{5}{6}+1}}{\frac{5}{6}+1} + \frac{2x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + c$$

$$\Rightarrow \frac{x^{\frac{1}{3}}}{\frac{1}{3}} + \frac{x^{\frac{11}{6}}}{\frac{11}{6}} + \frac{2x^{\frac{2}{3}}}{\frac{2}{3}} + c$$

$$\Rightarrow 3x^{\frac{1}{3}} + \frac{6x^{\frac{11}{6}}}{11} + 3x^{\frac{2}{3}} + c$$

14. Question

Evaluate the following integrals:

$$\int \frac{(1+\sqrt{x})^2}{\sqrt{x}}\,dx$$

Answer

Given:

$$\int \frac{(1+\sqrt{x})^2}{\sqrt{x}}dx$$

By applying $(a + b)^2 = a^2 + b^2 + 2ab$

$$\Rightarrow \int \frac{(1)^2 + (\sqrt{x})^2 + 2 \times 1 \times \sqrt{x}}{\sqrt{x}}dx$$

$$\Rightarrow \int \frac{1 + x + 2\sqrt{x}}{\sqrt{x}}dx$$

By Splitting, we get,

$$\Rightarrow \int (\frac{1}{\sqrt{x}} + \frac{x}{\sqrt{x}} + \frac{2\sqrt{x}}{\sqrt{x}})dx$$

$$\Rightarrow \int x^{-\frac{1}{2}}dx + \int x \times x^{-\frac{1}{2}}dx + 2\int dx$$

$$\Rightarrow \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + \int x^{1-\frac{1}{2}}dx + 2x + c$$

$$\Rightarrow \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \int x^{\frac{1}{2}}dx + 2x + c$$

$$\Rightarrow 2x^{\frac{1}{2}} + \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + 2x + c$$

$$\Rightarrow 2x^{\frac{1}{2}} + \frac{2x^{\frac{3}{2}}}{3} + 2x + c$$

15. Question

Evaluate the following integrals:

$\int\sqrt{x}(3 - 5x)\,dx$

Answer

Given:

$$\int \sqrt{x}(3-5x)dx$$

By multiplying $\sqrt{x}$ inside the bracket we get,

$$\Rightarrow\int (3\sqrt{x} - 5x\sqrt{x})dx$$

$$\Rightarrow \int \left(3x^{\frac{1}{2}} - 5x^1 \times x^{\frac{1}{2}}\right)dx$$

$$\Rightarrow \int (3x^{\frac{1}{2}} - 5x^{1+\frac{1}{2}})dx$$

$$\Rightarrow \int (3x^{\frac{1}{2}} - 5x^{\frac{3}{2}})dx$$

By Splitting, we get,

$\Rightarrow 3\int x^{\frac{1}{2}}dx - 5\int x^{\frac{3}{2}}dx$

By using the formula,

$\int x^n\,dx = \frac{x^{n+1}}{n+1}$

$\Rightarrow \frac{3x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{5x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + c$

$\Rightarrow \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{5x^{\frac{5}{2}}}{\frac{5}{2}} + c$

$\Rightarrow 2x^{\frac{3}{2}} - 2x^{\frac{5}{2}} + c$

16. Question

Evaluate the following integrals:

$\int \frac{(x+1)(x-2)}{\sqrt{x}}\,dx$

Answer

Given:

$\int \frac{(x+1)(x-2)}{\sqrt{x}}\,dx$

$\Rightarrow \int \frac{x^2 - 2x + x - 2}{\sqrt{x}}\,dx$

$\Rightarrow \int \frac{x^2 - x - 2}{\sqrt{x}}\,dx$

By Splitting,

$\Rightarrow \int \frac{x^2}{\sqrt{x}}dx - \int \frac{x}{\sqrt{x}}dx - \int \frac{2}{\sqrt{x}}dx$

$\Rightarrow \int x^2 \times x^{-\frac{1}{2}}dx - \int x \times x^{-\frac{1}{2}}dx - 2\int x^{-\frac{1}{2}}dx$

$\Rightarrow \int x^{2-\frac{1}{2}}dx - \int x^{1-\frac{1}{2}}dx - 2\int x^{-\frac{1}{2}}dx$

$\Rightarrow \int x^{\frac{3}{2}}dx - \int x^{\frac{1}{2}}dx - 2\int x^{-\frac{1}{2}}dx$

By applying the formula,

$\int x^n\,dx = \frac{x^{n+1}}{n+1}$

$\Rightarrow \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} - \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{2x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$

$\Rightarrow \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$$\Rightarrow \frac{2}{5}x^{\frac{5}{2}} - \frac{2}{3}x^{\frac{3}{2}} - 4x^{\frac{1}{2}} + c$$

17. Question

Evaluate the following integrals:

$$\int \frac{x^5 + x^{-2} + 2}{x^2} dx$$

Answer

Given:

$$\int \frac{x^5 + x^{-2} + 2}{x^2} dx$$

By Splitting, we get,

$$\Rightarrow \int \left(\frac{x^5}{x^2} + \frac{x^{-2}}{x^2} + \frac{2}{x^2}\right) dx$$

$$\Rightarrow \int (x^5 \times x^{-2} + x^{-2} \times x^{-2} + 2 \times x^{-2}) dx$$

By applying,

$$\Rightarrow \int (x^{5-2} + x^{-2-2} + 2x^{-2}) dx$$

$$\Rightarrow \int (x^3 + x^{-4} + 2x^{-2}) dx$$

By Splitting, we get,

$$\Rightarrow \int x^3 dx + \int x^{-4} dx + 2\int x^{-2} dx$$

By applying the formula,

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\Rightarrow \frac{x^{3+1}}{3+1} + \frac{x^{-4+1}}{-4+1} + \frac{2x^{-2+1}}{-2+1} + c$$

$$\Rightarrow \frac{x^4}{4} + \frac{x^{-3}}{-3} + \frac{2x^{-1}}{-1} + c$$

18. Question

Evaluate the following integrals:

$\int (3x + 4)^2 dx$

Answer

Given:

$$\int (3x + 4)^2 dx$$

By applying,

$(a + b)^2 = a^2 + b^2 + 2ab$

$\Rightarrow \int ((3x)^2 + 4^2 + 2 \times 3x \times 4)dx$

$\Rightarrow \int (9x^2 + 16 + 24x)dx$

By Splitting, we get,

$\Rightarrow \int 9x^2 dx + \int 16dx + \int 24xdx$

$\Rightarrow 9\int x^2 + 16\int dx + 24\int xdx$

By applying,

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\int kdx = kx + c$$

$\Rightarrow \frac{9x^{2+1}}{2+1} + 16x + \frac{24x^{1+1}}{1+1} + c$

$\Rightarrow \frac{9}{3}x^3 + 16x + \frac{24}{2}x^2 + c$

$\Rightarrow 3x^3 + 16x + 12x^2 + c$

19. Question

Evaluate the following integrals:

$$\int \frac{2x^4 + 7x^3 + 6x^2}{x^2 + 2x} dx$$

Answer

Given:

$$\int \frac{2x^4 + 7x^3 + 6x^2}{x^2 + 2x} dx$$

Take x is common on both numerator and denominator,

$\Rightarrow \int \frac{x(2x^3 + 7x^2 + 6x)}{x(x+2)} dx$

$\Rightarrow \int \frac{2x^3 + 7x^2 + 6x}{x+2} dx$

Splitting $7x^2$ into $4x^2$ and $3x^2$

$\Rightarrow \int \frac{2x^3 + 4x^2 + 3x^2 + 6x}{x+2} dx$

Common the $2x^2$ from first two elements and 3x from next elements,

$\Rightarrow \int \frac{2x^2(x+2) + 3x(x+2)}{x+2} dx$

Now common the $x + 2$ from the elements

$$\Rightarrow \int \frac{(x+2)(2x^2+3x)}{x+2}dx$$

$$\Rightarrow \int (2x^2+3x)dx$$

Now Splitting, we get,

$$\Rightarrow \int 2x^2 dx + \int 3x dx$$

Now applying the formula,

$$\Rightarrow \frac{2x^{2+1}}{2+1} + \frac{3x^{1+1}}{1+1} + c$$

$$\Rightarrow \frac{2x^3}{3} + 3x + c$$

20. Question

Evaluate the following integrals:

$$\int \frac{5x^4+12x^3+7x^2}{x^2+x}dx$$

Answer

Given:

$$\int \frac{5x^4+12x^3+7x^2}{x^2+x}dx$$

Now spilt $12x^3$ into $7x^3$ and $5x^3$

$$\Rightarrow \int \frac{5x^4+7x^3+5x^3+7x^2}{x^2+x}dx$$

Now common $5x^3$ from two elements 7x from other two elements,

$$\Rightarrow \int \frac{5x^2(x+1)+7x(x+1)}{x^2+x}dx$$

$$\Rightarrow \frac{\int (5x^2+7x)(x+1)}{x(x+1)}dx$$

$$\Rightarrow \int (5x^2+7x)dx$$

Now Splitting, we get,

$$\Rightarrow \int 5x^2 dx + \int 7x\, dx$$

$$\Rightarrow \frac{5x^{2+1}}{2+1} + \frac{7x^{1+1}}{1+1} + c$$

$$\Rightarrow \frac{5x^3}{3} + \frac{7x^2}{2} + c$$

21. Question

Evaluate the following integrals:

$$\int \frac{\sin^2 x}{1+\cos} dx$$

Answer

Given:

$$\int \frac{\sin^2 x}{1+\cos x} dx$$

We know that,

$\sin^2 x = 1 - \cos^2 x$

$$\Rightarrow \int \frac{1-\cos^2 x}{1+\cos x} dx$$

We treat $1 - \cos^2 x$ as $a^2 - b^2 = (a + b)(a - b)$

$$\Rightarrow \int \frac{(1)^2-(\cos x)^2}{1+\cos x} dx$$

$$\Rightarrow \int \frac{(1+\cos x)(1-\cos x)}{1+\cos x} dx$$

$$\Rightarrow \int (1-\cos x) dx$$

By Splitting, we get,

$$\Rightarrow \int dx - \int \cos x \, dx$$

We know that,

$$\int k dx = kx + c$$

$$\int \cos x \, dx = \sin x$$

$\Rightarrow x - \sin x + c$

22. Question

Evaluate the following integrals:

$\int (se^2 x + cosec^2 x)\, dx$

Answer

Given:

$$\int (\sec^2 x + cosec^2 x) dx$$

By Splitting, we get,

$$\Rightarrow \int \sec^2 x \, dx + \int cosec^2 x dx$$

By applying the formula,

$$\int \sec^2 x \, dx = \tan x$$

$$\int \text{codec}^2 x dx = -\cot x$$

⇒tan x – cot x + c

23. Question

Evaluate the following integrals:

$$\int \frac{\sin^3 x - \cos^3 x}{\sin^2 x \cos^2 x} dx$$

Answer

Given:

$$\int \frac{\sin^3 x - \cos^3 x}{\sin^2 x \cos^2 x} dx$$

By Splitting, we get,

$$\Rightarrow \int \left(\frac{\sin^3 x}{\sin^2 x\cos^2 x} - \frac{\cos^3 x}{\sin^2 x\cos^2 x}\right) dx$$

By cancelling the $\sin^2 x$ on first and $\cos^2 x$ on second,

$$\Rightarrow \int \left(\frac{\sin x}{\cos^2 x} - \frac{\cos x}{\sin^2 x}\right) dx$$

We know that,

$$\frac{\sin x}{\cos x} = \tan x$$

$$\frac{\cos x}{\sin x} = \cot x$$

$$\frac{1}{\cos x} = \sec x$$

$$\frac{1}{\sin x} = \operatorname{cosec} x$$

$$\Rightarrow \int (\tan x \sec x - \cot x \operatorname{cosec} x) dx$$

We know that,

$$\int \tan x \sec x \, dx = \sec x$$

$$\int \cot x \operatorname{cosec} x \, dx = -\cot x$$

⇒secx - (- cotx) + c

⇒secx + cotx + c

24. Question

Evaluate the following integrals:

$$\int \frac{5\cos^3 x + 6\sin^3 x}{2\sin^2 x \cos^2 x} dx$$

Answer

Given:

$$\int \frac{5\cos^3 x + 6\sin^3 x}{2\sin^2 x\cos^2 x} dx$$

By Splitting we get,

$$\Rightarrow \int \frac{5\cos^3 x}{2\sin^2 x\cos^2 x} dx + \int \frac{6\sin^3 x}{2\sin^2 x\cos^2 x} dx$$

$$\Rightarrow \frac{5}{2}\int \frac{\cos x\cos^2 x}{\sin^2 x\cos^2 x} dx + 3\int \frac{\sin^2 x\sin^1 x}{\sin^2 x\cos^2 x} dx$$

$$\Rightarrow \frac{5}{2}\int \frac{\cos x}{\sin^2 x} dx + 3\int \frac{\sin^1 x}{1\cos^2 x} dx$$

We know that,

$$\int 1\frac{\cos x}{\sin x} dx = \cot x$$

$$\int \frac{\sin x}{\cos x} dx = \tan x$$

$$\int 1\frac{1}{\sin x} dx = \sec x$$

$$\int 1\frac{1}{\sin x} dx = \operatorname{cosec} x$$

$$\Rightarrow \frac{5}{2}\int \cot x\operatorname{cosec} x\, dx + 3\int \sec x\tan x\, dx$$

We know that,

$$\int \cot x\operatorname{cosec} x\, dx = -\operatorname{cosec} x$$

$$\int \sec x\tan x\, dx = \sec x$$

$$\Rightarrow \frac{5}{2}(-\operatorname{cosec} x) + 3\sec x + c$$

$$I = -\frac{5}{2}\operatorname{cosec} x + 3\sec x + c$$

25. Question

Evaluate the following integrals:

$\int (\tan x + \cot x)^2\, dx$

Answer

Given:

$$I = \int (\tan x + \cot x)^2 dx$$

$$\Rightarrow \int (\tan^2 x + \cot^2 x + 2\tan x\cot x)^1 dx$$

We know that,

$\tan^2 x = \sec^2 x - 1$

$\cot^2 x = \text{cosec}^2 x - 1$

$\tan x = \frac{1}{\cot x}$

$\Rightarrow \int \left(\sec^2 x - 1 + \text{cosec}^2 - 1 + \frac{2}{\cot x} \cot x\right) dx$

$\Rightarrow \int (\sec^2 x + \text{cosec}^2 x - 2 + 2) dx$

$\Rightarrow \int (\sec^2 x + \text{cosec}^2 x) dx$

$\Rightarrow \int \sec^2 x + \int \text{cosec}^2 x dx$

We know that,

$\int \sec^2 x \, dx = \tan x$

$\int \text{cosec}^2 x dx = -\cot x$

I=tanx – cotx - c

26. Question

Evaluate the following integrals:

$\int \frac{1 - \cos 2x}{1 + \cos 2x} dx$

Answer

Let $I = \int \frac{1 - \cos 2x}{1 + \cos 2x} dx$

We know cos2θ = 1 – 2sin²θ = 2cos²θ – 1

Hence, in the numerator, we can write 1 – cos2x = 2sin²x

In the denominator, we can write 1 + cos2x = 2cos²x

Therefore, we can write the integral as

$I = \int \frac{2 \sin^2 x}{2 \cos^2 x} dx$

$\Rightarrow I = \int \frac{\sin^2 x}{\cos^2 x} dx$

$\Rightarrow I = \int \tan^2 x \, dx$

$\Rightarrow I = \int (\sec^2 x - 1) dx$ [∵ $\sec^2\theta - \tan^2\theta = 1$]

$\Rightarrow I = \int \sec^2 x \, dx - \int dx$

Recall $\int \mathbf{\sec^2 x \, dx = \tan x + c}$ and $\int \mathbf{dx = x + c}$

∴ I = tan x – x + c

Thus, $\int \frac{1 - \cos 2x}{1 + \cos 2x} dx = \tan x - x + c$

27. Question

Evaluate the following integrals:

$$\int \frac{\cos x}{1-\cos x} dx$$

Answer

Let $I = \int \frac{\cos x}{1-\cos x} dx$

On multiplying and dividing (1 + cos x), we can write the integral as

$$I = \int \frac{\cos x}{1-\cos x}\left(\frac{1+\cos x}{1+\cos x}\right) dx$$

$$\Rightarrow I = \int \frac{\cos x(1+\cos x)}{(1-\cos x)(1+\cos x)} dx$$

$$\Rightarrow I = \int \frac{\cos x+\cos^2 x}{1-\cos^2 x} dx$$

$\Rightarrow I = \int \frac{\cos x+\cos^2 x}{\sin^2 x} dx$ [$\because \sin^2\theta + \cos^2\theta = 1$]

$$\Rightarrow I = \int \left(\frac{\cos x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x}\right) dx$$

$$\Rightarrow I = \int \left(\frac{1}{\sin x} \times \frac{\cos x}{\sin x} + \frac{\cos^2 x}{\sin^2 x}\right) dx$$

$$\Rightarrow I = \int (\operatorname{cosec} x \cot x + \cot^2 x) dx$$

$\Rightarrow I = \int (\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x - 1) dx$ [$\because \operatorname{cosec}^2\theta - \cot^2\theta = 1$]

$$\Rightarrow I = \int \operatorname{cosec} x \cot x\, dx + \int \operatorname{cosec}^2 x\, dx - \int dx$$

Recall $\int \mathbf{cosec^2 x\, dx = -\cot x + c}$ and $\int \mathbf{dx = x + c}$

We also have $\int \mathbf{cosec\, x \cot x\, dx = -cosec\, x + c}$

$\therefore I = -\operatorname{cosec} x - \cot x - x + c$

Thus, $\int \frac{\cos x}{1-\cos x} dx = -\operatorname{cosec} x - \cot x - x + c$

28. Question

Evaluate the following integrals:

$$\int \frac{\cos^2 x - \sin^2 x}{\sqrt{1+\cos 4x}} dx$$

Answer

Let $I = \int \frac{\cos^2 x - \sin^2 x}{\sqrt{1+\cos 4x}} dx$

We know $\cos 2\theta = 2\cos^2\theta - 1 = \cos^2\theta - \sin^2\theta$

Hence, in the numerator, we can write $\cos^2 x - \sin^2 x = \cos 2x$

In the denominator, we can write $4x = 2 \times 2x$

$\Rightarrow 1 + \cos 4x = 1 + \cos(2\times 2x)$

$\Rightarrow 1 + \cos 4x = 2\cos^2 2x$

Therefore, we can write the integral as

$$I = \int \frac{\cos 2x}{\sqrt{2\cos^2 2x}} dx$$

$$\Rightarrow I = \int \frac{\cos 2x}{\sqrt{2}\cos 2x} dx$$

$$\Rightarrow I = \int \frac{1}{\sqrt{2}} dx$$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \int dx$$

Recall $\int \mathbf{dx = x + c}$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \times x + c$$

$$\therefore I = \frac{x}{\sqrt{2}} + c$$

Thus, $\int \frac{\cos^2 x - \sin^2 x}{\sqrt{1+\cos 4x}} dx = \frac{x}{\sqrt{2}} + c$

29. Question

Evaluate the following integrals:

$$\int \frac{1}{1-\cos x} dx$$

Answer

Let $I = \int \frac{1}{1-\cos x} dx$

On multiplying and dividing (1 + cos x), we can write the integral as

$$I = \int \frac{1}{1-\cos x}\left(\frac{1+\cos x}{1+\cos x}\right) dx$$

$$\Rightarrow I = \int \frac{1+\cos x}{(1-\cos x)(1+\cos x)} dx$$

$$\Rightarrow I = \int \frac{1+\cos x}{1-\cos^2 x} dx$$

$\Rightarrow I = \int \frac{1+\cos x}{\sin^2 x} dx$ [$\because \sin^2\theta + \cos^2\theta = 1$]

$$\Rightarrow I = \int \left(\frac{1}{\sin^2 x} + \frac{\cos x}{\sin^2 x}\right) dx$$

$$\Rightarrow I = \int \left(\frac{1}{\sin^2 x} + \frac{1}{\sin x} \times \frac{\cos x}{\sin x}\right) dx$$

$$\Rightarrow I = \int (\operatorname{cosec}^2 x + \operatorname{cosec} x \cot x) dx$$

$$\Rightarrow I = \int \operatorname{cosec}^2 x\, dx + \int \operatorname{cosec} x \cot x\, dx$$

Recall $\int \mathbf{cosec^2 x\, dx = -\cot x + c}$

We also have $\int \mathbf{cosec\, x \cot x\, dx = -cosec\, x + c}$

$\therefore I = -\cot x - \operatorname{cosec} x + c$

Thus, $\int \frac{1}{1-\cos x} dx = -\cot x - \operatorname{cosec} x + c$

30. Question

Evaluate the following integrals:

$$\int \frac{1}{1-\sin x} dx$$

Answer

Let $I = \int \frac{1}{1-\sin x} dx$

On multiplying and dividing (1 + sin x), we can write the integral as

$$I = \int \frac{1}{1-\sin x}\left(\frac{1+\sin x}{1+\sin x}\right) dx$$

$$\Rightarrow I = \int \frac{1+\sin x}{(1-\sin x)(1+\sin x)} dx$$

$$\Rightarrow I = \int \frac{1+\sin x}{1-\sin^2 x} dx$$

$\Rightarrow I = \int \frac{1+\sin x}{\cos^2 x} dx$ [∵ $\sin^2\theta + \cos^2\theta = 1$]

$$\Rightarrow I = \int \left(\frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\right) dx$$

$$\Rightarrow I = \int \left(\frac{1}{\cos^2 x} + \frac{1}{\cos x} \times \frac{\sin x}{\cos x}\right) dx$$

$$\Rightarrow I = \int (\sec^2 x + \sec x \tan x) dx$$

$$\Rightarrow I = \int \sec^2 x\, dx + \int \sec x \tan x\, dx$$

Recall $\int \mathbf{\sec^2 x\, dx = \tan x + c}$

We also have $\int \mathbf{\sec x \tan x\, dx = \sec x + c}$

∴ I = tan x + sec x + c

Thus, $\int \frac{1}{1-\sin x} dx = \tan x + \sec x + c$

31. Question

Evaluate the following integrals:

$$\int \frac{\tan x}{\sec x + \tan x} dx$$

Answer

Let $I = \int \frac{\tan x}{\sec x + \tan x} dx$

On multiplying and dividing (sec x – tan x), we can write the integral as

$$I = \int \frac{\tan x}{\sec x + \tan x}\left(\frac{\sec x - \tan x}{\sec x - \tan x}\right) dx$$

$$\Rightarrow I = \int \frac{\tan x (\sec x - \tan x)}{(\sec x + \tan x)(\sec x - \tan x)} dx$$

$$\Rightarrow I = \int \frac{\sec x \tan x - \tan^2 x}{\sec^2 x - \tan^2 x} dx$$

$\Rightarrow I = \int (\sec x \tan x - \tan^2 x) dx$ [∵ $\sec^2\theta - \tan^2\theta = 1$]

$$\Rightarrow I = \int \left(\sec x \tan x - (\sec^2 x - 1)\right) dx$$

$$\Rightarrow I = \int (\sec x \tan x - \sec^2 x + 1) dx$$

$$\Rightarrow I = \int \sec x \tan x \, dx - \int \sec^2 x \, dx + \int dx$$

Recall $\int \mathbf{\sec^2 x \, dx = \tan x + c}$ and $\int \mathbf{dx = x + c}$

We also have $\int \mathbf{\sec x \tan x \, dx = \sec x + c}$

∴ I = sec x – tan x + x + c

Thus, $\int \frac{\tan x}{\sec x + \tan x} dx = \sec x - \tan x + x + c$

32. Question

Evaluate the following integrals:

$$\int \frac{\operatorname{cosec} x}{\operatorname{cosec} x - \cot x} dx$$

Answer

Let $I = \int \frac{\operatorname{cosec} x}{\operatorname{cosec} x - \cot x} dx$

On multiplying and dividing (cosec x + cot x), we can write the integral as

$$I = \int \frac{\operatorname{cosec} x}{\operatorname{cosec} x - \cot x} \left(\frac{\operatorname{cosec} x + \cot x}{\operatorname{cosec} x + \cot x}\right) dx$$

$$\Rightarrow I = \int \frac{\operatorname{cosec} x (\operatorname{cosec} x + \cot x)}{(\operatorname{cosec} x - \cot x)(\operatorname{cosec} x + \cot x)} dx$$

$$\Rightarrow I = \int \frac{\operatorname{cosec}^2 x + \operatorname{cosec} x \cot x}{\operatorname{cosec}^2 x - \cot^2 x} dx$$

$\Rightarrow I = \int (\operatorname{cosec}^2 x + \operatorname{cosec} x \cot x) dx$ [∵ $\operatorname{cosec}^2\theta - \cot^2\theta = 1$]

$$\Rightarrow I = \int \operatorname{cosec}^2 x \, dx + \int \operatorname{cosec} x \cot x \, dx$$

Recall $\int \mathbf{\operatorname{cosec}^2 x \, dx = -\cot x + c}$

We also have $\int \mathbf{\operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c}$

∴ I = –cot x – cosec x + c

Thus, $\int \frac{\operatorname{cosec} x}{\operatorname{cosec} x - \cot x} dx = -\cot x - \operatorname{cosec} x + c$

33. Question

Evaluate the following integrals:

$$\int \frac{1}{1 + \cos 2x} dx$$

Answer

Let $I = \int \frac{1}{1+\cos 2x} dx$

We know $\cos 2\theta = 2\cos^2\theta - 1$

Hence, in the denominator, we can write $1 + \cos 2x = 2\cos^2 x$

Therefore, we can write the integral as

$$I = \int \frac{1}{2\cos^2 x} dx$$

$$\Rightarrow I = \frac{1}{2}\int \frac{1}{\cos^2 x} dx$$

$$\Rightarrow I = \frac{1}{2}\int \sec^2 x \, dx$$

Recall $\int \mathbf{sec^2 x \, dx = tan\, x + c}$

$$\therefore I = \frac{1}{2}\tan x + c$$

Thus, $\int \frac{1}{1+\cos 2x} dx = \frac{1}{2}\tan x + c$

34. Question

Evaluate the following integrals:

$$\int \frac{1}{1-\cos 2x} dx$$

Answer

Let $I = \int \frac{1}{1-\cos 2x} dx$

We know $\cos 2\theta = 1 - 2\sin^2\theta$

Hence, in the denominator, we can write $1 - \cos 2x = 2\sin^2 x$

Therefore, we can write the integral as

$$I = \int \frac{1}{2\sin^2 x} dx$$

$$\Rightarrow I = \frac{1}{2}\int \frac{1}{\sin^2 x} dx$$

$$\Rightarrow I = \frac{1}{2}\int \operatorname{cosec}^2 x \, dx$$

Recall $\int \mathbf{cosec^2 x \, dx = -\cot x + c}$

$$\Rightarrow I = \frac{1}{2}(-\cot x) + c$$

$$\therefore I = -\frac{1}{2}\cot x + c$$

Thus, $\int \frac{1}{1-\cos 2x} dx = -\frac{1}{2}\cot x + c$

35. Question

Evaluate the following integrals:

$$\int \tan^{-1}\left(\frac{\sin 2x}{1+\cos 2x}\right) dx$$

Answer

Let $I = \int \tan^{-1}\left(\frac{\sin 2x}{1+\cos 2x}\right) dx$

We know $\cos 2\theta = 2\cos^2\theta - 1$

Hence, in the denominator, we can write $1 + \cos 2x = 2\cos^2 x$

In the numerator, we have $\sin 2x = 2\sin x\cos x$

Therefore, we can write the integral as

$$I = \int \tan^{-1}\left(\frac{2\sin x\cos x}{2\cos^2 x}\right) dx$$

$$\Rightarrow I = \int \tan^{-1}\left(\frac{\sin x}{\cos x}\right) dx$$

$$\Rightarrow I = \int \tan^{-1}(\tan x)\, dx$$

$$\Rightarrow I = \int x dx$$

Recall $\int \mathbf{x^n dx} = \frac{\mathbf{x^{n+1}}}{\mathbf{n+1}} + \mathbf{c}$

$$\Rightarrow I = \frac{x^{1+1}}{1+1} + c$$

$$\therefore I = \frac{x^2}{2} + c$$

Thus, $\int \tan^{-1}\left(\frac{\sin 2x}{1+\cos 2x}\right) dx = \frac{x^2}{2} + c$

36. Question

Evaluate the following integrals:

$$\int \cos^{-1}(\sin x)\, dx$$

Answer

Let $I = \int \cos^{-1}(\sin x)\, dx$

We know $\sin\theta = \cos(90° - \theta)$

Therefore, we can write the integral as

$$I = \int \cos^{-1}\left[\cos\left(\frac{\pi}{2} - x\right)\right] dx$$

$$\Rightarrow I = \int \left(\frac{\pi}{2} - x\right) dx$$

$$\Rightarrow I = \int \frac{\pi}{2} dx - \int x dx$$

$$\Rightarrow I = \frac{\pi}{2}\int dx - \int x dx$$

Recall $\int \mathbf{x^n dx} = \frac{\mathbf{x^{n+1}}}{\mathbf{n+1}} + \mathbf{c}$ and $\int \mathbf{dx} = \mathbf{x} + \mathbf{c}$

$$\Rightarrow I = \frac{\pi}{2} \times x - \frac{x^{1+1}}{1+1} + c$$

$\therefore I = \frac{\pi x}{2} - \frac{x^2}{2} + c$

Thus, $\int \cos^{-1}(\sin x)\,dx = \frac{\pi x}{2} - \frac{x^2}{2} + c$

37. Question

Evaluate the following integrals:

$$\int \cot^{-1}\left(\frac{\sin 2x}{1-\cos 2x}\right)dx$$

Answer

Let $I = \int \cot^{-1}\left(\frac{\sin 2x}{1-\cos 2x}\right)dx$

We know $\cos 2\theta = 1 - 2\sin^2\theta$

Hence, in the denominator, we can write $1 - \cos 2x = 2\sin^2 x$

In the numerator, we have $\sin 2x = 2\sin x\cos x$

Therefore, we can write the integral as

$$I = \int \cot^{-1}\left(\frac{2\sin x\cos x}{2\sin^2 x}\right)dx$$

$$\Rightarrow I = \int \cot^{-1}\left(\frac{\cos x}{\sin x}\right)dx$$

$$\Rightarrow I = \int \cot^{-1}(\cot x)\,dx$$

$$\Rightarrow I = \int x\,dx$$

Recall $\int \mathbf{x^n dx} = \frac{\mathbf{x^{n+1}}}{\mathbf{n+1}} + \mathbf{c}$

$$\Rightarrow I = \frac{x^{1+1}}{1+1} + c$$

$$\therefore I = \frac{x^2}{2} + c$$

Thus, $\int \cot^{-1}\left(\frac{\sin 2x}{1-\cos 2x}\right)dx = \frac{x^2}{2} + c$

38. Question

Evaluate the following integrals:

$$\int \sin^{-1}\left(\frac{2\tan x}{1+\tan^2 x}\right)dx$$

Answer

Let $I = \int \sin^{-1}\left(\frac{2\tan x}{1+\tan^2 x}\right)dx$

We know $\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$

Therefore, we can write the integral as

$$I = \int \sin^{-1}(\sin 2x)\,dx$$

$\Rightarrow I = \int 2x dx$

$\Rightarrow I = 2\int x dx$

Recall $\int \mathbf{x^n dx} = \frac{\mathbf{x^{n+1}}}{\mathbf{n+1}} + \mathbf{c}$

$\Rightarrow I = 2 \times \frac{x^{1+1}}{1+1} + c$

$\Rightarrow I = 2 \times \frac{x^2}{2} + c$

$\therefore I = x^2 + c$

Thus, $\int \sin^{-1}\left(\frac{2\tan x}{1+\tan^2 x}\right) dx = x^2 + c$

39. Question

Evaluate the following integrals:

$$\int \frac{(x^3+8)(x-1)}{x^2-2x+4} dx$$

Answer

Let $I = \int \frac{(x^3+8)(x-1)}{x^2-2x+4} dx$

We know $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

Hence, in the numerator, we can write

$x^3 + 8 = x^3 + 2^3$

$\Rightarrow x^3 + 8 = (x + 2)(x^2 - x \times 2 + 2^2)$

$\Rightarrow x^3 + 8 = (x + 2)(x^2 - 2x + 4)$

Therefore, we can write the integral as

$$I = \int \frac{(x+2)(x^2-2x+4)(x-1)}{x^2-2x+4} dx$$

$\Rightarrow I = \int (x+2)(x-1) dx$

$\Rightarrow I = \int (x^2 + x - 2) dx$

$\Rightarrow I = \int x^2 dx + \int x dx - \int 2dx$

$\Rightarrow I = \int x^2 dx + \int x dx - 2\int dx$

Recall $\int \mathbf{x^n dx} = \frac{\mathbf{x^{n+1}}}{\mathbf{n+1}} + \mathbf{c}$ and $\int \mathbf{dx} = \mathbf{x} + \mathbf{c}$

$\Rightarrow I = \frac{x^{2+1}}{2+1} + \frac{x^{1+1}}{1+1} - 2 \times x + c$

$\therefore I = \frac{x^3}{3} + \frac{x^2}{2} - 2x + c$

Thus, $\int \frac{(x^3+8)(x-1)}{x^2-2x+4}dx = \frac{x^3}{3} + \frac{x^2}{2} - 2x + c$

40. Question

Evaluate the following integrals:

$$\int (a\tan x + b\cot x)^2 dx$$

Answer

Let $I = \int (a\tan x + b\cot x)^2 dx$

We know $(a + b)^2 = a^2 + 2ab + b^2$

Therefore, we can write the integral as

$$I = \int [(a\tan x)^2 + 2(a\tan x)(b\cot x) + (b\cot x)^2]dx$$

$$\Rightarrow I = \int (a^2\tan^2 x + 2ab\tan x\cot x + b^2\cot^2 x)dx$$

$$\Rightarrow I = \int (a^2\tan^2 x + 2ab + b^2\cot^2 x)dx \left[\because \cot\theta = \frac{1}{\tan\theta}\right]$$

We have $\sec^2\theta - \tan^2\theta = \operatorname{cosec}^2\theta - \cot^2\theta = 1$

$$\Rightarrow I = \int [a^2(\sec^2 x - 1) + 2ab + b^2(\operatorname{cosec}^2 x - 1)]dx$$

$$\Rightarrow I = \int (a^2\sec^2 x - a^2 + 2ab + b^2\operatorname{cosec}^2 x - b^2)dx$$

$$\Rightarrow I = \int (a^2\sec^2 x + b^2\operatorname{cosec}^2 x - a^2 + 2ab - b^2)dx$$

$$\Rightarrow I = \int \left(a^2\sec^2 x + b^2\operatorname{cosec}^2 x - (a^2 - 2ab + b^2)\right)dx$$

$$\Rightarrow I = \int (a^2\sec^2 x + b^2\operatorname{cosec}^2 x - (a-b)^2)dx$$

$$\Rightarrow I = \int a^2\sec^2 x\,dx + \int b^2\operatorname{cosec}^2 x\,dx - \int (a-b)^2 dx$$

$$\Rightarrow I = a^2\int \sec^2 x\,dx + b^2\int \operatorname{cosec}^2 x\,dx - (a-b)^2\int dx$$

Recall $\int \mathbf{\sec^2 x\,dx = \tan x + c}$ and $\int \mathbf{dx = x + c}$

We also have $\int \mathbf{\operatorname{cosec}^2 x\,dx = -\cot x + c}$

$\Rightarrow I = a^2\tan x + b^2(-\cot x) - (a - b)^2 \times x + c$

$\therefore I = a^2\tan x - b^2\cot x - (a - b)^2 x + c$

Thus, $\int (a\tan x + b\cot x)^2 dx = a^2\tan x - b^2\cot x - (a-b)^2 x + c$

41. Question

Evaluate the following integrals:

$$\int \frac{x^3 - 3x^2 + 5x - 7 + x^2 a^x}{2x^2} dx$$

Answer

Let $I = \int \frac{x^3-3x^2+5x-7+x^2a^x}{2x^2}dx$

$\Rightarrow I = \frac{1}{2}\int \frac{x^3-3x^2+5x-7+x^2a^x}{x^2}dx$

$\Rightarrow I = \frac{1}{2}\int \left(\frac{x^3}{x^2}-\frac{3x^2}{x^2}+\frac{5x}{x^2}-\frac{7}{x^2}+\frac{x^2a^x}{x^2}\right)dx$

$\Rightarrow I = \frac{1}{2}\int \left(x-3+\frac{5}{x}-\frac{7}{x^2}+a^x\right)dx$

$\Rightarrow I = \frac{1}{2}\int \left(x-3+\frac{5}{x}-7x^{-2}+a^x\right)dx$

$\Rightarrow I = \frac{1}{2}\left[\int xdx - \int 3dx + \int \frac{5}{x}dx - \int 7x^{-2}dx + \int a^x dx\right]$

$\Rightarrow I = \frac{1}{2}\left[\int xdx - 3\int dx + 5\int \frac{1}{x}dx - 7\int x^{-2}dx + \int a^x dx\right]$

Recall $\int \mathbf{x^n dx} = \frac{x^{n+1}}{n+1} + \mathbf{c}$ and $\int \mathbf{dx} = \mathbf{x} + \mathbf{c}$

We also have $\int \mathbf{a^x dx} = \frac{a^x}{\log a} + \mathbf{c}$ and $\int \frac{1}{x}\mathbf{dx} = \mathbf{\log x} + \mathbf{c}$

$\Rightarrow I = \frac{1}{2}\left[\frac{x^{1+1}}{1+1} - 3\times x + 5\times\log x - 7\left(\frac{x^{-2+1}}{-2+1}\right) + \frac{a^x}{\log a}\right] + c$

$\Rightarrow I = \frac{1}{2}\left[\frac{x^2}{2} - 3x + 5\log x + 7x^{-1} + \frac{a^x}{\log a}\right] + c$

$\therefore I = \frac{1}{2}\left[\frac{x^2}{2} - 3x + 5\log x + \frac{7}{x} + \frac{a^x}{\log a}\right] + c$

Thus, $\int \frac{x^3-3x^2+5x-7+x^2a^x}{2x^2}dx = \frac{1}{2}\left[\frac{x^2}{2} - 3x + 5\log x + \frac{7}{x} + \frac{a^x}{\log a}\right] + c$

42. Question

Evaluate the following integrals:

$\int \frac{\cos x}{1+\cos x}dx$

Answer

Let $I = \int \frac{\cos x}{1+\cos x}dx$

On multiplying and dividing (1 – cos x), we can write the integral as

$I = \int \frac{\cos x}{1+\cos x}\left(\frac{1-\cos x}{1-\cos x}\right)dx$

$\Rightarrow I = \int \frac{\cos x(1-\cos x)}{(1+\cos x)(1-\cos x)}dx$

$\Rightarrow I = \int \frac{\cos x - \cos^2 x}{1-\cos^2 x}dx$

$\Rightarrow I = \int \frac{\cos x - \cos^2 x}{\sin^2 x}dx$ [$\because \sin^2\theta + \cos^2\theta = 1$]

$\Rightarrow I = \int \left(\frac{\cos x}{\sin^2 x} - \frac{\cos^2 x}{\sin^2 x}\right)dx$

$\Rightarrow I = \int \left(\frac{1}{\sin x} \times \frac{\cos x}{\sin x} - \frac{\cos^2 x}{\sin^2 x}\right) dx$

$\Rightarrow I = \int (\operatorname{cosec} x \cot x - \cot^2 x) dx$

$\Rightarrow I = \int (\operatorname{cosec} x \cot x - \operatorname{cosec}^2 x + 1) dx$ [∵ $\operatorname{cosec}^2\theta - \cot^2\theta = 1$]

$\Rightarrow I = \int \operatorname{cosec} x \cot x \, dx - \int \operatorname{cosec}^2 x \, dx + \int dx$

Recall $\int \mathbf{cosec^2 x \, dx = -\cot x + c}$ and $\int \mathbf{dx = x + c}$

We also have $\int \mathbf{cosec\, x \cot x \, dx = -cosec\, x + c}$

$\Rightarrow I = -\operatorname{cosec} x - (-\cot x) + x + c$

$\Rightarrow I = -\operatorname{cosec} x + \cot x + x + c$

Thus, $\int \frac{\cos x}{1+\cos x} dx = -\operatorname{cosec} x + \cot x + x + c$

43. Question

Evaluate the following integrals:

$$\int \frac{1-\cos x}{1+\cos x} dx$$

Answer

Let $I = \int \frac{1-\cos x}{1+\cos x} dx$

We have $\cos x = \cos\left(2 \times \frac{x}{2}\right)$

We know $\cos 2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$

Hence, in the numerator, we can write $1 - \cos x = 2\sin^2\frac{x}{2}$

In the denominator, we can write $1 + \cos x = 2\cos^2\frac{x}{2}$

Therefore, we can write the integral as

$$I = \int \frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}} dx$$

$$\Rightarrow I = \int \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} dx$$

$$\Rightarrow I = \int \tan^2\frac{x}{2} dx$$

$\Rightarrow I = \int \left(\sec^2\frac{x}{2} - 1\right) dx$ [∵ $\sec^2\theta - \tan^2\theta = 1$]

$$\Rightarrow I = \int \sec^2\frac{x}{2} dx - \int dx$$

Recall $\int \mathbf{\sec^2 x \, dx = \tan x + c}$ and $\int \mathbf{dx = x + c}$

$$\Rightarrow I = \frac{\tan\frac{x}{2}}{\frac{1}{2}} - x + c$$

$\therefore I = 2\tan\frac{x}{2} - x + c$

Thus, $\int \frac{1-\cos x}{1+\cos x} dx = 2\tan\frac{x}{2} - x + c$

44. Question

Evaluate the following integrals:

$$\int\left\{3\sin x - 4\cos x + \frac{5}{\cos^2 x} - \frac{6}{\sin^2 x} + \tan^2 x - \cot^2 x\right\}dx$$

Answer

Let $I = \int \left\{3\sin x - 4\cos x + \frac{5}{\cos^2 x} - \frac{6}{\sin^2 x} + \tan^2 x - \cot^2 x\right\} dx$

$$\Rightarrow I = \int \{3\sin x - 4\cos x + 5\sec^2 x - 6\operatorname{cosec}^2 x + \tan^2 x - \cot^2 x\}dx$$

We have $\sec^2\theta - \tan^2\theta = \operatorname{cosec}^2\theta - \cot^2\theta = 1$

$$\Rightarrow I = \int \{3\sin x - 4\cos x + 5\sec^2 x - 6\operatorname{cosec}^2 x + (\sec^2 x - 1) - (\operatorname{cosec}^2 x - 1)\}dx$$

$$\Rightarrow I = \int \{3\sin x - 4\cos x + 5\sec^2 x - 6\operatorname{cosec}^2 x + \sec^2 x - 1 - \operatorname{cosec}^2 x + 1\}dx$$

$$\Rightarrow I = \int \{3\sin x - 4\cos x + 6\sec^2 x - 7\operatorname{cosec}^2 x\}dx$$

$$\Rightarrow I = \int 3\sin x\,dx - \int 4\cos x\,dx + \int 6\sec^2 x\,dx - \int 7\operatorname{cosec}^2 x\,dx$$

$$\Rightarrow I = 3\int \sin x\,dx - 4\int \cos x\,dx + 6\int \sec^2 x\,dx - 7\int \operatorname{cosec}^2 x\,dx$$

Recall $\int \mathbf{\sec^2 x\,dx = \tan x + c}$ and $\int \mathbf{\sin x\,dx = -\cos x + c}$

We also have $\int \mathbf{\operatorname{cosec}^2 x\,dx = -\cot x + c}$ and $\int \mathbf{\cos x\,dx = \sin x + c}$

$\Rightarrow I = 3(-\cos x) - 4(\sin x) + 6(\tan x) - 7(-\cot x) + c$

$\therefore I = -3\cos x - 4\sin x + 6\tan x + 7\cot x + c$

Thus, $\int \left\{3\sin x - 4\cos x + \frac{5}{\cos^2 x} - \frac{6}{\sin^2 x} + \tan^2 x - \cot^2 x\right\} dx = -3\cos x - 4\sin x + 6\tan x + 7\cot x + c$

45. Question

If $f'(x) = x - \frac{1}{x^2}$ and $f(1) = \frac{1}{2}$, find f(x).

Answer

Given $f'(x) = x - \frac{1}{x^2}$ and $f(1) = \frac{1}{2}$

On integrating the given equation, we have

$$\int f'(x)dx = \int \left(x - \frac{1}{x^2}\right)dx$$

We know $\int f'(x)dx = f(x)$

$\Rightarrow f(x) = \int \left(x - \frac{1}{x^2}\right) dx$

$\Rightarrow f(x) = \int (x - x^{-2}) dx$

$\Rightarrow f(x) = \int x dx - \int x^{-2} dx$

Recall $\int \mathbf{x^n dx} = \frac{\mathbf{x^{n+1}}}{\mathbf{n+1}} + \mathbf{c}$

$\Rightarrow f(x) = \frac{x^{1+1}}{1+1} - \frac{x^{-2+1}}{-2+1} + c$

$\Rightarrow f(x) = \frac{x^2}{2} - \frac{x^{-1}}{-1} + c$

$\Rightarrow f(x) = \frac{x^2}{2} + \frac{1}{x} + c$

On substituting x = 1 in f(x), we get

$f(1) = \frac{1^2}{2} + \frac{1}{1} + c$

$\Rightarrow \frac{1}{2} = \frac{1}{2} + 1 + c$

$\Rightarrow 0 = 1 + c$

$\Rightarrow 1 + c = 0$

$\therefore c = -1$

On substituting the value of c in f(x), we get

$f(x) = \frac{x^2}{2} + \frac{1}{x} + (-1)$

$\therefore f(x) = \frac{x^2}{2} + \frac{1}{x} - 1$

Thus, $f(x) = \frac{x^2}{2} + \frac{1}{x} - 1$

46. Question

If f’(x) = x + b, f(1) = 5, f(2) = 13, find f(x).

Answer

Given f’(x) = x + b, f(1) = 5 and f(2) = 13

On integrating the given equation, we have

$\int f'(x) dx = \int (x + b) dx$

We know $\int f'(x) dx = f(x)$

$\Rightarrow f(x) = \int (x + b) dx$

$\Rightarrow f(x) = \int x dx + \int b dx$

$\Rightarrow f(x) = \int x dx + b \int dx$

Recall $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ and $\int dx = x + c$

$\Rightarrow f(x) = \frac{x^{1+1}}{1+1} + b(x) + c$

$\Rightarrow f(x) = \frac{x^2}{2} + bx + c$

On substituting x = 1 in f(x), we get

$f(1) = \frac{1^2}{2} + b(1) + c$

$\Rightarrow 5 = \frac{1}{2} + b + c$

$\Rightarrow 5 - \frac{1}{2} = b + c$

$\Rightarrow b + c = \frac{9}{2}$ ..... (1)

On substituting x = 2 in f(x), we get

$f(2) = \frac{2^2}{2} + b(2) + c$

⇒ 13 = 2 + 2b + c

⇒ 13 – 2 = 2b + c

⇒ 2b + c = 11 ..... (2)

By subtracting equation (1) from equation (2), we have

$(2b + c) - (b + c) = 11 - \frac{9}{2}$

$\Rightarrow 2b + c - b - c = \frac{13}{2}$

$\therefore b = \frac{13}{2}$

On substituting the value of b in equation (1), we get

$\frac{13}{2} + c = \frac{9}{2}$

$\Rightarrow c = \frac{9}{2} - \frac{13}{2}$

∴ c = –2

On substituting the values of b and c in f(x), we get

$f(x) = \frac{x^2}{2} + \frac{13}{2}x + (-2)$

$\therefore f(x) = \frac{x^2}{2} + \frac{13}{2}x - 2$

Thus, $f(x) = \frac{x^2}{2} + \frac{13}{2}x - 2$

47. Question

If f’(x) = $8x^3$ – 2x, f(2) = 8, find f(x).

Answer

Given f'(x) = $8x^3 - 2x$ and f(2) = 8

On integrating the given equation, we have

$$\int f'(x)dx = \int (8x^3 - 2x)dx$$

We know $\int f'(x)dx = f(x)$

$$\Rightarrow f(x) = \int (8x^3 - 2x)dx$$

$$\Rightarrow f(x) = \int 8x^3dx - \int 2xdx$$

$$\Rightarrow f(x) = 8\int x^3dx - 2\int xdx$$

Recall $\int \mathbf{x^n dx} = \frac{\mathbf{x^{n+1}}}{\mathbf{n+1}} + \mathbf{c}$

$$\Rightarrow f(x) = 8\left(\frac{x^{3+1}}{3+1}\right) - 2\left(\frac{x^{1+1}}{1+1}\right) + c$$

$$\Rightarrow f(x) = 8\left(\frac{x^4}{4}\right) - 2\left(\frac{x^2}{2}\right) + c$$

$\Rightarrow$ f(x) = $2x^4 - x^2 + c$

On substituting x = 2 in f(x), we get

f(2) = $2(2^4) - 2^2 + c$

$\Rightarrow$ 8 = 32 – 4 + c

$\Rightarrow$ 8 = 28 + c

$\therefore$ c = –20

On substituting the value of c in f(x), we get

f(x) = $2x^4 - x^2 + (-20)$

$\therefore$ f(x) = $2x^4 - x^2 - 20$

Thus, f(x) = $2x^4 - x^2 - 20$

48. Question

If f'(x) = a sin x + b cos x and f'(0) = 4, f(0) = 3, $f\left(\frac{\pi}{2}\right) = 5$, find f(x).

Answer

Given f'(x) = a sin x + b cos x and f'(0) = 4

On substituting x = 0 in f'(x), we get

f'(0) = asin0 + bcos0

$\Rightarrow$ 4 = a × 0 + b × 1

$\Rightarrow$ 4 = 0 + b

$\therefore$ b = 4

Hence, f'(x) = a sin x + 4 cos x

On integrating this equation, we have

$\int f'(x)dx = \int (a\sin x + 4\cos x)dx$

We know $\int f'(x)dx = f(x)$

$\Rightarrow f(x) = \int (a\sin x + 4\cos x)dx$

$\Rightarrow f(x) = \int a\sin x\,dx + \int 4\cos x\,dx$

$\Rightarrow f(x) = a\int \sin x\,dx + 4\int \cos x\,dx$

Recall $\int \mathbf{\sin x\,dx = -\cos x + c}$ and $\int \mathbf{\cos x\,dx = \sin x + c}$

$\Rightarrow f(x) = a(-\cos x) + 4(\sin x) + c$

$\Rightarrow f(x) = -a\cos x + 4\sin x + c$

On substituting x = 0 in f(x), we get

f(0) = –acos0 + 4sin0 + c

⇒ 3 = -a × 1 + 4 × 0 + c

⇒ 3 = -a + c

⇒ c – a = 3 -------------- (1)

On substituting $x = \frac{\pi}{2}$ in f(x), we get

$f\left(\frac{\pi}{2}\right) = -a\cos\frac{\pi}{2} + 4\sin\frac{\pi}{2} + c$

⇒ 5 = -a × 0 + 4 × 1 + c

⇒ 5 = 0 + 4 + c

⇒ 5 = 4 + c

∴ c = 1

On substituting c = 1 in equation (1), we get

1 – a = 3

⇒ a = 1 – 3

∴ a = –2

On substituting the values of c and a in f(x), we get

f(x) = –(–2)cos x + 4sinx + 1

∴ f(x) = 2cosx + 4sinx + 1

Thus, f(x) = 2cosx + 4sinx + 1

49. Question

Write the primitive or anti-derivative of $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$.

Answer

Given $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$

Let $I = \int f(x)dx$

$$\Rightarrow I=\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx$$

$$\Rightarrow I=\int\left(x^{\frac{1}{2}}+\frac{1}{x^{\frac{1}{2}}}\right)dx$$

$$\Rightarrow I=\int\left(x^{\frac{1}{2}}+x^{-\frac{1}{2}}\right)dx$$

$$\Rightarrow I=\int x^{\frac{1}{2}}dx+\int x^{-\frac{1}{2}}dx$$

Recall $\int \mathbf{x^n dx}=\frac{\mathbf{x^{n+1}}}{\mathbf{n+1}}+\mathbf{c}$

$$\Rightarrow I=\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}+\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c$$

$$\Rightarrow I=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+\frac{x^{\frac{1}{2}}}{\frac{1}{2}}+c$$

$$\Rightarrow I=\frac{2}{3}x^{\frac{3}{2}}+2x^{\frac{1}{2}}+c$$

$$\therefore I=\frac{2}{3}x\sqrt{x}+2\sqrt{x}+c$$

Thus, the primitive of f(x) is $\frac{2}{3}x\sqrt{x}+2\sqrt{x}+c$

Exercise 19.3

1. Question

Evaluate: $\int (2x-3)^5+\sqrt{3x+2}\,dx$

Answer

Let I = $\int (2x-3)^5+\sqrt{3x+2}$ then,

I = $\int (2x-3)^5+(3x+2)^{\frac{1}{2}}$

$$=\frac{(2x-3)^{5+1}}{2(5+1)}+\frac{(3x+2)^{\frac{1}{2}+1}}{3(\frac{1}{2}+1)}$$

$$=\frac{(2x-3)^{6}}{2(6)}+\frac{(3x+2)^{\frac{3}{2}}}{3\left(\frac{3}{2}\right)}$$

$$=\frac{(2x-3)^{6}}{12}+\frac{2(3x+2)^{\frac{3}{2}}}{9}$$

Hence, I = $\frac{(2x-3)^{6}}{12}+\frac{2(3x+2)^{\frac{3}{2}}}{9}+C$

2. Question

Evaluate: $\int \frac{1}{(7x-5)^3}+\frac{1}{\sqrt{5x-4}}dx$

Answer

Let I $=\int \frac{1}{(7x-5)^3}+\frac{1}{\sqrt{5x-4}}dx$ then,

I$=\int (7x-5)^{-3}+(5x-4)^{-\frac{1}{2}}$

$=\frac{(7x-5)^{-3+1}}{7(-3+1)}+\frac{(5x-4)^{-\frac{1}{2}+1}}{5\left(-\frac{1}{2}+1\right)}$

$=\frac{(7x-5)^{-2}}{-14}+\frac{(5x-4)^{\frac{1}{2}}}{5\left(\frac{1}{2}\right)}$

Hence, I $=-\frac{1}{14}(7x-5)^{-2}+\frac{2}{5}\sqrt{5x-4}$ +C

3. Question

Evaluate: $\int \frac{1}{2-3x}+\frac{1}{\sqrt{3x-2}}dx$

Answer

Let I $=\int \frac{1}{2-3x}+\frac{1}{\sqrt{3x-2}}dx$

I$=\int \frac{1}{2-3x}+\frac{1}{\sqrt{3x-2}}dx$

We know $\int \frac{1}{x}dx=\log|x|$+C

$=\frac{\log|2-3x|}{-3}+\frac{2}{3}(3x-2)^{\frac{1}{2}}$

$=-\frac{1}{3}x.\log|2x-3|+\frac{2}{3}\sqrt{3x-3}+C$

4. Question

Evaluate: $\int \frac{x+3}{(x+1)^4}dx$

Answer

Let I $=\int \frac{x+3}{(x+1)^4}dx$

I $=\int \frac{x+3}{(x+1)^4}dx$

$=\int \frac{x+1}{x+1^4}dx+\int \frac{2}{(x+1)^4}dx$

$=\int \frac{1}{(x+1)^3}dx+\int \frac{2}{(x+1)^4}dx$

$=\int (x+1)^{-3}\,dx+\int 2(x+1)^{-4}\,dx$

$=\frac{[x+1]^{-3+1}}{-3+1}+\frac{2(x+1)^{-4+1}}{-4+1}$

$=\frac{[x+1]^{-2}}{-2}+\frac{2(x+1)^{-3}}{-3}$

Hence, I $=-\frac{1}{2(x+1)^2}-\frac{2}{3(x+1)^3}+C$

5. Question

Evaluate: $\int \frac{1}{\sqrt{x+1}+\sqrt{x}} dx$

Answer

Let I $=\int \frac{1}{\sqrt{x+1}+\sqrt{x}} dx$

$= \int \frac{1}{\sqrt{x+1}+\sqrt{x}} dx$

Now Multiply with the conjugate, we get

$= \int \frac{1}{\sqrt{x+1}+\sqrt{x}} \cdot \frac{\sqrt{x+1}-\sqrt{x}}{\sqrt{x+1}-\sqrt{x}} dx$

$= \int \frac{\sqrt{x+1}-\sqrt{x}}{x+1-x} dx$

$= \int \sqrt{x+1} - \sqrt{x}$ dx

$= \int (x+1)^{\frac{1}{2}} - x^{\frac{1}{2}}$

$= \frac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}}$

Hence I= $\frac{2}{3}(x+1)^{\frac{3}{2}} - \frac{2}{3}(x)^{\frac{3}{2}}$+C

6. Question

Evaluate: $\int \frac{1}{\sqrt{2x+3}+\sqrt{2x-3}} dx$

Answer

Let I = $\int \frac{1}{\sqrt{2x+3}+\sqrt{2x-3}}$ dx

I = $\int \frac{1}{\sqrt{2x+3}+\sqrt{2x-3}}$ dx

Now, Multiply with the conjugate, we get

$= \int \frac{1}{\sqrt{2x+3}+\sqrt{2x-3}} \times \frac{(\sqrt{2x+3}-\sqrt{2x-3})}{\sqrt{2x+3}-\sqrt{2x-3}} dx$

$=\int \frac{(\sqrt{2x+3}-\sqrt{2x-3})}{(\sqrt{2x+3})^2-(\sqrt{2x-3})^2} dx$

$= \int \frac{(\sqrt{2x+3}-\sqrt{2x-3})}{2x+3-2x+3} dx$

$= \int \frac{\sqrt{2x+3}}{6} dx - \int \frac{\sqrt{2x-3}}{6} dx$

$= \frac{1}{6}\int (2x+3)^{\frac{1}{2}} dx - \frac{1}{6}\int (2x-3)^{\frac{1}{2}} dx$

$= \frac{1}{6}\left(\frac{2x+3}{2}\right)^{\frac{1}{2}+1} - \frac{1}{6}[\frac{2x-3}{2}]^{\frac{1}{2}+1}$

$= \frac{1}{6}\left(\frac{2x+3}{2\times\frac{3}{2}}\right)^{\frac{3}{2}} - \frac{1}{6}\left(\frac{2x-3}{2\times\frac{3}{2}}\right)^{\frac{3}{2}}$

Hence, I= $\frac{1}{18}(2x+3)^{\frac{3}{2}} - \frac{1}{18}(2x-3)^{\frac{2}{3}} + C$

7. Question

Evaluate: $\int \frac{2x}{(2x+1)^2} dx$

Answer

Let I = $\int \frac{2x}{(2x+1)^2} dx$

$= \int \frac{2x+1-1}{(2x+1)^2} dx$

$= \int \frac{2x+1}{(2x+1)^2} - \frac{1}{(2x+1)^2} dx$

$= \int \frac{1}{(2x+1)} - (2x+1)^{-2} dx$

$= \frac{1}{2}\log|2x+1| - \frac{(2x+1)^{-2+1}}{-2+1(2)}$

$= \frac{1}{2}\log|2x+1| - \frac{(2x+1)^{-1}}{-2}$

Hence, I= $\frac{1}{2}\log|2x+1| + \frac{1}{2(2x+1)}$+C

8. Question

Evaluate: $\int \frac{1}{\sqrt{x+a}+\sqrt{x+b}} dx$

Answer

Let I = $\int \frac{1}{\sqrt{x+a}+\sqrt{x+b}}$ dx

$= \int \frac{1}{\sqrt{x+a}+\sqrt{x+b}}$ dx

Now, Multiply with conjugate, we get

$= \int \frac{1}{\sqrt{x+a}+\sqrt{x+b}} \times \frac{(\sqrt{x+a}-\sqrt{x+b})}{\sqrt{x+a}-\sqrt{(x+b)}}$ dx

$= \int \frac{(\sqrt{x+a}-\sqrt{x+b})}{(\sqrt{x+a})^2-\sqrt{(x+b)}^2}$dx

$= \int \frac{(\sqrt{x+a}-\sqrt{x+b})}{a-b}$dx

$= \frac{1}{a-b}\left[\frac{2}{3}(x+a)^{\frac{3}{2}} - \frac{2}{3}(x+b)^{\frac{3}{2}}\right]$

Hence, I= $\frac{2}{3(a-b)}\left[(x+a)^{\frac{3}{2}} - (x+b)^{\frac{3}{2}}\right]$+C

9. Question

Evaluate: $\int \sin x\sqrt{1+\cos 2x}\, dx$

Answer

Let I = $\int \sin x \sqrt{(1+\cos 2x)} dx$

$= \int \sin x \sqrt{(1+\cos 2x)} dx$

$= \int \sin x \sqrt{2\cos^2 x} dx$

$= \int \sin x \sqrt{2} \cos x \, dx$

$= \sqrt{2} \int \sin x \ \cos x \, dx$

Now, Multiply and Divide by 2 we get,

$= \frac{\sqrt{2}}{2} \int 2 \sin x \ \cos x \, dx$

$= \frac{\sqrt{2}}{2} \int \ \sin 2x \, dx$

$= \frac{\sqrt{2}}{2} \frac{-\cos 2x}{2}$

Hence, I= $-\frac{1}{2\sqrt{2}} \cos 2x + C$

10. Question

Evaluate: $\int \frac{1+\cos x}{1-\cos x} dx$

Answer

Let I $= \int \frac{1+\cos x}{1-\cos x} dx$

$\Rightarrow \int \frac{1+\cos x}{1-\cos x} dx$

$\Rightarrow \int \frac{2\cos^2 \frac{x}{2}}{2\sin^2 \frac{x}{2}} dx$

$\Rightarrow \int \cot^2 \frac{x}{2} dx$

$\Rightarrow \int \left(\operatorname{cosec}^2 \frac{x}{2} - 1\right) dx$

$\Rightarrow \frac{\left(-\cot \frac{x}{2}\right)}{\frac{1}{2}} - x$

Hence, I= $-2 \cot \frac{x}{2} - x + C$

11. Question

Evaluate: $\int \frac{1-\cos x}{1+\cos x} dx$

Answer

Let I $= \int \frac{(1-\cos x)}{(1+\cos x)} dx$

$= \int \frac{(1-\cos x)}{(1+\cos x)} dx$

$= \int \frac{\left(2\sin^2 \frac{x}{2}\right)}{2\cos^2 \frac{x}{2}}$

$= \int \tan^2 \frac{x}{2} \, dx$

$= \int (\sec^2 \frac{x}{2} - 1) \, dx$

$= \frac{\left(\tan\frac{x}{2}\right)}{\frac{1}{2}} - x$

Hence, I= $2\tan\frac{x}{2} - x + C$

12. Question

Evaluate: $\int \frac{1}{1-\sin\frac{x}{2}} dx$

Answer

Let I = $\frac{1}{1-\sin\frac{x}{2}} dx$

$= \frac{1}{1-\sin\frac{x}{2}} dx$

Now, Multiply with the conjugate we get,

$= \int \frac{1}{1-\sin\frac{x}{2}} \times \frac{1+\sin\frac{x}{2}}{1+\sin\frac{x}{2}} dx$

$= \int \frac{1+\sin\frac{x}{2}}{1-\sin^2\frac{x}{2}} dx$

$= \int \frac{1+\sin\frac{x}{2}}{\cos^2\frac{x}{2}} dx$

$= \int \frac{1}{\cos^2\frac{x}{2}} dx + \int \frac{\sin\frac{x}{2}}{\cos^2\frac{x}{2}} dx$

$= \int \sec^2\frac{x}{2} dx + \int \tan\frac{x}{2} . \sec\frac{x}{2} dx$

$= \frac{\left(\tan\frac{x}{2}\right)}{\frac{1}{2}} + \frac{\left(\sec\frac{x}{2}\right)}{\frac{1}{2}}$

Hence, I= $2\tan\frac{x}{2} + 2\sec\frac{x}{2} + C$

13. Question

Evaluate: $\int \frac{1}{1+\cos 3x} dx$

Answer

Let I = $\int \frac{1}{1+\cos 3x} dx$

$= \int \frac{1}{1+\cos 3x} dx$

Now Multiply with Conjugate,

$= \int \frac{1}{1+\cos 3x} \times \frac{1-\cos 3x}{1-\cos 3x} dx$

$= \int \frac{1-\cos 3x}{1-\cos^2 3x} dx$

$= \int \frac{1-\cos 3x}{\sin^2 3x} dx$

$= \int \frac{1}{\sin^2 3x} dx - \int \frac{\cos 3x}{\sin^2 3x} dx$

$= \int (\text{cosec}^2 3x - \text{cosec}\, 3x.\cot 3x)\, dx$

$= -\frac{\cot 3x}{3} + \frac{\text{cosec}\, 3x}{3}$

$= -\frac{1}{3}.\frac{\cos 3x}{\sin 3x} + \frac{1}{3}.\frac{1}{\sin 3x}$

Hence, I= $\frac{1-\cos 3x}{3\sin 3x}$ +C

14. Question

Evaluate:∫ $(e^x + 1)^2 e^x$ dx

Answer

Let I = ∫ $(e^x + 1)^2 e^x$ dx

Let e^x +1= t = e^x dx = dt

I = ∫ $(e^x + 1)^2 e^x$ dx

= ∫ t^2 dt

$= \frac{t^3}{3}$

Now, substitute the value of t

Hence, I= $\frac{(e^x+1)^3}{3}$+C

15. Question

Evaluate: $\int \left(e^x + \frac{1}{e^x}\right)^2 dx$

Answer

Let I = $\int \left(e^x + \frac{1}{e^x}\right)^2$

$= \int \left(e^{2x} + \frac{1}{e^{2x}} + 2\right)$

$= \frac{e^{2x}}{2} - \frac{1}{2}e^{-2x} + 2x$

Hence, I = $\frac{1}{2}e^x + 2x - \frac{1}{2}e^{-2x}$+C

16. Question

Evaluate: $\int \frac{1+\cos 4x}{\cot x - \tan x} dx$

Answer

Let I = $\int \frac{1+\cos 4x}{\cot x - \tan x}$dx

$= \int \frac{1+\cos 4x}{\cot x - \tan x}$dx

$= \int \frac{1+\cos^2 2x}{\frac{\cos x}{\sin x} - \frac{\sin x}{\cos x}}$dx

$= \int \frac{2\cos^2 2x}{\frac{\cos^2 x - \sin^2 x}{\sin x \cos x}}$dx

$= \int \frac{2\cos^2 2x.\sin x.\cos x}{\cos^2 x - \sin^2 x} dx$

$= \int \frac{\cos^2 2x.\sin 2x.}{\cos^2 2x} dx$

$= \int \cos 2x. \sin 2x dx$

$= \frac{1}{2} \int [2 \sin 2x \cos 2x]\ dx$

$= \frac{1}{2} \int \sin(2x + 2x) + \sin(2x - 2x)\ dx$

$= \frac{1}{2} \int \sin 4x + 0\ dx$

$= \frac{1}{2} - \frac{\cos 4x}{4}$

Hence, $I = -\frac{1}{8} \cos 4x + C$

17. Question

Evaluate: $\int \frac{1}{\sqrt{x+3} - \sqrt{x+2}} dx$

Answer

Let $I = \int \frac{1}{\sqrt{x+3} - \sqrt{x+2}} dx$

$= \int \frac{1}{\sqrt{x+3} - \sqrt{x+2}} dx$

Now, Multiply with the conjugate

$= \int \frac{1}{\sqrt{x+3} - \sqrt{x+2}} \times \frac{\sqrt{x+3} + \sqrt{x+2}}{\sqrt{x+3} + \sqrt{x+2}} dx$

$= \int \frac{\sqrt{x+3} + \sqrt{x+2}}{(\sqrt{x+3})^2 - (\sqrt{x+2})^2} dx$

$= \int \frac{\sqrt{x+3} + \sqrt{x+2}}{x+3-x-2} dx$

$= \int (x+3)^{\frac{1}{2}} + (x+2)^{\frac{1}{2}} dx$

$= \frac{(x+3)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{(x+2)^{\frac{3}{2}}}{\frac{3}{2}}$

Hence, $I = \frac{2}{3}(x+3)^{\frac{3}{2}} + \frac{2}{3}(x+2)^{\frac{3}{2}} + C$

18. Question

$\int \tan^2(2x - 3) dx$

Answer

Let $I = \int \tan^2(2x - 3) dx$

$= \int \tan^2(2x - 3) dx$

$= \int \sec^2(2x - 3) - 1\ dx$

Let 2x -3 = t dx = dt/2

$= \frac{1}{2}\int \sec^2 t - 1\ dt$

$= \frac{1}{2}$ tan t –x

Substitute the value of t

Hence, I= $\frac{1}{2}\tan(2x - 3) - x$+C

19. Question

Evaluate: $\int \frac{1}{\cos^2 x(1 - \tan x)^2} dx$

Answer

Let I = $\int \frac{1}{\cos^2 x\,(1-\tan x)^2} dx$

$= \int \frac{1}{\cos^2 x\,(1-\tan x)^2} dx$

$= \int \frac{1}{\cos^2 x\left(1-\frac{\sin x}{\cos x}\right)^2} dx$

$= \int \frac{1}{(\cos x - \sin x)^2} dx$

$= \int \frac{1}{1-\sin 2x} dx$

$= \int \frac{1}{1+\cos\left(\frac{\pi}{2}+2x\right)} dx$

$= \int \frac{1}{2\cos^2\left(\frac{\pi}{4}+x\right)} dx$

$= \frac{1}{2}\int \sec^2\left(\frac{\pi}{4}+x\right) dx$

Hence, I = $\frac{1}{8}\left[\tan\left(\frac{\pi}{4}+x\right)\right] + 1$+C

Exercise 19.4

1. Question

Evaluate: $\int \frac{x^2 + 5x + 2}{x + 2} dx$

Answer

By doing long division of the given equation we get

Quotient = x + 3

Remainder = –4

∴ We can write the above equation as

⇒ x + 3$-\frac{4}{x+2}$

∴ The above equation becomes

⇒ $\int x + 3 - \frac{4}{x+2}\ dx$

⇒ $\int x\,dx + 3\int dx - 4\int \frac{1}{x+2} dx$

We know $\int x\,dx = \frac{x^n}{n+1}$; $\int \frac{1}{x}dx = \ln x$

$\Rightarrow \frac{x^2}{2} + 3x - 4\ln(x+2)$ +c. (Where c is some arbitrary constant)

2. Question

Evaluate: $\int \frac{x^3}{x-2}dx$

Answer

By doing long division of the given equation we get

Quotient = x^2+2x+4

Remainder = 8

∴ We can write the above equation as

$\Rightarrow x^2+2x+4+\frac{8}{x-2}$

∴ The above equation becomes

$\Rightarrow \int x^2 + 2x + 4 + \frac{8}{x-2}\,dx$

$\Rightarrow \int x^2\,dx + 2\int x\,dx + 4\int dx + 8\int \frac{1}{x-2}dx$

We know $\int x\,dx = \frac{x^n}{n+1}$; $\int \frac{1}{x}dx = \ln x$

$\Rightarrow \frac{x^3}{3} + 2\frac{x^2}{2} + 4x + 8\ln(x-2) + c$

$\Rightarrow \frac{x^3}{3} + x^2 + 4x + 8\ln(x-2) + c$. (Where c is some arbitrary constant)

3. Question

Evaluate: $\int \frac{x^2+x+5}{3x+2}dx$

Answer

By doing long division of the given equation we get

Quotient $= \frac{x}{3} + \frac{1}{9}$

Remainder $= \frac{43}{9}$

∴ We can write the above equation as

$\Rightarrow \frac{x}{3} + \frac{1}{9} + \frac{43}{9}\left(\frac{1}{3x+2}\right)$

∴ The above equation becomes

$\Rightarrow \int \frac{x}{3} + \frac{1}{9} + \frac{43}{9}\left(\frac{1}{3x+2}\right)dx$

$\Rightarrow \frac{1}{3}\int x dx + \frac{1}{9}\int dx + \frac{43}{9}\int \frac{1}{3x+2}dx$

We know $\int x\,dx = \frac{x^n}{n+1}$; $\int \frac{1}{x}dx = \ln x$

$\Rightarrow \frac{1}{3} \times \frac{x^3}{2} + \frac{1}{9} \times \frac{x^2}{2} + \frac{43}{9}\ln(3x+2) + c$

$\Rightarrow \frac{x^3}{6}+\frac{x^2}{18}+\frac{43}{9}\ln(3x+2)+c$. (Where c is some arbitrary constant)

4. Question

Evaluate: $\int \frac{2x+3}{(x-1)^2}dx$

Answer

The above equation can be written as

$\Rightarrow \int \frac{2x-2+2+3}{(x-1)^2}$

$\Rightarrow \int \frac{2(x-1)+5}{(x-1)^2}$

$\Rightarrow 2\int \frac{1.dx}{(x-1)}+5\int \frac{1.dx}{(x-1)^2}$

We know $\int x\,dx = \frac{x^n}{n+1}; \int \frac{1}{x}dx = \ln x$

$\Rightarrow 2\ln(x-1)+5\int (x-1)^{-2}dx$

$\Rightarrow 2\ln(x-1)+5\int \frac{(x-1)^{-1}}{-1}dx$

$\Rightarrow 2\ln(x-1)-\frac{5}{(x-1)}$ + c. (Where c is an arbitrary constant)

5. Question

Evaluate: $\int \frac{x^2+3x-1}{(x+1)^2}dx$

Answer

$\Rightarrow \int \frac{x^2+x+2x-1}{(x+1)^2}dx$

$\Rightarrow \int \frac{x(x+1)+2x-1}{(x+1)^2}dx$

$\Rightarrow \int \frac{x(x+1)}{(x+1)^2}dx+\int \frac{2x-1}{(x+1)^2}dx$

$\Rightarrow \int \frac{x}{x+1}dx+\int \frac{2x+2-2-1}{(x+1)^2}dx$

$\Rightarrow \int \frac{x+1-1}{x+1}dx+\int \frac{2(x+1)-3}{(x+1)^2}dx$

$\Rightarrow \int dx-\int \frac{1}{x+1}dx+\int \frac{2}{x+1}dx-\int \frac{3}{(x+1)^2}dx$

We know $\int x\,dx = \frac{x^n}{n+1}; \int \frac{1}{x}dx = \ln x$

$\Rightarrow x-\ln(x+1)+2\ln(x+1)-\int 3(x+1)^{-2}dx$

$\Rightarrow x-\ln(x+1)+2\ln(x+1)+\frac{3}{x+1}+c$

$\Rightarrow x+\ln(x+1)+\frac{3}{x+1}+c$. (Where c is some arbitrary constant)

6. Question

Evaluate: $\int \frac{2x-1}{(x-1)^2} dx$

Answer

In this question degree of denominator is larger than that of numerator so we need to manipulate numerator.

$\Rightarrow \int \frac{2x+2-2-1}{(x-1)^2}$

$\Rightarrow \int \frac{2(x-1)-1}{(x-1)^2}$

$\Rightarrow \int \frac{2}{x-1} dx - \frac{1}{(x-1)^2} dx$

We know $\int x\, dx = \frac{x^n}{n+1}; \int \frac{1}{x} dx = \ln x$

$\Rightarrow 2\ln(x-1) - \int (x-1)^{-2} dx$

$\Rightarrow 2\ln(x-1) - \frac{1}{x-1} + c.$ (where c is some arbitrary constant)

Exercise 19.5

1. Question

Evaluate: $\int \frac{x+1}{\sqrt{2x+3}} dx$

Answer

In these questions, little manipulation makes the questions easier to solve

Here multiply and divide by 2 we get

$\Rightarrow \frac{1}{2}\int \frac{2x+2}{\sqrt{2x+3}} dx$

Add and subtract 1 from the numerator

$\Rightarrow \frac{1}{2}\int \frac{2x+2+1-1}{\sqrt{2x+3}} dx$

$\Rightarrow \frac{1}{2}\int \frac{2x+3-1}{\sqrt{2x+3}} dx$

$\Rightarrow \frac{1}{2}\int \frac{2x+3}{\sqrt{2x+3}} dx - \frac{1}{2}\int \frac{1}{\sqrt{2x+3}} dx$

$\Rightarrow \frac{1}{2}\left(\int \sqrt{2x+3}\, dx - \int (2x+3)^{\frac{-1}{2}} dx\right)$

$\Rightarrow \frac{1}{2} \times \frac{(2x+3)^{\frac{3}{2}}}{2\times\frac{3}{2}} - \frac{1}{2} \times \frac{(2x+3)^{\frac{1}{2}}}{2\times\frac{1}{2}} + c$

$\Rightarrow \frac{(2x+3)^{\frac{3}{2}}}{6} - \frac{(2x+3)^{\frac{1}{2}}}{2} + c$

2. Question

Evaluate: $\int x\sqrt{x+2}\, dx$

Answer

Here Add and subtract 2 from x

We get

$\Rightarrow \int (x + 2 - 2)\sqrt{x + 2}dx$

$\Rightarrow \int (x + 2)^{\frac{3}{2}}dx - \int 2\sqrt{x + 2}dx$

$\Rightarrow \frac{2(x+2)^{\frac{5}{2}}}{5} - \frac{4(x+2)^{\frac{3}{2}}}{3} + c$

3. Question

Evaluate: $\int \frac{x-1}{\sqrt{x+4}}dx$

Answer

In these questions, little manipulation makes the questions easier to solve

Add and subtract 5 from the numerator

$\Rightarrow \int \frac{x+5-5-1}{\sqrt{x+4}}dx$

$\Rightarrow \int \frac{x+4-5}{\sqrt{x+4}}dx$

$\Rightarrow \int \frac{x+4}{\sqrt{x+4}}dx - \int \frac{5}{\sqrt{x+4}}dx$

$\Rightarrow \left(\int \sqrt{x + 4}\,dx - 5\int (x + 4)^{\frac{-1}{2}}dx\right)$

$\Rightarrow \frac{(x+4)^{\frac{3}{2}}}{\frac{3}{2}} - 5 \times \frac{(x+4)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$\Rightarrow \frac{2(x+4)^{\frac{3}{2}}}{3} - 10(x + 4)^{\frac{1}{2}} + c$

4. Question

Evaluate: $\int (x+2)\sqrt{3x+5}\ dx$

Answer

Here multiply and divide the question by 3

We get

$\Rightarrow \frac{1}{3}\int 3(x + 2)\sqrt{3x + 5}\,dx$

$\Rightarrow \frac{1}{3}\int (3x + 6)\sqrt{3x + 5}\,dx$

Add and subtract 1 from above equation

$\Rightarrow \frac{1}{3}\int (3x + 6 + 1 - 1)\sqrt{3x + 5}\,dx$

$\Rightarrow \frac{1}{3}\int (3x + 5 - 1)\sqrt{3x + 5}\,dx$

$\Rightarrow \frac{1}{3}\int (3x + 5)^{\frac{3}{2}}dx - \int \frac{1}{3}\sqrt{3x + 5}dx$

$\Rightarrow \frac{1}{3} \times \frac{2(3x+5)^{\frac{5}{2}}}{3\times5} - \frac{2(3x+5)^{\frac{3}{2}}}{3\times3} + c$

$\Rightarrow \frac{2(3x+5)^{\frac{5}{2}}}{45} - \frac{2(3x+5)^{\frac{3}{2}}}{9} + c$

5. Question

Evaluate: $\int \frac{2x+1}{\sqrt{3x+2}} dx$

Answer

Let 2x + 1 = λ(3x + 2) + μ

2x + 1 = 3xλ + 2λ + μ

comparing coefficients we get

3λ = 2 ; 2λ + μ = 1

$\Rightarrow \lambda = \frac{2}{3}; \mu = \frac{-1}{3}$

Replacing 2x + 1 by λ(3x + 2) + μ in the given equation we get

$\Rightarrow \int \frac{\lambda(3x+2)+\mu}{\sqrt{3x+2}} dx$

$\Rightarrow \lambda \int \frac{3x+2}{\sqrt{3x+2}} dx + \mu \int \frac{1}{\sqrt{3x+2}} dx$

$\Rightarrow \left(\lambda \int \sqrt{3x+2}\, dx - \mu \int (3x+2)^{\frac{-1}{2}} dx\right)$

$\Rightarrow \frac{2}{3} \times \frac{(3x+2)^{\frac{3}{2}}}{3\times\frac{3}{2}} - \frac{1}{3} \times \frac{(3x+2)^{\frac{1}{2}}}{3\times\frac{1}{2}} + c$

$\Rightarrow \frac{4(3x+2)^{\frac{3}{2}}}{27} - \frac{2(3x+2)^{\frac{1}{2}}}{9} + c$

6. Question

Evaluate: $\int \frac{3x+5}{\sqrt{7x+9}} dx$

Answer

Let 3x + 5 = λ(7x + 9) + μ

3x + 5 = 7xλ + 9λ + μ

comparing coefficients, we get

7λ = 3 ; 9λ + μ = 1

$\Rightarrow \lambda = \frac{3}{7}; \mu = \frac{8}{7}$

Replacing 3x + 5 by λ(7x + 9) + μ in the given equation we get

$\Rightarrow \int \frac{\lambda(7x+9)+\mu}{\sqrt{7x+9}} dx$

$\Rightarrow \lambda \int \frac{7x+9}{\sqrt{7x+9}} dx + \mu \int \frac{1}{\sqrt{7x+9}} dx$

$\Rightarrow \left(\lambda \int \sqrt{7x+9}\, dx + \mu \int (7x+9)^{\frac{-1}{2}} dx\right)$

$\Rightarrow \frac{3}{7} \times \frac{(7x+9)^{\frac{3}{2}}}{7\times\frac{3}{2}} + \frac{8}{7} \times \frac{(7x+9)^{\frac{1}{2}}}{7\times\frac{1}{2}} + c$

$\Rightarrow \frac{6(7x+9)^{\frac{3}{2}}}{147} - \frac{16(7x+9)^{\frac{1}{2}}}{49} + c$

7. Question

Evaluate: $\int \frac{x}{\sqrt{x+4}} dx$

Answer

In these questions, little manipulation makes the questions easier to solve

Add and subtract 4 from the numerator

$\Rightarrow \int \frac{x+4-4}{\sqrt{x+4}} dx$

$\Rightarrow \int \frac{x+4-4}{\sqrt{x+4}} dx$

$\Rightarrow \int \frac{x+4}{\sqrt{x+4}} dx - \int \frac{4}{\sqrt{x+4}} dx$

$\Rightarrow \left(\int \sqrt{x+4}\, dx - 4\int (x+4)^{\frac{-1}{2}} dx\right)$

$\Rightarrow \frac{(x+4)^{\frac{3}{2}}}{\frac{3}{2}} - 4 \times \frac{(x+4)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$\Rightarrow \frac{2(x+4)^{\frac{3}{2}}}{3} - 8(x+4)^{\frac{1}{2}} + c$

8. Question

Evaluate: $\int \frac{2-3x}{\sqrt{1+3x}} dx$

Answer

Let 2 – 3x = λ(3x + 1) + μ

2 – 3x = 3xλ + λ + μ

comparing coefficients we get

3λ = – 3 ; λ + μ = 2

$\Rightarrow \lambda = -1; \mu = 3$

Replacing 2 – 3x by λ(3x + 1) + μ in given equation we get

$\Rightarrow \int \frac{\lambda(3x+1)+\mu}{\sqrt{3x+1}} dx$

$\Rightarrow \lambda \int \frac{3x+1}{\sqrt{3x+1}} dx + \mu \int \frac{1}{1} dx$

$\Rightarrow \left(\lambda \int \sqrt{3x+1}\, dx + \mu \int (3x+1)^{\frac{-1}{2}} dx\right)$

$\Rightarrow -1 \times \frac{(3x+1)^{\frac{3}{2}}}{3 \times \frac{3}{2}} + 3 \times \frac{(3x+1)^{\frac{1}{2}}}{3 \times \frac{1}{2}} + c$

$\Rightarrow \frac{-2(3x+1)^{\frac{3}{2}}}{9} - 2(3x+1)^{\frac{1}{2}} + c$

9. Question

Evaluate: $\int (5x+3)\sqrt{2x-1}\, dx$

Answer

Let 5x + 3 = λ(2x – 1) + μ

5x + 3 = 2xλ – λ + μ

comparing coefficients we get

2λ = 5 ; – λ + μ = 3

$\Rightarrow \lambda = \frac{5}{2}; \mu = \frac{11}{2}$

Replacing 5x + 3 by λ(2x – 1) + μ in the given equation we get

$\Rightarrow \int \sqrt{2x-1}\ \lambda(2x-1) + \mu dx$

$\Rightarrow \lambda \int (2x-1)\sqrt{2x-1}\,dx + \int \sqrt{2x-1}\,\mu dx$

$\Rightarrow \left(\lambda \int (2x-1)^{\frac{3}{2}}dx - \mu \int (2x-1)^{\frac{1}{2}}dx\right)$

$\Rightarrow \frac{5}{2} \times \frac{(2x-1)^{\frac{5}{2}}}{2\times\frac{5}{2}} - \frac{11}{2} \times \frac{(2x-1)^{\frac{3}{2}}}{2\times\frac{3}{2}} + c$

$\Rightarrow \frac{(2x-1)^{\frac{5}{2}}}{2} - \frac{11(2x-1)^{\frac{3}{2}}}{6} + c$

10. Question

Evaluate: $\int \frac{x}{\sqrt{x+a} - \sqrt{x+b}}dx$

Answer

Rationalise the given equation we get

$\Rightarrow \int \frac{x}{\sqrt{x+a}-\sqrt{x-b}} \times \frac{\sqrt{x+a}+\sqrt{x-b}}{\sqrt{x+a}+\sqrt{x-b}}dx$

$\Rightarrow \int \frac{x(\sqrt{x+a}-\sqrt{x-b})}{x+a-x-b}dx$

$\Rightarrow \int \frac{x(\sqrt{x+a}-\sqrt{x-b})}{a-b}dx$

$\Rightarrow \frac{1}{a-b}\int x(\sqrt{x+a} - \sqrt{x-b})dx$

Assume x = √t

$\Rightarrow dx = \frac{dt}{2\sqrt{t}}$

Substituting t and dt

$\Rightarrow \int \sqrt{t}\frac{(\sqrt{\sqrt{t}+a}-\sqrt{\sqrt{t}-b})}{2\sqrt{t}(a-b)}dt$

$\Rightarrow \frac{1}{2(a-b)}\int (\sqrt{\sqrt{t}+a} - \sqrt{\sqrt{t}-b})dt$

$\Rightarrow \frac{1}{2(a-b)}\int (\sqrt{t}+a)^{1\backslash 2}dt - \int (\sqrt{t}-b)^{1/2}dt$

$\Rightarrow \frac{1}{2(a-b)}\left(\frac{4}{3}(\sqrt{t}+a^2)^{\frac{3}{2}} - \frac{4}{3}(t-a^2)^{\frac{3}{2}}\right)$

But x = √t

$\Rightarrow \frac{1}{2(a-b)}\left(\frac{2}{3}(x+a)^{\frac{3}{2}} - \frac{2}{3}(x-b)^{\frac{3}{2}}\right)$

Exercise 19.6

1. Question

Evaluate: $\int \sin^2(2x + 5)\, dx$

Answer

$\sin^2 x = \frac{1-\cos 2x}{2}$

∴ The given equation becomes,

$\Rightarrow \int \frac{1-\cos 2(2x+5)}{2} dx$

We know $\int \cos ax\, dx = \frac{1}{a}\sin ax + c$

$\Rightarrow \frac{1}{2}\int dx - \frac{1}{2}\int \cos(4x + 10)\, dx$

$\Rightarrow \frac{x}{2} - \frac{1}{8}\sin(4x + 10) + c$

2. Question

Evaluate: $\int \sin^3(2x + 1)\, dx$

Answer

We know $\sin 3x = -4\sin^3 x + 3\sin x$

$\Rightarrow 4\sin^3 x = 3\sin x - \sin 3x$

$\Rightarrow \sin^3 x = \frac{3\sin x - \sin 3x}{4}$

$\Rightarrow \int \sin^3(2x + 1)dx = \int \frac{3\sin(2x+1) - \sin 3(2x+1)}{4} dx$

⇒ We know $\int \sin ax\, dx = \frac{-1}{a}\cos ax + c$

$\Rightarrow \frac{3}{8}\int \sin(2x + 1)dx - \frac{1}{4}\int \sin(6x + 3)dx$

$\Rightarrow \frac{-3}{8}\cos(2x + 1) + \frac{1}{24}\cos(6x + 3) + c.$

3. Question

Evaluate: $\int \cos^4 2x\, dx$

Answer

$\cos^4 2x = (\cos^2 2x)^2$

$\Rightarrow \cos^2 x = \frac{1+\cos 2x}{2}$

$\Rightarrow (\cos^2 2x)^2 = \left(\frac{1+\cos 4x}{2}\right)^2$

$\Rightarrow \left(\frac{1+\cos 4x}{2}\right)^2 = \left(\frac{1+2\cos 4x+\cos^2 4x}{4}\right)$

$\Rightarrow \cos^2 4x = \frac{1+\cos 8x}{2}$

$\Rightarrow \left(\frac{1+2\cos 4x+\cos^2 4x}{4} = \frac{1}{4} + \frac{\cos 4x}{2} + \frac{1+\cos 8x}{8}\right)$

Now the question becomes

$\Rightarrow \frac{1}{4}\int dx + \frac{1}{2}\int \cos 4x\, dx + \frac{1}{8}\int dx + \frac{1}{8}\int \cos 8x\, dx$

We know $\int \cos ax\,dx = \frac{1}{a}\sin ax + c$

$\Rightarrow \frac{x}{4} + \frac{1}{8}\sin 4x + \frac{x}{8} + \frac{\sin 8x}{64} + c$

$\Rightarrow \frac{24x+8\sin 4x+\sin 8x}{64} + c$

4. Question

Evaluate: $\int \sin^2 b\,x\,dx$

Answer

$\sin^2 x = \frac{1-\cos 2x}{2}$

∴ The given equation becomes,

$\Rightarrow \int \frac{1-\cos 2b}{2} dx$

We know $\int \cos ax\,dx = \frac{1}{a}\sin ax + c$

$\Rightarrow \frac{1}{2}\int dx - \frac{1}{2}\int \cos(2b)\,dx$

$\Rightarrow \frac{x}{2} - \frac{1}{4b}\sin(2bx) + c$

5. Question

Evaluate: $\int \sin^2 \frac{x}{2} dx$

Answer

$\sin^2 x = \frac{1-\cos 2x}{2}$

∴ The given equation becomes,

$\Rightarrow \int \frac{1-\cos 2\frac{x}{2}}{2} dx = \int \frac{1-\cos x}{2} dx$

We know $\int \cos ax\,dx = \frac{1}{a}\sin ax + c$

$\Rightarrow \frac{1}{2}\int dx - \frac{1}{2}\int \cos(x)\,dx$

$\Rightarrow \frac{x}{2} - \frac{1}{2}\sin(x) + c$

6. Question

Evaluate: $\int \cos^2 \frac{x}{2} dx$

Answer

We know, $\cos^2 x = \frac{1+\cos 2x}{2}$

∴ The given equation becomes,

$\Rightarrow \int \frac{1+\cos 2\frac{x}{2}}{2} dx = \int \frac{1+\cos x}{2} dx$

We know $\int \cos ax\,dx = \frac{1}{a}\sin ax + c$

$\Rightarrow \frac{1}{2}\int dx + \frac{1}{2}\int \cos(x)\,dx$

$\Rightarrow \frac{x}{2} + \frac{1}{2}\sin(x) + c$

7. Question

Evaluate: $\int \cos^2 nx \, dx$

Answer

We know, $\cos^2 x = \frac{1+\cos 2x}{2}$

∴ The given equation becomes,

$\Rightarrow \int \frac{1+\cos nx}{2} dx = \int \frac{1+\cos 2nx}{2} dx$

We know $\int \cos ax \, dx = \frac{1}{a}\sin ax + c$

$\Rightarrow \frac{1}{2}\int dx + \frac{1}{2}\int \cos(2nx)\, dx$

$\Rightarrow \frac{x}{2} + \frac{1}{4n}\sin(2nx) + c$

8. Question

Evaluate: $\int \sin x \sqrt{1-\cos 2x}\, dx$

Answer

$\Rightarrow 2\sin^2 x = 1 - \cos 2x$

We can substitute the above result in the given equation

∴ The given equation becomes

$\Rightarrow \int \sin x \sqrt{2\sin^2 x}$

$\Rightarrow \int \sqrt{2}\sin^2 x$

$\sin^2 x = \frac{1-\cos 2x}{2}$

$\Rightarrow \frac{\sqrt{2}}{2}\int 1 - \cos 2x \, dx$

$\Rightarrow \frac{1}{\sqrt{2}}\int dx - \frac{1}{\sqrt{2}}\int \cos 2x \, dx$

$\Rightarrow \frac{x}{\sqrt{2}} - \frac{1}{2\sqrt{2}}\sin(2x) + c$

Exercise 19.7

1. Question

$\int \sin 4x \cos 7x \, dx$

Answer

We know $2\sin A\cos B = \sin(A + B) + \sin(A - B)$

∴ $\sin 4x\cos 7x = \frac{\sin 11x + \sin(-3x)}{2}$

We know $\sin(-\theta) = -\sin\theta$

∴ $\sin(-3x) = -\sin 3x$

∴ The above equation becomes

$\Rightarrow \int \frac{1}{2}(\sin 11x - \sin 3x)dx$

$\Rightarrow \frac{1}{2}(\int \sin 11x\, dx - \int \sin 3x\, dx)$

We know $\int \sin ax\, dx = \frac{-1}{a}\cos ax + c$

$\Rightarrow \frac{1}{2}\left(\frac{-1}{11}\cos 11x + \frac{1}{3}\cos 3x\right)$

$\Rightarrow \frac{11\cos 3x - 3\cos 11x}{66} + c$

2. Question

$\int \cos 3x \cos 4x\, dx$

Answer

We know 2cosAcosB = cos(A – B) + cos(A + B)

$\therefore$ cos4xcos3x = $\frac{\cos x + \cos 7x}{2}$

$\therefore$ The above equation becomes

$\Rightarrow \int \frac{1}{2}(\cos x - \cos 7x)dx$

$\Rightarrow \frac{1}{2}(\int \cos x\, dx - \int \cos 7x\, dx)$

We know $\int \cos ax\, dx = \frac{1}{a}\sin ax + c$

$\Rightarrow \frac{1}{2}\left(\sin x - \frac{1}{7}\sin 7x\right)$

$\Rightarrow \frac{7\sin x - \sin 7x}{14} + c$

3. Question

$\int \cos mx \cos nx\, dx$, $m \neq n$

Answer

We know 2cosAcosB = cos(A – B) + cos(A + B)

$\therefore$ cosmxcosnx = $\frac{\cos(m-n)x + \cos(m+n)x}{2}$

$\therefore$ The above equation becomes

$\Rightarrow \int \frac{1}{2}(\cos(m-n)x + \cos(m+n)x)dx$

We know $\int \cos ax\, dx = \frac{1}{a}\sin ax + c$

$\Rightarrow \frac{1}{2}\left(\frac{1}{m-n}\sin(m-n)x + \frac{1}{m+n}\sin(m+n)x\right)$

$\Rightarrow \frac{1}{2}\left(\frac{(m+n)\sin(m-n)x + (m-n)\sin(m+n)x}{m^2-n^2}\right) + c$

4. Question

$\int \sin mx \cos nx\, dx$, $m \neq n$

Answer

We know 2sinAcosB = sin(A + B) + sin(A – B)

$\therefore$ sinmxcosnx = $\frac{\sin(m+n)x + \sin(m-n)x}{2}$

∴ The above equation becomes

$\Rightarrow \int \frac{1}{2}(\sin(m + n)x + \sin(m - n)x)dx$

We know $\int \sin ax\, dx = \frac{-1}{a}\cos ax + c$

$\Rightarrow \frac{1}{2}\left(\frac{-1}{m+n}\cos(m + n)x - \frac{1}{(m-n)}\cos(m-n)x\right)$

$\Rightarrow \frac{1}{2}\left(\frac{-(m-n)\cos(m+n)x-(m+n)\cos(m-n)x}{m^2-n^2}\right)$

5. Question

$\int$ sin 2x sin 4x sin 6x dx

Answer

We need to simplify the given equation to make it easier to solve

We know 2sinAsinB = cos(A – B) – cos(A + B)

∴ sin4xsin2x $= \frac{\cos 2x - \cos 6x}{2}$

∴ The above equation becomes

$\Rightarrow \int \frac{1}{2}(\cos 2x - \cos 6x)\sin 6x\, dx$

$\Rightarrow \frac{1}{2}\int((\cos 2x \sin 6x) - (\cos 6x \sin 6x))dx$

We know 2sinAcosB = sin(A + B) + sin(A – B)

∴ sin6xcos2x $= \frac{\sin 8x + \sin 4x}{2}$

Also 2sinx.cosx = sin2x

∴ sin6xcos6x $= \frac{\sin 12x}{2}$

∴ The above equation simplifies to

$\Rightarrow \frac{1}{2}\int \frac{1}{2}(\sin 8x + \sin 4x)dx - \int \frac{1}{2}\sin 12x\, dx$

$\Rightarrow \frac{1}{4}(\int \sin 8x\, dx + \int \sin 4x\, dx - \int \sin 12x\, dx)$

We know $\int \sin ax\, dx = \frac{-1}{a}\cos ax + c$

$\Rightarrow \frac{1}{4}\left(\frac{-1}{8}\cos 8x + \frac{(-1)}{4}\cos 4x + \frac{1}{12}\cos\ 12x + c\right)$

$\Rightarrow \frac{1}{4}\left(\frac{2\cos 12x - 3\cos 8x - 6\cos 4x}{24} + c\right)$

$\Rightarrow \frac{2\cos 12x - 3\cos 8x - 6\cos 4x}{96} + c$ (where c is some arbitrary constant)

6. Question

$\int$ sin x cos 2x sin 3x dx

Answer

We know 2sinAcosB = sin(A + B) + sin(A – B)

∴ sin3xcos2x $= \frac{\sin 5x + \sin x}{2}$

∴ The given equation becomes

$\Rightarrow \int \frac{1}{2}(\sin 5x - \sin x) \sin x \, dx$

$\Rightarrow \int \frac{1}{2}(\sin 5x \sin x \, dx - \sin^2 x \, dx)$

We know 2sinAsinB = cos(A – B) – cos(A + B)

∴ sin5xsinx = $\frac{\cos 4x - \cos 6x}{2}$

Also $\sin^2 x = \frac{1-\cos 2x}{2}$

∴ Above equation can be written as

$\Rightarrow \int \frac{1}{2}(\frac{1}{2}(\cos 4x - \cos 6x)dx - \frac{1}{2}(1 - \cos 2x)dx)$

$\Rightarrow \frac{1}{4}\int \cos 4x \, dx - \int \cos 6x \, dx - \int dx + \int \cos 2x \, dx$

We know $\int \cos ax \, dx = \frac{1}{a}\sin ax + c$

$\Rightarrow \frac{1}{4}\left(\frac{1}{4}\sin 4x - \frac{1}{6}\sin 6x - x + \frac{1}{2}\sin 2x + c\right)$

$\Rightarrow \frac{1}{4}\left(\frac{3\sin 4x - 2\sin 6x - 12 + 6\sin 2x}{12} + c\right)$

$\Rightarrow \frac{3\sin 4x - 2\sin 6x - 12 + 6\sin 2x}{48} + c$

NOTE: – Whenever you are solving integral questions having trigonometric functions in the product then the first thing that should be done is convert them in the form of addition or subtraction.

Exercise 19.8

1. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{1-\cos 2x}} dx$$

Answer

In the given equation cos 2x = $\cos^2 x - \sin^2 x$

Also we know $\cos^2 x + \sin^2 x = 1$.

∴Substituting the values in the above equation we get

$\Rightarrow \int \frac{1}{\sqrt{\sin^2 x + \cos^2 x - (-\sin^2 x + \cos^2 x)}} dx$

$\Rightarrow \int \frac{1}{\sqrt{\sin^2 x + \cos^2 x + \sin^2 x - \cos^2 x}} dx$

$\Rightarrow \int \frac{1}{\sqrt{2\sin^2 x}} dx$

$\Rightarrow \int \frac{1}{\sqrt{2}\sin x} dx$

$\Rightarrow \int \frac{\csc x}{\sqrt{2}} dx$

$\Rightarrow \frac{1}{\sqrt{2}}\int \csc x \, dx$

$\Rightarrow \frac{1}{\sqrt{2}}\log\left|\frac{\tan x}{2}\right| + c$

2. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{1+\cos x}} dx$$

Answer

In the given equation

$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$

Also, $\cos^2\frac{x}{2} + \sin^2\frac{x}{2} = 1$

Substituting in the above equation we get,

$$\Rightarrow \int \frac{1}{\sqrt{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + \left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)}} dx$$

$$\Rightarrow \int \frac{1}{\sqrt{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + \cos^2\frac{x}{2} - \sin^2\frac{x}{2}}} dx$$

$$\Rightarrow \int \frac{1}{\sqrt{2\cos^2\frac{x}{2}}} dx$$

$$\Rightarrow \int \frac{1}{\sqrt{2}\cos\frac{x}{2}} dx$$

$$\Rightarrow \frac{1}{\sqrt{2}} \int \sec\frac{x}{2} dx$$

$$\Rightarrow \frac{1}{\sqrt{2}} \ln\left|\sec\frac{x}{2} + \tan\frac{x}{2}\right| + c$$

3. Question

Evaluate the following integrals:

$$\int \sqrt{\frac{1+\cos 2x}{1-\cos 2x}} dx$$

Answer

$1 + \cos 2x = 2\cos^2 x$

$1 - \cos 2x = 2\sin^2 x$

(both of them are trigonometric formuales)

$$\Rightarrow \int \sqrt{\frac{2\cos^2 x}{2\sin^2 x}} dx$$

$$\Rightarrow \int \sqrt{\cot^2 x}\, dx$$

$$\Rightarrow \int \cot x\, dx$$

$\Rightarrow \ln|\sin x| + c$

4. Question

Evaluate the following integrals:

$$\int \sqrt{\frac{1-\cos x}{1+\cos x}} dx$$

Answer

1 – cosx = $2\sin^2\frac{x}{2}$

1 + cosx = $2\cos^2\frac{x}{2}$

$\Rightarrow \int\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}dx$

$\Rightarrow \int\sqrt{\tan^2\frac{x}{2}}dx$

$\Rightarrow \int\tan\frac{x}{2}dx$

$\Rightarrow -2\ln\left|\cos\frac{x}{2}\right| + c$

5. Question

Evaluate the following integrals:

$\int\frac{\sec x}{\sec 2x}dx$

Answer

Here first of all convert secx in terms of cosx

∴ We get

$\Rightarrow \sec x = \frac{1}{\cos x}, \sec 2x = \frac{1}{\cos 2x}$

∴ We get

$\Rightarrow \frac{\frac{1}{\cos x}}{\frac{1}{\cos 2x}}$

$= \frac{\cos 2x}{\cos x}$

∴ The equation now becomes

$\Rightarrow \int\frac{\cos 2x}{\cos x}dx$

We know

Cos 2x = 2 $\cos^2$x - 1

∴ We can write the above equation as

$\Rightarrow \int\frac{2\cos^2 x-1}{\cos x}dx$

$\Rightarrow \int 2\cos x\,dx - \int\frac{1}{\cos x}dx$

⇒ 2 sin x – $\int\sec x\,dx$

($\int\sec x\,dx = \ln|\sec x + \tan x| + c$

⇒ 2 sin x – ln|sec x + tan x| + c

6. Question

Evaluate the following integrals:

$$\int \frac{\cos 2x}{(\cos x + \sin x)^2} dx$$

Answer

Expanding $(\cos x + \sin x)^2 = \cos^2 x + \sin^2 x + 2 \sin x \cos x$

We know $\cos^2 x + \sin^2 x = 1$, $2\sin x\cos x = \sin 2x$

$\therefore (\cos x + \sin x)^2 = 1 + \sin 2x$

$\therefore$ we can write the given equation as

$\Rightarrow \int \frac{\cos 2x}{1 + \sin 2x} dx$

Assume $1 + \sin 2x = t$

$\Rightarrow \frac{d(1 + \sin 2x)}{dx} = \frac{dt}{dx}$

$\Rightarrow 2\cos 2x \, dx = dt$

$\therefore \cos 2x dx = \frac{dt}{2}$

Substituting these values in the above equation we get

$\Rightarrow \int \frac{1}{2t} dt$

$\Rightarrow \frac{1}{2} \ln t + c$

substituting $t = 1 + 2 \sin x$ in above equation

$\Rightarrow \frac{1}{2} \ln(1 + 2 \sin x) + c$

7. Question

Evaluate the following integrals:

$$\int \frac{\sin(x - a)}{\sin(x - b)} dx$$

Answer

While solving these types of questions, it is better to eliminate the denominator.

$\Rightarrow \int \frac{\sin(x-a)}{\sin(x-b)} dx$

Add and subtract b in (x - a)

$\Rightarrow \int \frac{\sin(x-a+b-b)}{\sin(x-b)} dx$

$\Rightarrow \int \frac{\sin(x-b+b-a)}{\sin(x-b)}$

Numerator is of the form $\sin(A + B) = \sin A\cos B + \cos A\sin B$

Where $A = x - b$; $B = b - a$

$\Rightarrow \int \frac{\sin(x-b)\cos(b-a) + \cos(x-b)\sin(b-a)}{\sin(x-b)} dx$

$\Rightarrow \int \frac{\sin(x-b)\cos(b-a)}{\sin(x-b)} dx + \int \frac{\cos(x-b)\sin(b-a)}{\sin(x-b)} dx$

$\Rightarrow \int \cos(b - a)\, dx + \int \cot(x - b) \sin(b - a)\, dx$

$\Rightarrow \cos(b-a)\int dx + \sin(b-a)\int \cot(x-b)\,dx$

As $\int \cot(x)\,dx = \ln|\sin x|$

⇒ cos(b - a)x + sin(b - a)ln|sin(x - b)|

8. Question

Evaluate the following integrals:

$$\int \frac{\sin(x-\alpha)}{\sin(x+\alpha)}dx$$

Answer

Add and subtract α in the numerator

$\Rightarrow \int \frac{\sin(x-\alpha+\alpha-\alpha)}{\sin(x+\alpha)}dx$

$\Rightarrow \int \frac{\sin(x+\alpha-2\alpha)}{\sin(x+\alpha)}$

Numerator is of the form sin(A - B) = sinAcosB - cosAsinB

Where A = x + α ; B = 2α

$\Rightarrow \int \frac{\sin(x+\alpha)\cos(2\alpha)-\cos(x+\alpha)\sin(2\alpha)}{\sin(x+\alpha)}dx$

$\Rightarrow \int \frac{\sin(x+\alpha)\cos(2\alpha)}{\sin(x+\alpha)}dx + \int \frac{\cos(x+\alpha)\sin(2\alpha)}{\sin(x+\alpha)}dx$

$\Rightarrow \int \cos(2\alpha)\,dx + \int \cot(x+\alpha)\sin(2\alpha)\,dx$

$\Rightarrow \cos(2\alpha)\int dx + \sin(2\alpha)\int \cot(x+\alpha)\,dx$

As $\int \cot(x)\,dx = \ln|\sin x|$

⇒ cos(2α)x + sin(2α)ln|sin(x + α)|

9. Question

Evaluate the following integrals:

$$\int \frac{1+\tan x}{1-\tan x}dx$$

Answer

Convert tanx in form of sinx and cosx.

$\Rightarrow \tan x = \frac{\sin x}{\cos x}$

∴ The equation now becomes

$\Rightarrow \int \frac{1+\frac{\sin x}{\cos x}}{1-\frac{\sin x}{\cos x}}dx$

$\Rightarrow \int \frac{\frac{\cos x+\sin x}{\cos x}}{\frac{\cos x-\sin x}{\cos x}}dx$

$\Rightarrow \int \frac{\cos x+\sin x}{\cos x-\sin x}dx$

Let cosx - sinx = t

$\therefore \frac{d(\cos x-\sin x)}{dx} = \frac{dt}{dx}$

⇒ - (cosx + sinx)dx =dt

Substituting dt and t

We get

$\Rightarrow \int -\frac{dt}{t}$

⇒ - ln t + c

t = cosx – sinx

∴ - ln|cosx – sinx| + c

10. Question

Evaluate the following integrals:

$$\int \frac{\cos x}{\cos(x-a)} dx$$

Answer

Add and subtract a from x in the numerator

∴ The equation becomes

$\Rightarrow \int \frac{\cos(x-a+a)}{\cos(x-a)}$

Numerator is of the form cos(A + B)= cosAcosB - sinAsinB

Where A= x - a ; B= a

$\Rightarrow \int \frac{\cos(x-a)\cos a}{\cos(x-a)} dx - \int \frac{\sin(x-a)\sin a}{\cos(x-a)} dx$

$\Rightarrow \cos a \int dx - \sin a \int \tan(x-a)\, dx$

As $\int \tan x = \ln|\sec x| + c$

⇒ xcosa – sina $\frac{\ln|\sec(x-a)|}{(x-a)}$ + c

11. Question

Evaluate the following integrals:

$$\int \sqrt{\frac{1-\sin 2x}{1+\sin 2x}} dx$$

Answer

We know $\cos^2x + \sin^2x = 1$.

Also, 2sinxcosx = sin2x

$1 + \sin 2x = \cos^2x + \sin^2x + 2\sin x\cos x = (\cos x + \sin x)^2$

$1 - \sin 2x = \cos^2x + \sin^2x - 2\sin x\cos x = (\cos x - \sin x)^2$

∴ The equation becomes

$\Rightarrow \int \sqrt{\frac{(\cos x-\sin x)^2}{(\cos x+\sin x)^2}} dx$

$\Rightarrow \int \frac{(\cos x-\sin x)}{(\cos x+\sin x)} dx$

Assume cosx + sinx = t

∴ d(cosx + sinx) = dt

= cosx - sinx

∴ dt = cosx – sinx

$\Rightarrow \int \frac{dt}{t}$

= ln|t| + c

But t = cosx + sinx

∴ ln|cosx + sinx| + c.

12. Question

Evaluate the following integrals:

$$\int \frac{e^{3x}}{e^{3x}+1} dx$$

Answer

Assume $e^{3x} + 1 = t$

$\Rightarrow d(e^{3x} + 1) = dt$

$\Rightarrow 3e^{3x} = dt$

$\Rightarrow e^{3x} = \frac{dt}{3}$

Substituting t and dt in the given equation we get

$\Rightarrow \int \frac{dt}{3t}$

$\Rightarrow \frac{1}{3} \int \frac{dt}{t}$

$\Rightarrow \frac{1}{3} \ln |t| + c$

But $t = e^{3x} + 1$

∴ The above equation becomes

$\Rightarrow \frac{1}{3} \ln |e^{3x} + 1| + c.$

13. Question

Evaluate the following integrals:

$$\int \frac{\sec x \tan x}{3 \sec x + 5} dx$$

Answer

Assume 3secx + 5=t

d(3secx + 5)=dt

3secxtanx=dt

Secxtanx=$\frac{dt}{3}$

Substitute t and dt

We get

$\Rightarrow \frac{1}{3}\int \frac{dt}{t}$

$\Rightarrow \frac{1}{3}\ln|t| + c$

But t = 3secx + 5

∴ the equation becomes

$\Rightarrow \frac{1}{3}\ln|3\sec x + 5| + c.$

14. Question

Evaluate the following integrals:

$$\int \frac{1-\cot x}{1+\cot x} dx$$

Answer

Convert cotx in form of sinx and cosx.

$\Rightarrow \cot x = \frac{\cos x}{\sin x}$

∴ The equation now becomes

$\Rightarrow \int \frac{1-\frac{\cos x}{\sin x}}{1+\frac{\cos x}{\sin x}} dx$

$\Rightarrow \int \frac{\frac{\cos x-\sin x}{\sin x}}{\frac{\cos x+\sin x}{\sin x}} dx$

$\Rightarrow \int \frac{\cos x-\sin x}{\cos x+\sin x} dx$

Assume cosx + sinx = t

∴ d(cosx + sinx) = dt

= cosx - sinx

∴ dt = cosx – sinx

$\Rightarrow \int \frac{dt}{t}$

= ln|t| + c

But t = cosx + sinx

∴ ln|cosx + sinx| + c.

15. Question

Evaluate the following integrals:

$$\int \frac{\sec x \operatorname{cosec} x}{\log(\tan x)} dx$$

Answer

Assume log(tanx) = t

d(log(tanx)) = dt

$\Rightarrow \frac{\sec^2 x}{\tan x} dx = dt$

$\Rightarrow$ secx.cosecx.dx=dt

Put t and dt in given equation we get

$\Rightarrow \int \frac{dt}{t}$

= ln|t| + c.

But t = log(tanx)

= ln|log(tanx)| + c.

16. Question

Evaluate the following integrals:

$$\int \frac{1}{x(3+\log x)} dx$$

Answer

Assume 3 + logx = t

d(3 + logx) =dt

$\Rightarrow \frac{1}{x} dx = dt$

Put t and dt in given equation we get

$\Rightarrow \int \frac{dt}{t}$

= ln|t| + c.

But t = 3 + logx

= ln|3 + logx| + c

17. Question

Evaluate the following integrals:

$$\int \frac{e^x + 1}{e^x + x} dx$$

Answer

Assume e^x + x =t

d(e^x + x)=dt

e^x + 1 = dt

Put t and dt in given equation we get

$\Rightarrow \int \frac{dt}{t}$

= ln|t| + c.

But t = e^x + x

= ln| e^x + 1| + c

18. Question

Evaluate the following integrals:

$$\int \frac{1}{x \log x} dx$$

Answer

Assume logx =t

d(logx)=dt

$\frac{1}{x}dx$ = dt

Put t and dt in given equation we get

$\Rightarrow \int \frac{dt}{t}$

= ln|t| + c.

But t = logx

= ln| logx| + c

19. Question

Evaluate the following integrals:

$$\int \frac{\sin 2x}{a\cos^2 x + b\sin^2 x} dx$$

Answer

Assume $a\cos^2x + b\sin^2x = t$

$d(a\cos^2x + b\sin^2x) = dt$

(- 2acosx.sinx + 2bsinx.cosx)dx = dt

(bsin2x - asin2x)dx=dt

(b - a)sin2xdx=dt

Sin2xdx $=\frac{dt}{(b-a)}$

Put t and dt in given equation we get

$\Rightarrow \frac{1}{(b-a)} \int \frac{dt}{t}$

$= \frac{1}{b-a}$ln|t| + c.

But $t = a\cos^2x + b\sin^2x$

$= \frac{1}{b-a}\ln| a\cos^2x + b\sin^2x | + c.$

20. Question

Evaluate the following integrals:

$$\int \frac{\cos x}{2 + 3\sin x} dx$$

Answer

Assume 2 + 3sinx = t

d(2 + 3sinx)=dt

3cosxdx = dt

$\cos x dx = \frac{dt}{3}$

Put t and dt in given equation we get

$\Rightarrow \frac{1}{3}\int \frac{dt}{t}$

$= \frac{1}{3}\ln|t| + c$

But t =2 + 3sinx

$= \frac{1}{3}\ln|2 + 3\sin x| + c.$

21. Question

Evaluate the following integrals:

$$\int \frac{1-\sin x}{x+\cos x} dx$$

Answer

Assume x + cosx = t

d(x + cosx) =dt

⇒ 1 - sinx dx =dt

Put t and dt in given equation we get

$\Rightarrow \int \frac{dt}{t}$

= ln|t| + c.

But t = x + cosx

= ln| x + cosx | + c

22. Question

Evaluate the following integrals:

$$\int \frac{a}{b+ce^x} dx$$

Answer

First of all take e^x common from denominator so we get

$\Rightarrow \int \frac{a}{e^x\left(\frac{b}{e^x}+c\right)} . dx$

$\Rightarrow \int \frac{a.e^{-x}}{be^{-x}+c} dx$

Assume $be^{-x} + c = t$

$d(be^{-x} + c) = dt$

$\Rightarrow -be^{-x}dx = dt$

$\Rightarrow e^{-x}dx = \frac{-dt}{b}$

Substituting t and dt we get

$\Rightarrow \int \frac{-adt}{bt}$

$\Rightarrow \frac{-a}{b} \ln|t| + c$

But t =(be $^{-x}$ + c)

$\Rightarrow \frac{-a}{b} \ln|be^{-x} + c| + c$

23. Question

Evaluate the following integrals:

$$\int \frac{1}{e^x + 1} dx$$

Answer

First of all, take e^x common from the denominator, so we get

$\Rightarrow \int \frac{1}{e^x\left(\frac{1}{e^x} + 1\right)} . dx$

$\Rightarrow \int \frac{1.e^{-x}}{e^{-x} + 1} dx$

Assume e $^{-x}$ + 1 =t

d(e $^{-x}$ + 1) = dt

$\Rightarrow$ - e $^{-x}$dx= dt

$\Rightarrow$ e $^{-x}$dx= - dt

Substituting t and dt we get

$\Rightarrow \int \frac{-dt}{t}$

$\Rightarrow \ln|t| + c$

But t =(e $^{-x}$ + 1)

$\Rightarrow \ln|e^{-x} + 1| + c.$

24. Question

Evaluate the following integrals:

$$\int \frac{\cot x}{\log \sin x} dx$$

Answer

Assume log(sinx)= t

d(log(sinx)) =dt

$\Rightarrow \frac{\cos x}{\sin x}$ dx =dt

$\Rightarrow$ cotx dx = dt

Put t and dt in given equation we get

$\Rightarrow \int \frac{dt}{t}$

= ln|t| + c.

But t = log(sinx)

= ln| log(sinx) | + c

25. Question

Evaluate the following integrals:

$$\int \frac{e^{2x}}{e^{2x}-2} dx$$

Answer

Assume e^{2x} - 2= t

d(e^{2x} - 2) =dt

⇒ $2e^{2x}dx$ =dt

⇒ $e^{2x}dx = \frac{dt}{2}$

Put t and dt in the given equation we get

$\Rightarrow \frac{1}{2}\int \frac{dt}{t}$

$= \frac{1}{2}\ln|t| + c$

But t = e^{2x} - 2

$= \frac{1}{2}\ln|e^{2x}-2| + c$

26. Question

Evaluate the following integrals:

$$\int \frac{2\cos x - 3\sin x}{6\cos x + 4\sin x} dx$$

Answer

Taking 2 common in denominator we get

$\Rightarrow \int \frac{2\cos x - 3\sin x}{2(3\cos x + 2\sin x)} dx$

Now assume

3cosx + 2sinx = t

(- 3sinx + 2cosx)dx=dt

Put t and dt in given equation we get

$\Rightarrow \frac{1}{2}\int \frac{dt}{t}$

$= \frac{1}{2}\ln|t| + c$

But t = 3cosx + 2sinx

$= \frac{1}{2}\ln|3\cos x + 2\sin x| + c$

27. Question

Evaluate the following integrals:

$$\int \frac{\cos 2x + x + 1}{x^2 + \sin 2x + 2x} dx$$

Answer

Assume $x^2 + \sin 2x + 2x = t$

$d(x^2 + \sin 2x + 2x) = dt$

$(2x + 2\cos 2x + 2)dx = dt$

$2(x + \cos 2x + 1)dx = dt$

$(x + \cos 2x + 1)dx = \frac{1}{2} dt$

Put t and dt in given equation we get

$\Rightarrow \frac{1}{2} \int \frac{dt}{t}$

$= \frac{1}{2} \ln|t| + c$

But $t = x^2 + \sin 2x + 2x$

$= \frac{1}{2} \ln|x^2 + \sin 2x + 2x| + c$

28. Question

Evaluate the following integrals:

$$\int \frac{1}{\cos(x+a)\cos(x+b)} dx$$

Answer

Let $I = \int \frac{1}{\cos(x+a)\cos(x+b)} dx$

Dividing and multiplying I by sin (a – b) we get,

$I = \frac{1}{\sin(a-b)} \int \frac{\sin(a-b)}{\cos(x+a)\cos(x+b)} dx$

$I = \frac{1}{\sin(a-b)} \int \frac{\sin\{(x+a)-(x+b)\}}{\cos(x+a)\cos(x+b)} dx$

$I = \frac{1}{\sin(a-b)} \int \frac{\sin(x+a)\cos(x+b) - \cos(x+a)\sin(x+b)}{\cos(x+a)\cos(x+b)} dx$

$I = \frac{1}{\sin(a-b)} \int \{\tan(x+a) - \tan(x+b)\} dx$

We know that,

$$\int \tan x \, dx = |\log \sec x| + c$$

Therefore,

$I = \frac{1}{\sin(a-b)} \left\{ \frac{\log(\sec(x+a))}{x+a} - \frac{\log(\sec(x+b))}{x+b} \right\} + c$

29. Question

Evaluate the following integrals:

$$\int \frac{-\sin x + 2\cos x}{2\sin x + \cos x} dx$$

Answer

Assume 2sinx + cosx =t

d(2sinx + cosx) =dt

(2cosx - sinx)dx= dt

Put t and dt in given equation we get

$\Rightarrow \int \frac{dt}{t}$

$= \ln|t| + c$

But t = 2sinx + cosx

= ln|2sinx + cosx| + c.

30. Question

Evaluate the following integrals:

$$\int \frac{\cos 4x - \cos 2x}{\sin 4x - \sin 2x} dx$$

Answer

Assume sin4x – sin2x =t

d(sin4x – sin2x) =dt

(cos4x – cos2x)dx= dt

Put t and dt in given equation we get

$\Rightarrow \int \frac{dt}{t}$

$= \ln|t| + c$

But t = sin4x – sin2x

= ln| sin4x – sin2x | + c.

31. Question

Evaluate the following integrals:

$$\int \frac{\sec x}{\log(\sec x + \tan x)} dx$$

Answer

Assume log(secx + tanx) =t

d(log(secx + tanx)) =dt

(use chain rule to differentiate first differentiate log(secx + tanx) then (secx + tanx)

$\Rightarrow \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} dx$=dt

$\Rightarrow \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x} dx = dt$

⇒ secx dx =dt

Put t and dt in the given equation we get

$\Rightarrow \int \frac{dt}{t}$

$= \ln|t| + c$

But t = log(secx + tanx)

= ln| log(secx + tanx) | + c.

32. Question

Evaluate the following integrals:

$$\int \frac{\operatorname{cosec} x}{\log \tan \frac{x}{2}} dx$$

Answer

Assume $\log(\tan\frac{x}{2})$ =t

$d(\log(\tan\frac{x}{2}))$ =dt

(use chain rule to differentiate)

$\Rightarrow \frac{\sec^2\frac{x}{2}}{\tan\frac{x}{2}} dx$=dt

$\Rightarrow \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}} dx = dt$

$\Rightarrow \frac{1}{\sin x} dx = dt$

⇒ cosecx dx =dt

Put t and dt in the given equation we get

$\Rightarrow \int \frac{dt}{t}$

$= \ln|t| + c$

But t = $\log(\tan\frac{x}{2})$

= ln| $\log(\tan\frac{x}{2})$ | + c.

33. Question

Evaluate the following integrals:

$$\int \frac{1}{x \log x \log(\log x)} dx$$

Answer

Assume log(logx) =t

d(log(logx)) =dt

(use chain rule to differentiate first)

$\Rightarrow \frac{1}{x\log x} dx$=dt

Put t and dt in given equation we get

$\Rightarrow \int \frac{dt}{t}$

$= \ln|t| + c$

But t = log(log(x))

= ln| log(log(x)) | + c.

34. Question

Evaluate the following integrals:

$$\int \frac{\operatorname{cosec}^2 x}{1+\cot x} dx$$

Answer

Assume 1 + cotx =t

d(1 + cotx) =dt

⇒cosec^2x=dt

Put t and dt in given equation we get

$\Rightarrow \int \frac{dt}{t}$

$= \ln|t| + c$

But t = 1 + cotx

= ln|1 + cotx| + c.

35. Question

Evaluate the following integrals:

$$\int \frac{10x^9 + 10^x \log_e 10}{10^x + x^{10}} dx$$

Answer

Assume $10^x + x^{10}$ =t

d($10^x + x^{10}$) =dt

$a^x = \log_e a$

⇒ $10x^9 + 10^x \log_e 10$=dt

Put t and dt in given equation we get

$\Rightarrow \int \frac{dt}{t}$

$= \ln|t| + c$

But t = $10^x + x^{10}$

= ln|$10^x + x^{10}$| + c.

36. Question

Evaluate the following integrals:

$$\int \frac{1-\sin 2x}{x+\cos^2 x} dx$$

Answer

Assume x + $\cos^2$x =t

$d(x + \cos^2 x) = dt$

$(1 + (-2\cos x.\sin x))dx = dt$

$2\sin x.\cos x = \sin 2x$

$(1 - \sin 2x)dx = dt$

Put t and dt in given equation we get

$\Rightarrow \int \frac{dt}{t}$

$= \ln|t| + c$

But $t = x + \cos^2 x$

$= \ln| x + \cos^2 x | + c.$

37. Question

Evaluate the following integrals:

$$\int \frac{1 + \tan x}{x + \log x \sec x} dx$$

Answer

Assume $x + \log x \sec x = t$

$d(x + \log x \sec x) = dt$

$1 + \frac{\sec x \tan x}{\sec x} dx = dt$

$(1 + \tan x)dx = dt$

Put t and dt in given equation we get

$\Rightarrow \int \frac{dt}{t}$

$= \ln|t| + c$

But $t = x + \log x \sec x$

$= \ln| x + \log x \sec x | + c.$

38. Question

Evaluate the following integrals:

$$\int \frac{\sin 2x}{a^2 + b^2 \sin^2 x} dx$$

Answer

Assume $a^2 + b^2 \sin^2 x = t$

$d(a^2 + b^2 \sin^2 x) = dt$

$2b^2.\sin x.\cos x.dx = dt$

$(2\sin x.\cos x = \sin 2x)$

$\text{Sin}2x dx = \frac{dt}{b^2}$

Put t and dt in the given equation we get

$\Rightarrow \frac{1}{b^2}\int \frac{dt}{t}$

$= \frac{1}{b^2}\ln|t| + c$

But $t = a^2 + b^2\sin^2 x$

$= \frac{1}{b^2}\ln|a^2 + b^2\sin^2 x| + c.$

39. Question

Evaluate the following integrals:

$$\int \frac{x+1}{x(x+\log x)}dx$$

Answer

Assume $x + \log x = t$

$d(x + \log x) = dt$

$\Rightarrow \left(1 + \frac{1}{x}\right)dx = dt$

$\Rightarrow \left(\frac{x+1}{x}\right)dx = dt$

Put t and dt in the given equation we get

$\Rightarrow \int \frac{dt}{t}$

$= \ln|t| + c$

But $t = x + \log x$

$= \ln|x + \log x| + c.$

40. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{1-x^2}\,(2+3\sin^{-1}x)}dx$$

Answer

Assume $2 + 3\sin^{-1}x = t$

$d(2 + 3\sin^{-1}x) = dt$

$\Rightarrow \frac{3}{\sqrt{1-x^2}}dx = dt$

$\Rightarrow \frac{dx}{\sqrt{1-x^2}} = \frac{dt}{3}$

Put t and dt in the given equation we get

$\Rightarrow \frac{1}{3}\int \frac{dt}{t}$

$= \frac{1}{3}\ln|t| + c$

But $t = 2 + 3\sin^{-1}x$

$= \frac{1}{b^2}\ln|2 + 3\sin^{-1}x| + c.$

41. Question

Evaluate the following integrals:

$$\int \frac{\sec^2 x}{\tan x + 2} dx$$

Answer

Assume tanx + 2 =t

d(tanx + 2) =dt

($\sec^2 x dx$) = dt

Put t and dt in given equation we get

$\Rightarrow \int \frac{dt}{t}$

$= \ln|t| + c$

But t = tanx + 2

= ln| tanx + 2 | + c.

42. Question

Evaluate the following integrals:

$$\int \frac{2\cos 2x + \sec^2 x}{\sin 2x + \tan x - 5} dx$$

Answer

Assume sin2x + tanx - 5 =t

d(tanx + sin2x - 5) =dt

($2\cos 2x + \sec^2 x$)dx = dt

Put t and dt in given equation we get

$\Rightarrow \int \frac{dt}{t}$

$= \ln|t| + c$

But t = sin2x + tanx - 5

= ln|sin2x + tanx - 5| + c.

43. Question

Evaluate the following integrals:

$$\int \frac{\sin 2x}{\sin 5x \sin 3x} dx$$

Answer

sin2x can be written as sin(5x - 3x)

∴ The equation now becomes

$\Rightarrow \int \frac{\sin(5x-3x)}{\sin 5x \sin 3x} dx$

sin(A - B) = sinAcosB - cosAsinB

$\Rightarrow \int \frac{\sin 5x \cos 3x - \cos 5x \sin 3x}{\sin 5x \sin 3x} dx$

$\Rightarrow \int \frac{\sin 5x \cos 3x}{\sin 5x \sin 3x} dx - \int \frac{\cos 5x \sin 3x}{\sin 5x \sin 3x} dx$

$\Rightarrow \int \frac{\cos 3x}{\sin 3x} dx - \int \frac{\cos 5x}{\sin 5x} dx$

$\Rightarrow \int \cot 3x\, dx - \int \cot 5x\, dx$

$\Rightarrow \frac{1}{3} \ln|\sin 3x| - \frac{1}{5} \ln|\sin 5x| + c.$

44. Question

Evaluate the following integrals:

$$\int \frac{1 + \cot x}{x + \log \sin x} dx$$

Answer

Assume x + log(sinx) =t

d(x + log(sinx)) =dt

$1 + \frac{\cos x}{\sin x} dx = dt$

(1 + cot)dx = dt

Put t and dt in given equation we get

$\Rightarrow \int \frac{dt}{t}$

$= \ln|t| + c$

But t = x + log(sinx)

= ln| x + log(sinx) | + c.

45. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{x}\left(\sqrt{x} + 1\right)} dx$$

Answer

Assume √ x + 1 =t

d(√ x + 1) = dt

$\Rightarrow \frac{1}{2\sqrt{x}} dx = dt$

$\Rightarrow \frac{1}{\sqrt{x}} dx = 2dt$

Put t and dt in given equation we get

$\Rightarrow \int 2 \frac{dt}{t}$

$= \ln|t| + c$

But t = √ x + 1

=2 ln| √ x + 1 | + c.

46. Question

Evaluate the following integrals:

$\int \tan 2x \tan 3x \tan 5x\, dx$

Answer

We know tan5x = tan(2x + 3x)

$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$\therefore \tan(2x + 3x) = \frac{\tan 2x + \tan 3x}{1 - \tan 2x \tan 3x}$

$\therefore \tan(5x) = \frac{\tan 2x + \tan 3x}{1 - \tan 2x \tan 3x}$

$\Rightarrow \tan(5x)(1 - \tan 2x.\tan 3x) = \tan(2x) + \tan(3x)$

$\Rightarrow \tan(5x) - \tan 2x.\tan 3x.\tan 5x = \tan(2x) + \tan(3x)$

$\Rightarrow \tan(5x) - \tan(2x) - \tan(3x) = \tan 2x.\tan 3x.\tan 5x$

Substituting the above result in given equation we get

$\Rightarrow \int \tan 5x - \tan 3x - \tan 2x\, dx$

$\Rightarrow \int \tan 5x\, dx - \int \tan 3x\, dx - \int \tan 2x\, dx$

$\Rightarrow \frac{-1}{5} \ln|\cos 5x| - \frac{(-1)}{3} \ln|\cos 3x| - \frac{(-1)}{2} \ln|\cos 2x| + c.$

$\Rightarrow \frac{-1}{5} \ln|\cos 5x| + \frac{1}{3} \ln|\cos 3x| + \frac{1}{2} \ln|\cos 2x| + c.$

47. Question

Evaluate the following integrals:

$\int \{1 + \tan x \tan (x + \theta)\}\, dx$

Answer

$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

$\therefore \tan(x - (x + \theta)) = \frac{\tan x - \tan(x + \theta)}{1 + \tan x \tan(x + \theta)}$

$\therefore \tan(\theta) = \frac{\tan x - \tan(x + \theta)}{1 + \tan x \tan(x + \theta)}$

$\Rightarrow \tan(\theta)(1 + \tan x.\tan(x + \theta)) = \tan(x) - \tan(x + \theta)$

$\Rightarrow (1 + \tan x.\tan(x + \theta)) = \frac{1}{\tan\theta}(\tan x - \tan(x + \theta)$

$\Rightarrow \int \frac{1}{\tan\theta}(\tan x - \tan(x + \theta)).dx$

$\Rightarrow \frac{1}{\tan\theta} \int \tan x\, dx - \int \tan(x + \theta)\, dx$

$\Rightarrow \frac{1}{\tan\theta}(-\ln|\cos x| - (-\ln|\cos(x + \theta)| + c.$

$\Rightarrow \frac{1}{\tan\theta}(-\ln|\cos x| + \ln|\cos(x + \theta)| + c.$

48. Question

Evaluate the following integrals:

$$\int \frac{\sin 2x}{\sin\left(x-\frac{\pi}{6}\right)\sin\left(x+\frac{\pi}{6}\right)}dx$$

Answer

sin(A - B) = sinAcosB - cosAsinB

∴ We can write $\sin\left(x-\frac{\pi}{6}\right) = \sin x\cos\frac{\pi}{6} - \cos x\sin\frac{\pi}{6}$

sin(A + B) = sinAcosB + cosAsinB

∴ We can write $\sin\left(x+\frac{\pi}{6}\right) = \sin x\cos\frac{\pi}{6} + \cos x\sin\frac{\pi}{6}$

∴ The given equation becomes

$\Rightarrow \int \frac{\sin 2x}{\left(\sin x\cos\frac{\pi}{6}-\cos x\sin\frac{\pi}{6}\right)\left(\sin x\cos\frac{\pi}{6}+\cos x\sin\frac{\pi}{6}\right)}dx$

$\Rightarrow \int \frac{\sin 2x}{\left(\sin x\frac{\sqrt{3}}{2}-\cos x\frac{1}{2}\right)\left(\sin x\frac{\sqrt{3}}{2}+\cos x\frac{1}{2}\right)}dx$

Denominator is of the form (a - b)(a + b) = a^2 - b^2

$\Rightarrow \int \frac{\sin 2x}{\left(\frac{3}{4}\sin^2 x-\cos^2 x\frac{1}{4}\right)}dx$....(1)

We know $\sin^2 x + \cos^2 x = 1$

∴ $\sin^2 x = 1 - \cos^2 x$

Substituting the above result in (1) we get

$\Rightarrow \int \frac{\sin 2x}{\left(\frac{3}{4}(1-\cos^2 x)-\cos^2 x\frac{1}{4}\right)}dx$

$\Rightarrow \int \frac{\sin 2x}{\left(\frac{3}{4}-\cos^2 x\right)}dx$...(2)

Let us assume $\left(\frac{3}{4}-\cos^2 x\right) = t$

$\Rightarrow d\left(\frac{3}{4}-\cos^2 x\right) = dt$

⇒ 2sinx.cosx.dx=dt

⇒ sin2x.dx=dt

Substituting dt and t in (2) we get

$\Rightarrow \int \frac{dt}{t}$

= ln|t| + c

But t = $\left(\frac{3}{4}-\cos^2 x\right)$

∴ ln|$\left(\frac{3}{4}-\cos^2 x\right)$| + c.

49. Question

Evaluate the following integrals:

$$\int \frac{e^{x-1}+x^{e-1}}{e^x+x^e}dx$$

Answer

Multiplying and dividing the numerator by e we get the given as

$\Rightarrow \frac{1}{e}\int \frac{e^x + ex^{e-1}}{e^x + x^e}dx$...(1)

Assume $e^x + x^e = t$

$\Rightarrow d(e^x + x^e) = dt$

$\Rightarrow e^x + ex^{e-1} = dt$

Substituting t and dt in equation 1 we get

$\Rightarrow \frac{1}{e}\int \frac{dt}{t}$

= ln|t| + c

But $t = e^x + x^e$

$\therefore \ln|e^x + x^e| + c.$

50. Question

Evaluate the following integrals:

$$\int \frac{1}{\sin x \cos^2 x}dx$$

Answer

We know $\sin^2 x + \cos^2 x = 1$

$\Rightarrow \int \frac{\sin^2 x + \cos^2 x}{\sin x \cos^2 x}$

$\Rightarrow \int \frac{\sin^2 x}{\sin x \cos^2 x}dx + \int \frac{\cos^2 x}{\sin x \cos^2 x}dx$

$\Rightarrow \int \frac{\sin x}{\cos^2 x}dx + \int \frac{1}{\sin x}dx$

$\Rightarrow \int \tan x \sec x\, dx + \int \csc x\, dx$

d(secx) = tanx.secx

$\therefore \int \tan x \sec x\, dx = \sec x + c$

$\therefore \int \tan x \sec x\, dx + \int \csc x\, dx$

$\because \int \csc x\, dx = \log\left|\tan\frac{x}{2}\right| + c$

$\Rightarrow \sec x + \log|\tan\frac{x}{2}| + c.$

51. Question

Evaluate the following integrals:

$$\int \frac{1}{\cos 3x - \cos x}dx$$

Answer

The denominator is of the form cosC - cosD = $-2\sin\left(\frac{c+d}{2}\right).\sin\left(\frac{c-d}{2}\right)$

$\therefore$ cos3x - cosx= $-2\sin\left(\frac{3+1}{2}x\right)\sin\left(\frac{3-1}{2}x\right)$

$\therefore$ cos3x - cosx= - 2sin2x.sinx

- 2sin2x.sinx = - 2.2.sinx.cosx.sinx

- 2sin2x.sinx= - 4sin^{2}x.cosx

Also sin^{2}x + cos^{2}x = 1

$\Rightarrow \int \frac{\sin^2 x + \cos^2 x}{-4\sin^2 x \cos x} dx$

$\Rightarrow \frac{-1}{4}\int \frac{\sin^2 x}{\sin^2 x.\cos x} dx + \frac{-1}{4}\int \frac{\cos^2 x}{\sin^2 x.\cos x} dx$

$\Rightarrow \frac{-1}{4}\left(\int \frac{1}{\cos x} dx + \int \frac{\cos x}{\sin^2 x} dx\right)$

$\Rightarrow \frac{-1}{4}\int \sec x\, dx + \int \csc x . \cot x\, dx$

d(cscx) = cscx.cotx

$\therefore \int \csc x \cot x\, dx = \csc x + c$

$\therefore \int \sec x\, dx + \int \csc x . \cot x\, dx$

$\because \int \sec x\, dx = \log|\sec x + \tan x| + c$

$\Rightarrow \frac{-1}{4}(\csc x + \log|\sec x + \tan x|) + c$

Exercise 19.9

1. Question

Evaluate the following integrals:

$\int \frac{\log x}{x} dx$

Answer

Assume logx = t

$\Rightarrow$ d(logx) = dt

$\Rightarrow \frac{1}{x} dx = dt$

Substituting t and dt in above equation we get

$\Rightarrow \int t . dt$

$\Rightarrow \frac{t^2}{2} + c$

But t = log(x)

$\Rightarrow \frac{\log^2 x}{2} + c.$

2. Question

Evaluate the following integrals:

$\int \frac{\log\left(1+\frac{1}{x}\right)}{x(1+x)} dx$

Answer

Assume $\log\left(1+\frac{1}{x}\right) = t$

$\Rightarrow d(\log\left(1+\frac{1}{x}\right)) = dt$

$\Rightarrow \frac{1}{1+\frac{1}{x}} \times \frac{-1}{x^2}dx = dt$

$\Rightarrow \frac{x}{x+1} \times \frac{-1}{x^2}dx = dt$

$\Rightarrow \frac{-1.dx}{x(x+1)} = dt$

$\Rightarrow \frac{dx}{x(x+1)} = -dt$

∴ Substituting t and dt in the given equation we get

$\Rightarrow \int -t.dt$

$\Rightarrow -\int t.dt$

$\Rightarrow \frac{-t^2}{2} + c$

But $\log\left(1+\frac{1}{x}\right) = t$

$\Rightarrow -\frac{1}{2}\log^2\left(1+\frac{1}{x}\right) + c$

3. Question

Evaluate the following integrals:

$$\int \frac{\left(1+\sqrt{x}\right)^2}{\sqrt{x}}dx$$

Answer

Assume $1 + \sqrt{x} = t$

$\Rightarrow d(1 + \sqrt{x}) = dt$

$\Rightarrow \frac{1}{2\sqrt{x}}dx = dt$

$\Rightarrow \frac{1}{\sqrt{x}}dx = 2dt$

∴ Substituting t and dt in the given equation we get

$\Rightarrow \int 2t^2.dt$

$\Rightarrow 2\int t^2.dt$

$\Rightarrow \frac{2t^3}{3} + c$

But $1 + \sqrt{x} = t$

$\Rightarrow \frac{2(1+\sqrt{x})^3}{3} + c.$

4. Question

Evaluate the following integrals:

$$\int \sqrt{1+e^x}\, e^x\, dx$$

Answer

Assume $1 + e^x = t$

$\Rightarrow d(1 + e^x) = dt$

$\Rightarrow e^x dx = dt$

∴ Substituting t and dt in given equation we get

$\Rightarrow \int \sqrt{t}.dt$

$\Rightarrow \int t^{1/2}.dt$

$\Rightarrow \frac{2t^{\frac{3}{2}}}{3} + c$

But $1 + e^x = t$

$\Rightarrow \frac{2(1+e^x)^{3/2}}{3} + c.$

5. Question

Evaluate the following integrals:

$$\int \sqrt[3]{\cos^2 x} \sin x \, dx$$

Answer

Assume $\cos x = t$

$\Rightarrow d(\cos x) = dt$

$\Rightarrow -\sin x dx = dt$

$\Rightarrow dx = \frac{-dt}{\sin x}$

∴ Substituting t and dt in the given equation we get

$\Rightarrow \int \sqrt[3]{t^2} \sin x . \frac{dt}{\sin x}$

$\Rightarrow \int t^{3/2}.dt$

$\Rightarrow \frac{2t^{\frac{3}{2}}}{3} + c$

But $\cos x = t$

$\Rightarrow \frac{2(\cos x)^{3/2}}{3} + c.$

6. Question

Evaluate the following integrals:

$$\int \frac{e^x}{(1+e^x)^2} dx$$

Answer

Assume $1 + e^x = t$

$\Rightarrow d(1 + e^x) = dt$

$\Rightarrow e^x dx = dt$

∴ Substituting t and dt in given equation we get

$\Rightarrow \int \frac{1}{t^2} dt$

$\Rightarrow \int t^{-2}.dt$

$\Rightarrow \frac{-1}{t} + c$

But 1 + e^x = t

$\Rightarrow \frac{-1}{1+e^x} + c.$

7. Question

Evaluate the following integrals:

$\int \cot^3 x \operatorname{cosec}^2 x\, dx$

Answer

Assume cotx = t

⇒ d(cotx) = dt

⇒ - cosec^2x.dx = dt

$\Rightarrow dt = \frac{-dt}{\csc^2 x}$

∴ Substituting t and dt in the given equation we get

$\Rightarrow \int t^3 \csc^2 x . \frac{-dt}{\csc^2 x}$

$\Rightarrow \int -t^3 . dt$

$\Rightarrow -\int t^3 . dt$

$\Rightarrow \frac{-t^4}{4} + c$

But t = cotx

$\Rightarrow \frac{-\cot^4 x}{4} + c.$

8. Question

Evaluate the following integrals:

$$\int \frac{\left\{e^{\sin^{-1} x}\right\}^2}{\sqrt{1-x^2}} dx$$

Answer

Assume $\sin^{-1}x$ = t

⇒ d($\sin^{-1}x$) = dt

$\Rightarrow \frac{dx}{\sqrt{1-x^2}} = dt$

∴ Substituting t and dt in the given equation we get

$\Rightarrow \int e^{t^2} dt$

$\Rightarrow \int e^{2t} . dt$

$\Rightarrow \frac{e^{2t}}{2} + c$

But t = $\sin^{-1}x$

$\Rightarrow \frac{e^{2(\sin^{-1}x)}}{2} + c$

9. Question

Evaluate the following integrals:

$$\int \frac{1+\sin x}{\sqrt{x-\cos x}} dx$$

Answer

Assume x - cosx = t

⇒ d(x - cosx) = dt

⇒ (1 + sinx)dx = dt

∴ Substituting t and dt in given equation we get

$\Rightarrow \int \frac{1}{\sqrt{t}} dt$

$\Rightarrow \int t^{-1\backslash 2} . dt$

$\Rightarrow 2t^{1\backslash 2} + c$

But t = x – cosx.

⇒ $2(x - \cos x)^{1/2}$ + c.

10. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{1-x^2}\left(\sin^{-1}x\right)^2} dx$$

Answer

Assume $\sin^{-1}x$ = t

⇒ $d(\sin^{-1}x)$ = dt

$\Rightarrow \frac{dx}{\sqrt{1-x^2}} = dt$

∴ Substituting t and dt in the given equation we get

$\Rightarrow \int \frac{1}{t^2} dt$

$\Rightarrow \int t^{-2} . dt$

$\Rightarrow \frac{t^{-1}}{-1} + c$

But t = $\sin^{-1}x$

$\Rightarrow \frac{-1}{\sin^{-1}x} + c$

11. Question

Evaluate the following integrals:

$$\int \frac{\cot x}{\sqrt{\sin x}} dx$$

Answer

We know d(sinx) = cosx, and cot can be written in terms of cos and sin

$\therefore \cot x = \frac{\cos x}{\sin x}$

∴ The given equation can be written as

$\Rightarrow \int \frac{\cos x}{\sin x \sqrt{\sin x}} dx$

$\Rightarrow \int \frac{\cos x}{\sin^{3\backslash 2} x} dx$

Now assume sinx = t

d(sinx) = dt

cosx dx = dt

Substitute values of t and dt in above equation

$\Rightarrow \int \frac{dt}{t^{3\backslash 2}}$

$\Rightarrow \int t^{-3\backslash 2} dt$

$\Rightarrow -2t^{-1\backslash 2} + c$

$\Rightarrow -2\sin^{-1\backslash 2} x + c$

$\Rightarrow \frac{-2}{\sqrt{\sin x}} + c$

12. Question

Evaluate the following integrals:

$$\int \frac{\tan x}{\sqrt{\cos x}} dx$$

Answer

We know d(cosx) = sinx, and tan can be written interms of cos and sin

$\therefore \tan x = \frac{\sin x}{\cos x}$

∴ The given equation can be written as

$\Rightarrow \int \frac{\sin x}{\cos x \sqrt{\cos x}} dx$

$\Rightarrow \int \frac{\sin x}{\cos^{3\backslash 2} x} dx$

Now assume cosx = t

d(cosx) = - dt

sinx dx = - dt

Substitute values of t and dt in above equation

$\Rightarrow \int \frac{-dt}{t^{3\backslash 2}}$

$\Rightarrow -\int t^{-3\backslash 2} dt$

$\Rightarrow 2t^{-1\backslash 2} + c$

$\Rightarrow 2\cos^{-1\backslash 2} x + c$

$\Rightarrow \frac{2}{\sqrt{\cos x}} + c$

13. Question

Evaluate the following integrals:

$$\int \frac{\cos^3 x}{\sqrt{\sin x}} dx$$

Answer

In this equation, we can manipulate numerator

$\text{Cos}^3 x = \cos^2 x.\cos x$

∴ Now the equation becomes,

$\Rightarrow \int \frac{\cos^2 x.\cos x}{\sqrt{\sin x}} dx$

$\text{Cos}^2 x = 1 - \sin^2 x$

$\Rightarrow \int \frac{1-\sin^2 x.\cos x}{\sqrt{\sin x}} dx$

Now,

Let us assume sinx = t

d(sinx) = dt

cosx dx = dt

Substitute values of t and dt in the above equation

$\Rightarrow \int \frac{1-t^2.}{\sqrt{t}} dt$

$\Rightarrow \int \frac{1}{\sqrt{t}} dt - \int \frac{t^2}{\sqrt{t}} dt$

$\Rightarrow \int t^{-1\backslash 2} dt - \int t^{3\backslash 2} dt$

$\Rightarrow 2t^{1\backslash 2} - \frac{2}{5} t^{\frac{5}{2}} + c$

But t = sinx

$\Rightarrow 2 \sin x^{1\backslash 2} - \frac{2}{5} \sin x^{\frac{5}{2}} + c\cdot$

14. Question

Evaluate the following integrals:

$$\int \frac{\sin^3 x}{\sqrt{\cos x}} dx$$

Answer

In this equation, we can manipulate numerator

$\sin^3 x = \sin^2 x.\sin x$

∴ Now the equation becomes,

$\Rightarrow \int \frac{\sin^2 x.\sin x}{\sqrt{\cos x}} dx$

$\sin^2 x = 1 - \cos^2 x$

$\Rightarrow \int \frac{1-\cos^2 x.\sin x}{\sqrt{\cos x}} dx$

Now ,

Let us assume cosx = t

d(cosx) = dt

- sinx dx = dt

Substitute values of t and dt in above equation

$\Rightarrow -\int \frac{1-t^2}{\sqrt{t}} dt$

$\Rightarrow -\int \frac{1}{\sqrt{t}} dt - \int \frac{t^2}{\sqrt{t}} dt$

$\Rightarrow -\int t^{-1\backslash 2} dt + \int t^{3\backslash 2} dt$

$\Rightarrow -2t^{1\backslash 2} + \frac{2}{5} t^{\frac{5}{2}} + c$

But t = cosx

$\Rightarrow -2\cos x^{1\backslash 2} + \frac{2}{5} \cos x^{\frac{5}{2}} + c$

15. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{\tan^{-1} x}\,(1+x^2)} dx$$

Answer

Assume $\tan^{-1} x = t$

$d(\tan^{-1} x) = dt$

$\Rightarrow \frac{1}{1+x^2} dx = dt$

Substituting t and dt in above equation we get

$\Rightarrow \int \frac{1}{\sqrt{t}} dt$

$\Rightarrow \int t^{-1\backslash 2} . dt$

$\Rightarrow 2t^{1\backslash 2} + c$

But $t = \tan^{-1} x$

$\Rightarrow 2(\tan^{-1} x)^{1/2} + c.$

16. Question

Evaluate the following integrals:

$$\int \frac{\sqrt{\tan x}}{\sin x \cos x} dx$$

Answer

Multiply and divide by cosx

$\Rightarrow \int \frac{\sqrt{\tan x}.\cos x}{\sin x.\cos x.\cos x} dx$

$\Rightarrow \int \frac{\sqrt{\tan x.}}{\tan x.\cos^2 x} dx$

$\Rightarrow \int \frac{\sec^2 x.}{\sqrt{\tan x.}} dx$

Assume tanx = t

d(tanx) = dt

$\sec^2$x dx = dt

Substituting t and dt in above equation we get

$\Rightarrow \int \frac{1}{\sqrt{t}} dt$

$\Rightarrow \int t^{-1\backslash 2}.dt$

$\Rightarrow 2t^{1\backslash 2} + c$

But t = tanx

⇒ 2(tanx)$^{1/2}$ + c.

17. Question

Evaluate the following integrals:

$$\int \frac{1}{x}(\log x)^2 dx$$

Answer

Assume logx = t

d(log(x)) = dt

$\Rightarrow \frac{1}{x} dx = dt$

∴ Substituting t and dt in given equation we get

$\Rightarrow \int t^2.dt$

$\Rightarrow \int t^2.dt$

$\Rightarrow \frac{t^3}{3} + c$

But logx = t

$\Rightarrow \frac{(\log(x))^3}{3} + c.$

18. Question

Evaluate the following integrals:

$\int \sin^5 x \cos x\, dx$

Answer

Assume sinx = t

d(sinx) = dt

cosxdx = dt

∴ Substituting t and dt in given equation we get

$\Rightarrow \int t^5 dt$

$\Rightarrow \frac{t^6}{6} + c$

But t = sinx

$\Rightarrow \frac{\sin^6 x}{6} + c.$

19. Question

Evaluate the following integrals:

$\int \tan^{3/2} x \sec^2 x\, dx$

Answer

Assume tanx = t

d(tanx) = dt

$\sec^2 x dx = dt$

∴ Substituting t and dt in given equation we get

$\Rightarrow \int t^{\frac{3}{2}} dt$

$\Rightarrow \frac{2t^{\frac{5}{2}}}{5} + c$

But t = tanx

$\Rightarrow \frac{2\tan^{\frac{5}{2}} x}{5} + c$

20. Question

Evaluate the following integrals:

$$\int \frac{x^3}{\left(x^2+1\right)^3} dx$$

Answer

Assume $x^2 + 1 = t$

$\Rightarrow d(x^2 + 1) = dt$

$\Rightarrow 2x\, dx = dt$

$\Rightarrow xdx = \frac{dt}{2}$

x^3 can be write as $x^2.x$

∴ Now the given equation becomes

$\Rightarrow \int \frac{x^2.xdx}{(x^2+1)^3}$

$x^2 + 1 = t \Rightarrow x^2 = t - 1$

$\Rightarrow \int \frac{(t-1)dt}{2t^3}$

$\Rightarrow \frac{1}{2}\int \frac{t}{t^3}dt - \int \frac{1}{t^3}dt$

$\Rightarrow \frac{1}{2}\int t^{-2}dt - \int t^{-3}dt$

$\Rightarrow \frac{1}{2}(-1t^{-1} + \frac{1}{2}t^{-2}) + c$

But t = $(x^2 + 1)$

$\Rightarrow \frac{1}{2}(-1(x^2 + 1)^{-1} + \frac{1}{2}(x^2 + 1)^{-2}) + c$

$\Rightarrow \frac{-1}{2(x^2+1)} + \frac{1}{4(1+x^2)^2} + c$

$\Rightarrow \frac{-4(1+x^2)^2 + 2(1+x^2)}{8(1+x^2)^3} + c$

21. Question

Evaluate the following integrals:

$$\int (4x+2)\sqrt{x^2+x+1}\,dx$$

Answer

Here (4x + 2) can be written as 2(2x + 1).

Now assume, $x^2 + x + 1 = t$

$d(x^2 + x + 1) = dt$

(2x + 1)dx = dt

$\Rightarrow \int 2(2x + 1)\sqrt{x^2 + x + 1}dx$

$\Rightarrow \int 2\sqrt{t}dt$

$\Rightarrow \int 2t^{1/2}.dt$

$\Rightarrow \frac{4t^{\frac{3}{2}}}{3} + c$

But t = $x^2 + x + 1$

$\Rightarrow \frac{4(x^2+x+1)^{3/2}}{3} + c.$

22. Question

Evaluate the following integrals:

$$\int \frac{4x+3}{\sqrt{2x^2+3x+1}}dx$$

Answer

Assume, $2x^2 + 3x + 1 = t$

$d(x^2 + x + 1) = dt$

(4x + 3)dx = dt

Substituting t and dt in above equation we get

$\Rightarrow \int \frac{1}{\sqrt{t}} dt$

$\Rightarrow \int t^{-1\backslash 2} . dt$

$\Rightarrow 2t^{1\backslash 2} + c$

But $t = 2x^2 + 3x + 1$

$\Rightarrow 2(2x^2 + 3x + 1)^{1/2} + c.$

23. Question

Evaluate the following integrals:

$$\int \frac{1}{1+\sqrt{x}} dx$$

Answer

$x = t^2$

$d(x) = 2t.dt$

$dx = 2t.dt$

Substituting t and dt we get

$\Rightarrow \int \frac{2t.dt}{1+t}$

$\Rightarrow 2\int \frac{t.dt}{1+t}$

Add and subtract 1 from numerator

$\Rightarrow 2\int \frac{t+1-1}{1+t} dt$

$\Rightarrow 2\left(\int \frac{t+1}{t+1} dt - \int \frac{1}{1+t} dt\right)$

$\Rightarrow 2\left(\int dt - \int \frac{1}{1+t} dt\right)$

$\Rightarrow 2(t - \ln|1 + t|)$

But $t = \sqrt{x}$

$\Rightarrow 2(\sqrt{x} - \ln|1 + \sqrt{x}|) + c$

24. Question

Evaluate the following integrals:

$$\int e^{\cos^2 x} \sin 2x \, dx$$

Answer

Assume $\cos^2 x = t$

$d(\cos^2 x) = dt$

$- 2\sin x \cos x dx = dt$

$- \sin 2x.dx = dt$

Substituting t and dt

$\Rightarrow \int e^t . dt$

$\Rightarrow e^t + c.$

But $t = \cos^2 x$

$\Rightarrow e^{\cos 2x} + c$

25. Question

Evaluate the following integrals:

$$\int \frac{1+\cos x}{(x+\sin x)^3} dx$$

Answer

Assume x + sinx = t

d(x + sinx) = dt

(1 + cosx)dx = dt

Substituting t and dt in given equation

$\Rightarrow \int \frac{dt}{t^3}$

$\Rightarrow \int t^{-3} dt$

$\Rightarrow \frac{t^{-2}}{-2} + c$

$\Rightarrow \frac{-1}{2t^2} + c$

But t = x + sinx

$\Rightarrow \frac{-1}{2(x+\sin x)^2} + c$

26. Question

Evaluate the following integrals:

$$\int \frac{\cos x - \sin x}{1+\sin 2x} dx$$

Answer

We know $\cos^2 x + \sin^2 x = 1$, 2sinxcosx = sin2x

∴ Denominator can be written as

$\cos^2 x + \sin^2 x + 2\sin x\cos x = (\sin x + \cos x)^2$

∴ Now the given equation becomes

$\Rightarrow \int \frac{\cos x - \sin x}{(\sin x + \cos x)^2} dx$

Assume cosx + sinx = t

∴ d(cosx + sinx) = dt

= cosx - sinx

∴ dt = cosx – sinx

$\Rightarrow \int \frac{dt}{t^2}$

$\Rightarrow \int \frac{1}{t^2} dt$

$\Rightarrow \int t^{-2}.dt$

$\Rightarrow \frac{t^{-1}}{-1} + c$

But t = cosx + sinx

$\Rightarrow \frac{-1}{\cos x + \sin x} + c$

27. Question

Evaluate the following integrals:

$$\int \frac{\sin 2x}{(a + b\cos 2x)^2} dx$$

Answer

Assume a + bcos2x = t

d(a + bcos2x) = dt

- 2bsin2x dx = dt

Sin2xdx = $\frac{-dt}{2b}$

$\Rightarrow \frac{-1}{2b}\int \frac{dt}{t^2}$

$\Rightarrow \frac{-1}{2b}\int \frac{1}{t^2}dt$

$\Rightarrow \frac{-1}{2b}\int t^{-2}.dt$

$\Rightarrow \frac{t^{-1}}{2b} + c$

But t = a + bcos2x

$\Rightarrow \frac{1}{2b(a + b\cos 2x)} + c.$

28. Question

Evaluate the following integrals:

$$\int \frac{\log x^2}{x} dx$$

Answer

Assume log x = t

⇒ d(logx) = dt

$\Rightarrow \frac{1}{x}dx = dt$

Substituting the values oft and dt we get

$\Rightarrow \int t^2 dt$

$\Rightarrow \frac{t^3}{3} + c$

But t = logx

$\Rightarrow \frac{\log^3 x}{3} + c.$

29. Question

Evaluate the following integrals:

$$\int \frac{\sin x}{(1+\cos x)^2} dx$$

Answer

Assume 1 + cosx = t

⇒ d(1 + cosx) = dt

⇒ - sinx.dx = dt

Substituting the values oft and dt we get

$\Rightarrow -\int \frac{dt}{t^2}$

$\Rightarrow -\int \frac{1}{t^2} dt$

$\Rightarrow -\int t^{-2}.dt$

$\Rightarrow \frac{t^{-1}}{1} + c$

But t = 1 + cosx

$\Rightarrow \frac{+1}{1+\cos x} + c$

30. Question

Evaluate the following integrals:

∫ cotx log sin x dx

Answer

Assume log(sinx) = t

d(log(sinx)) = dt

$\Rightarrow \frac{\cos x}{\sin x} dx = dt$

⇒ cot x dx = dt

Substituting the values oft and dt we get

$\Rightarrow \int tdt$

$\Rightarrow \frac{t^2}{2} + c$

But t = log(sinx)

$\Rightarrow \frac{\log(\sin x)^2 x}{2} + c.$

31. Question

Evaluate the following integrals:

∫ sec x log (sec x + tan x) dx

Answer

Assume log(secx + tanx) = t

d(log(secx + tanx)) = dt

(use chain rule to differentiate first differentiate log(secx + tanx) then (secx + tanx)

$\Rightarrow \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} dx$ = dt

$\Rightarrow \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x} dx = dt$

⇒ secx dx = dt

Put t and dt in given equation we get

Substituting the values oft and dt we get

$\Rightarrow \int tdt$

$\Rightarrow \frac{t^2}{2} + c$

But t = log(secx + tanx)

$\Rightarrow \frac{\log^2(\sec x + \tan x)}{2} + c.$

32. Question

Evaluate the following integrals:

∫ cosec x log (cosec x – cot x) dx

Answer

Assume log(cosec x – cot x) = t

d(log(cosec x – cot x)) = dt

(use chain rule to differentiate first differentiate log(secx + tanx) then (secx + tanx)

$\Rightarrow \frac{-\csc x \cot x + \csc^2 x}{\operatorname{cosec} x - \cot x} dx$ = dt

$\Rightarrow \frac{\csc x(\csc x - \cot x)}{\operatorname{cosec} x - \cot x} dx = dt$

⇒ cscx dx = dt

Put t and dt in given equation we get

Substituting the values oft and dt we get

$\Rightarrow \int tdt$

$\Rightarrow \frac{t^2}{2} + c$

But t = log(cosec x – cot x)

$\Rightarrow \frac{\log^2(\operatorname{cosec} x - \cot x)}{2} + c.$

33. Question

Evaluate the following integrals:

∫ x^3 cos x^4 dx

Answer

Assume x^4 = t

d(x^4) = dt

$4x^3$dx = dt

$x^3dx = \frac{dt}{4}$

Substituting t and dt

$\Rightarrow \int \frac{1}{4}\cos t \, dt$

$\Rightarrow \frac{1 \sin t}{4} + c$

But $t = x^4$

$\Rightarrow \frac{1}{4}\sin x^4 + c.$

34. Question

Evaluate the following integrals:

$\int x^3 \sin x^4 \, dx$

Answer

Assume $x^4 = t$

$d(x^4) = dt$

$4x^3dx = dt$

$x^3dx = \frac{dt}{4}$

Substituting t and dt

$\Rightarrow \int \frac{1}{4}\sin t \, dt$

$\Rightarrow \frac{-1 \cos t}{4} + c$

But $t = x^4$

$\Rightarrow \frac{-1}{4}\cos x^4 + c.$

35. Question

Evaluate the following integrals:

$$\int \frac{x \sin^{-1} x^2}{\sqrt{1-x^4}} dx$$

Answer

Assume $\sin^{-1}x^2 = t$

$\Rightarrow d(\sin^{-1}x) = dt$

$\Rightarrow \frac{2xdx}{\sqrt{1-x^4}} = dt$

$\Rightarrow \frac{xdx}{\sqrt{1-x^4}} = \frac{dt}{2}$

∴ Substituting t and dt in given equation we get

$\Rightarrow \int \frac{t}{2} dt$

$\Rightarrow \frac{1}{2}\int t.dt$

$\Rightarrow \frac{t^2}{4} + c$

But $t = \sin^{-1}x$

$\Rightarrow \frac{(\sin^{-1}x^2)^2}{4} + c.$

36. Question

Evaluate the following integrals:

$\int x^3 \sin(x^4 + 1)\, dx$

Answer

Assume $x^4 + 1 = t$

$d(x^4 + 1) = dt$

$4x^3 dx = dt$

$x^3 dx = \frac{dt}{4}$

Substituting t and dt

$\Rightarrow \int \frac{1}{4} \sin t\, dt$

$\Rightarrow \frac{-1 \cos t}{4} + c$

But $t = x^4 + 1$

$\Rightarrow \frac{-1}{4} \cos(x^4 + 1) + c.$

37. Question

Evaluate the following integrals:

$\int \frac{(x+1)e^x}{\cos^2(xe^x)} dx$

Answer

Assume $xe^x = t$

$d(xe^x) = dt$

$(e^x + xe^x)\, dx = dt$

$e^x(1 + x)\, dx = dt$

Substituting t and dt

$\Rightarrow \int \frac{dt}{\cos^2 t}$

$\Rightarrow \int \sec^2 t\, dt$

$\Rightarrow \tan t + c$

But $t = xe^x + 1$

$\Rightarrow \tan(xe^x) + c.$

38. Question

Evaluate the following integrals:

$$\int x^2 e^{x^3} \cos\left(e^{x^3}\right) dx$$

Answer

Assume $e^{x^3} = t$

$\Rightarrow d\left(e^{x^3}\right) = dt$

$\Rightarrow 3x^2 . e^{x^3} dx = dt$

$\Rightarrow x^2 . e^{x^3} dx = \frac{dt}{3}$

Substituting t and dt

$\Rightarrow \int \frac{1}{3} \cos t . dt$

$\Rightarrow \frac{1}{3} \sin t + c$

But t = e^{x^3}

$\Rightarrow \frac{1}{3} \sin e^{x^3} + c$

39. Question

Evaluate the following integrals:

$\int 2x \sec^3 (x^2 + 3) \tan (x^2 + 3) dx$

Answer

$\sec^3 (x^2 + 3)$ can be written as $\sec^2 (x^2 + 3). \sec (x^2 + 3)$

Now the question becomes

$\Rightarrow \int 2x. \sec^2(x^2 + 3) \sec(x^2 + 3) \tan(x^2 + 3) dx$

Assume $\sec (x^2 + 3) = t$

$d(\sec (x^2 + 3)) = dt$

$2x \sec (x^2 + 3) \tan (x^2 + 3)dx = dt$

Substituting t and dt in the given equation

$\Rightarrow \int t^2 dt$

$\Rightarrow \frac{t^3}{3} + c$

$\Rightarrow \frac{1}{3}(\sec(x^2 + 3)^3) + c.$

40. Question

Evaluate the following integrals:

$$\int \left(\frac{x+1}{x}\right)(x + \log x)^2 dx$$

Answer

Assume $(x + \log x) = t$

$d(x + \log x) = dt$

$\Rightarrow \left(1 + \frac{1}{x}\right) dx = dt$

$\Rightarrow \frac{x+1}{x} dx = dt$

Substituting t and dt

$\Rightarrow \int t^2 dt$

$\Rightarrow \frac{t^3}{3} + c.$

But t = x + logx

$\Rightarrow \frac{(x+\log x)^3}{3} + c.$

41. Question

Evaluate the following integrals:

$$\int \tan x \sec^2 x \sqrt{1-\tan^2 x}\, dx$$

Answer

Assume 1 - $\tan^2 x$ = t

d(1 - $\tan^2 x$) = dt

2.tanx.$\sec^2 x$dx = dt

Substituting t and dt we get

$\Rightarrow \Rightarrow \int \frac{1}{2} \sqrt{t}\, dt$

$\Rightarrow \int \frac{1}{2} t^{1/2} . dt$

$\Rightarrow \frac{4t^{\frac{3}{2}}}{6} + c$

But t = 1 - $\tan^2 x$

$\Rightarrow \frac{-2\left(1-\tan^2 x\right)^{3/2}}{3} + c.$

42. Question

Evaluate the following integrals:

$$\int \log x \frac{\sin\left\{1+(\log x)^2\right\}}{x} dx$$

Answer

Assume 1 + $(\log x)^2$ = t

d(1 + $(\log x)^2$) = dt

$\Rightarrow \frac{2\log x}{x} dx = dt$

$\Rightarrow \frac{\log x}{x} dx = \frac{dt}{2}$

$\Rightarrow \int \sin t \frac{dt}{2}$

$\Rightarrow \frac{1}{2}\int \sin t\, dt$

$\Rightarrow \frac{-1}{2}\cos t + c$

But t = 1 + $(\log x)^2$

$\Rightarrow \frac{-1}{2}\cos(1 + \log x^2) + c.$

43. Question

Evaluate the following integrals:

$$\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx$$

Answer

Assume $\frac{1}{x} = t$

$\Rightarrow \frac{1}{x^2}dx = dt$

Substituting t and dt we get

$\Rightarrow \int \cos^2 t\, dt$

$\Rightarrow \cos^2 x = \frac{1 + \cos 2x}{2}$

∴ The given equation becomes,

$\Rightarrow \int \frac{1 - \cos 2t}{2}dx$

We know $\int \cos ax\, dx = \frac{1}{a}\sin ax + c$

$\Rightarrow \frac{1}{2}\int dxt - \frac{1}{2}\int \cos(2t)\, dt$

$\Rightarrow \frac{t}{2} - \frac{1}{4}\sin(t) + c$

But $\frac{1}{x} = t$

$\Rightarrow \frac{1}{2x} - \frac{1}{4}\sin\left(\frac{1}{x}\right) + c.$

44. Question

Evaluate the following integrals:

$\int \sec^4 x \tan x\, dx$

Answer

Put tanx = t

d(tanx) = dt

$\sec^2 x dx$ = dt

$\Rightarrow dx = \frac{dt}{\sec^2 x}$

We can write $\sec^4 x = \sec^2 x. \sec^2 x$

Now ,the question becomes

$\Rightarrow \int \sec^2 x . \sec^2 x . \tan x \frac{dt}{\sec^2 x}$

$\Rightarrow \int \sec^2 x . \tan x \, dt$

$Tan^2x + 1 = sec^2x$

tanx = t

$t^2 + 1 = sec^2x$

$\Rightarrow \int (t^2 + 1) t \, dt$

$\Rightarrow \int t^3 dt + \int t . dt$

$\Rightarrow \frac{t^4}{4} + \frac{t^2}{2} + c$

But t = tanx

$\Rightarrow \frac{\tan^4 x}{4} + \frac{\tan^2 x}{2} + c$

45. Question

Evaluate the following integrals:

$$\int \frac{e^{\sqrt{x}} \cos\left(e^{\sqrt{x}}\right)}{\sqrt{x}} dx$$

Answer

Assume $e^{\sqrt{x}} = t$

$d(e^{\sqrt{x}}) = dt$

$\Rightarrow \frac{e^{\sqrt{x}}}{2\sqrt{x}} dx = dt$

$\Rightarrow \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2dt$

Substituting t and dt

$\Rightarrow 2 \int \cos t \, dt$

= 2sint + c

But $t = e^{\sqrt{x}}$

$\Rightarrow 2 \sin(e^{\sqrt{x}}) + c.$

46. Question

Evaluate the following integrals:

$$\int \frac{1}{x^2} \cos^2\left(\frac{1}{x}\right) dx$$

Answer

Assume $\frac{1}{x} = t$

$\Rightarrow \frac{1}{x^2} dx = dt$

Substituting t and dt we get

$\Rightarrow \int \cos^2 t\, dt$

$\Rightarrow \cos^2 x = \frac{1 + \cos 2x}{2}$

∴ The given equation becomes,

$\Rightarrow \int \frac{1-\cos 2t}{2} dx$

We know $\int \cos ax\, dx = \frac{1}{a} \sin ax + c$

$\Rightarrow \frac{1}{2} \int dxt - \frac{1}{2} \int \cos(2t)\, dt$

$\Rightarrow \frac{t}{2} - \frac{1}{4} \sin(t) + c$

But $\frac{1}{x} = t$

$\Rightarrow \frac{1}{2x} - \frac{1}{4} \sin\left(\frac{1}{x}\right) + c.$

47. Question

Evaluate the following integrals:

$$\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$$

Answer

Assume $\sqrt{x} = t$

$d(\sqrt{x}) = dt$

$\Rightarrow \frac{1}{2\sqrt{x}} dx = dt$

$\Rightarrow \frac{1}{\sqrt{x}} dx = 2dt$

Substituting t and dt

$\Rightarrow 2 \int \sin t\, dt$

$= - 2\cos t + c$

But $\sqrt{x} = t$

$\Rightarrow 2 \cos(\sqrt{x}) + c.$

48. Question

Evaluate the following integrals:

$$\int \frac{(x+1)e^x}{\sin^2(xe^x)} dx$$

Answer

Assume $xe^x = t$

$d(xe^x) = dt$

$(e^x + xe^x)\, dx = dt$

$e^x(1 + x)\, dx = dt$

Substituting t and dt

$\Rightarrow \int \frac{dt}{\sin^2 t}$

$\Rightarrow \int \csc^2 t\, dt$

$\Rightarrow$ - cot t + c

But t = $xe^x + 1$

$\Rightarrow$ - cot (xe^x) + c.

49. Question

Evaluate the following integrals:

$$\int 5^{x+\tan^{-1}x}\left(\frac{x^2+2}{x^2+1}\right)dx$$

Answer

Assume x + $\tan^{-1}x$ = t

d(x + $\tan^{-1}x$) = dt

$\Rightarrow 1 + \frac{1}{x^2+1} = dt$

$\Rightarrow \frac{2+x^2}{x^2+1} = dt$

Substituting t and dt

$\Rightarrow \int 5^t dt$

$\Rightarrow \frac{5^t}{\log 5} + c$

But t = x + $\tan^{-1}x$

$\Rightarrow \frac{5^{x+\tan^{-1}x}}{\log 5} + c.$

50. Question

Evaluate the following integrals:

$$\int \frac{e^{m\sin^{-1}x}}{\sqrt{1-x^2}}dx$$

Answer

Assume $\sin^{-1}x$ = t

d($\sin^{-1}x$) = dt

$\Rightarrow \frac{dx}{\sqrt{1-x^2}} = dt$

∴ Substituting t and dt in given equation we get

$\Rightarrow \int e^{mt} dt$

$\Rightarrow \frac{e^{mt}}{m} + c$

But t = $\sin^{-1}x$

$\Rightarrow \frac{e^{m\sin^{-1}x}}{m} + c$

51. Question

Evaluate the following integrals:

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$$

Answer

Assume √x = t

d(√x) = dt

$\Rightarrow \frac{1}{2\sqrt{x}}dx = dt$

$\Rightarrow \frac{1}{\sqrt{x}}dx = 2dt$

Substituting t and dt

$\Rightarrow 2\int \cos t\, dt$

= 2sint + c

But √x = t

⇒2 sin(√x) + c.

52. Question

Evaluate the following integrals:

$$\int \frac{\sin\left(\tan^{-1}x\right)}{1+x^2}dx$$

Answer

Assume $\tan^{-1}x = t$

d($\tan^{-1}x$) = dt

$\Rightarrow \frac{1}{x^2+1} = dt$

Substituting t and dt

$\Rightarrow \int \sin t\, dt$

= - cost + c

But t = $\tan^{-1}x$

⇒ - cos($\tan^{-1}x$) + c.

53. Question

Evaluate the following integrals:

$$\int \frac{\sin(\log x)}{x}dx$$

Answer

Assume logx = t

d(logx) = dt

$\Rightarrow \frac{1}{x}dx = dt$

Substituting t and dt

$\Rightarrow \int \sin t \, dt$

= - cost + c

But t = logx

⇒ cos(logx) + c.

54. Question

Evaluate the following integrals:

$$\int \frac{e^{m \tan^{-1} x}}{1+x^2} dx$$

Answer

Assume $\tan^{-1}x = t$

$d(\tan^{-1}x) = dt$

$\Rightarrow \frac{1}{x^2+1} = dt$

Substituting t and dt

$\Rightarrow \int e^{mt} dt$

$\Rightarrow \frac{e^{mt}}{m} + c$

But $t = \tan^{-1}x$

$\Rightarrow \frac{e^{m\tan^{-1}x}}{m} + c.$

55. Question

Evaluate the following integrals:

$$\int \frac{x}{\sqrt{x^2+a^2}+\sqrt{x^2-a^2}} dx$$

Answer

Rationlize the given equation we get

$\Rightarrow \int \frac{x}{\sqrt{x^2+a^2}+\sqrt{x^2-a^2}} \times \frac{\sqrt{x^2+a^2}-\sqrt{x^2-a^2}}{\sqrt{x^2+a^2}-\sqrt{x^2-a^2}} dx$

$\Rightarrow \int \frac{x(\sqrt{x^2+a^2}-\sqrt{x^2-a^2})}{2a^2} dx$

Assume $x^2 = t$

2x.dx = dt

$\Rightarrow dx = \frac{dt}{2x}$

Substituting t and dt

$\Rightarrow \int \frac{(\sqrt{t+a^2}-\sqrt{t-a^2})}{4a^2} dt$

$\Rightarrow \frac{1}{4a^2} \int (\sqrt{t+a^2} - \sqrt{t-a^2}) dt$

$\Rightarrow \frac{1}{4a^2}\int (t + a^2)^{1\backslash 2}dt - \int (t - a^2)^{1/2}dt$

$\Rightarrow \frac{1}{4a^2}\left(\frac{2}{3}(t + a^2)^{\frac{3}{2}} - \frac{2}{3}(t - a^2)^{\frac{3}{2}}\right)$

But t = x^2

$\Rightarrow \frac{1}{4a^2}\left(\frac{2}{3}(x^2 + a^2)^{\frac{3}{2}} - \frac{2}{3}(x^2 - a^2)^{\frac{3}{2}}\right)$

56. Question

Evaluate the following integrals:

$$\int \frac{x\tan^{-1}x^2}{1+x^4}dx$$

Answer

Assume $\tan^{-1}x^2 = t$

$d(\tan^{-1}x^2) = dt$

$\Rightarrow \frac{2x}{x^4 + 1} = dt$

$\Rightarrow \frac{x}{x^4 + 1} = \frac{dt}{2}$

Substituting t and dt

$\Rightarrow \frac{1}{2}\int tdt$

$\Rightarrow \frac{t^2}{4} + c$

But $t = \tan^{-1}x^2$

$\Rightarrow \frac{(\tan^{-1}x^2)^2}{4} + c.$

57. Question

Evaluate the following integrals:

$$\int \frac{\left(\sin^{-1}x\right)^3}{\sqrt{1-x^2}}dx$$

Answer

Assume $\sin^{-1}x = t$

$d(\sin^{-1}x) = dt$

$\Rightarrow \frac{dx}{\sqrt{1-x^2}} = dt$

∴ Substituting t and dt in given equation we get

$\Rightarrow \int t^3dt$

$\Rightarrow \frac{t^4}{4} + c$

But $t = \sin^{-1}x$

$\Rightarrow \frac{(\sin^{-1}x)^4}{4} + c.$

58. Question

Evaluate the following integrals:

$$\int \frac{\sin(2+3\log x)}{x} dx$$

Answer

Assume 2 + 3logx = t

d(2 + 3logx) = dt

$\Rightarrow \frac{3}{x} dx = dt$

$\Rightarrow \frac{1}{x} dx = \frac{dt}{3}$

Substituting t and dt

$\Rightarrow \frac{1}{3} \int \sin t \, dt$

= - cost + c

But t = 2 + 3logx

$\Rightarrow \frac{-1}{3} \cos(2 + 3\log x) + c.$

59. Question

Evaluate the following integrals:

$$\int x e^{x^2} dx$$

Answer

Assume $x^2 = t$

⇒ 2x.dx = dt

$\Rightarrow x.dx = \frac{dt}{2}$

Substituting t and dt

$\Rightarrow \int e^t . \frac{dt}{2}$

$\Rightarrow \frac{1}{2} e^t + c$

But $x^2 = t$

$\Rightarrow \frac{e^{x^2}}{2} + c.$

60. Question

Evaluate the following integrals:

$$\int \frac{e^{2x}}{1+e^x} dx$$

Answer

Assume $1 + e^x = t$

$e^x = t - 1$

$d(1 + e^x) = dt$

$e^x\,dx = dt$

$dx = \frac{dt}{e^x}$

Substitute t and dt we get

$\Rightarrow \int e^{2x}.\frac{dt}{e^x}$

$\Rightarrow \int e^x.dt$

$\Rightarrow \int (t-1)dt$

$\Rightarrow \int t.dt - \int dt$

$\Rightarrow \frac{t^2}{2} - t + c$

But $t = 1 + e^x$

$\Rightarrow \frac{(1+e^x)^2}{2} - (1 + e^x) + c$

61. Question

Evaluate the following integrals:

$\int \frac{\sec^2\sqrt{x}}{\sqrt{x}}dx$

Answer

Assume $\sqrt{x} = t$

$d(\sqrt{x}) = dt$

$\Rightarrow \frac{1}{2\sqrt{x}}dx = dt$

$\Rightarrow \frac{1}{\sqrt{x}}dx = 2dt$

Substituting t and dt

$\Rightarrow 2\int \sec^2 t\,dt$

$= 2\tan t + c$

But $\sqrt{x} = t$

$\Rightarrow 2\tan(\sqrt{x}) + c.$

62. Question

Evaluate the following integrals:

$\int \tan^3 2x \sec 2x\,dx$

Answer

$\tan^3 2x.\sec 2x = \tan^2 2x.\tan 2x.\sec 2x.dx$

$\tan^2 2x = \sec^2 2x - 1$

$\Rightarrow \tan^2 2x.\tan 2x.\sec 2x.dx = (\sec^2 2x - 1).\tan 2x.\sec 2x.dx$

$\Rightarrow \sec^2 2x\,\tan 2x.\sec 2xdx - \tan 2x.\sec 2xdx$

$\therefore \int \sec^2 2x . \tan 2x . \sec 2x \, dx - \int \tan 2x . \sec 2x . dx$

$\Rightarrow \int \sec^2 2x . \tan 2x . \sec 2x . dx - \frac{\sec 2x}{2} + c$

Assume sec2x = t

d(sec2x) = dt

sec2x.tan2x.dx = dt

$\Rightarrow \int t^2 . dt - \frac{\sec 2x}{2} + c$

$\Rightarrow \frac{t^3}{3} - \frac{\sec 2x}{2} + c$

But t = sec2x

$\Rightarrow \frac{(\sec 2x)^3}{3} - \frac{\sec 2x}{2} + c.$

63. Question

Evaluate the following integrals:

$$\int \frac{x + \sqrt{x+1}}{x+2} dx$$

Answer

The given equation can be written as

$\Rightarrow \int \frac{x}{x+2} dx + \int \frac{\sqrt{x+1}}{x+2} dx$

First integration be I1 and second be I2.

⇒ For I1

Add and subtract 2 from the numerator

$\Rightarrow \int \frac{x+2-2}{x+2}$

$\Rightarrow \int \frac{x+2}{x+2} . dx - \int \frac{2}{x+2} . dx$

$\Rightarrow \int dx - 2 \int \frac{dx}{x+2}$

⇒ x - 2ln|x + 2| + c1

∴ I1 = x - 2ln|x + 2| + c1

For I2

$\Rightarrow \int \frac{\sqrt{x+1}}{x+2} dx$

Assume x + 1 = t

dt = dx

$\Rightarrow \int \frac{\sqrt{t}}{t+1} dt$

Substitute u = √t

dt = 2√t.du

$t = u^2$

$\Rightarrow 2\int \frac{u^2}{u^2+1}du$

Add and subtract 1 in the above equation:

$\Rightarrow 2\int \frac{u^2+1-1}{u^2+1}du$

$\Rightarrow 2\int \frac{u^2+1}{u^2+1}du - \int \frac{1}{u^2+1}du$

$\Rightarrow 2\int du - \int \frac{1}{u^2+1}du$

⇒ $2u - \tan^{-1}(u) + c2$

But $u = \sqrt{t}$

∴ $2\sqrt{t} - \tan^{-1}(\sqrt{t}) + c2$

Also $t = x + 1$

∴ $2\sqrt{(x + 1)} - \tan^{-1}(x + 1) + c2$

$I = I1 + I2$

∴ $I = x - 2\ln|x + 2| + c1 + 2\sqrt{(x + 1)} - \tan^{-1}(x + 1) + c2$

$I = x - 2\ln|x + 2| + 2\sqrt{(x + 1)} - \tan^{-1}(x + 1) + c.$

64. Question

Evaluate the following integrals:

$$\int 5^{5^{5^x}} 5^{5^x} 5^x dx$$

Answer

Assume $5^{5^{5^x}} = t$

$\Rightarrow d\left(5^{5^{5^x}}\right) = dt$

$\Rightarrow 5^{5^{5^x}}.5^{5^x}5^x(\log 5^3)dx = dt$

Substituting t and dt

$\Rightarrow 5^{5^{5^x}}.5^{5^x}5^x.dx = \frac{dt}{(\log 5^3)}$

$\Rightarrow \int \frac{dt}{(\log 5^3)}$

$\Rightarrow \frac{1}{(\log 5^3)}\int dt + c$

$\Rightarrow \frac{t}{(\log 5^3)} + c$

But $t = 5^{5^{5^x}}$

$\Rightarrow \frac{5^{5^{5^x}}}{(\log 5^3)} + c$

65. Question

Evaluate the following integrals:

$$\int \frac{1}{x\sqrt{x^4-1}} dx$$

Answer

Assume $x^2 = t$

$2x.dx = dt$

$\Rightarrow dx = \frac{dt}{2x}$

Substituting t and dt

$\Rightarrow \int \frac{dt}{2x} \times \frac{1}{x \times \sqrt{t^2-1}}$

$\Rightarrow \int \frac{dt}{2x^2} \times \frac{1}{\sqrt{t^2-1}}$

$\Rightarrow \frac{1}{2}\int \frac{dt}{t\sqrt{t^2-1}}$

$\Rightarrow \frac{1}{2}\sec^{-1} t + c$

But $t = x^2$

$\Rightarrow \frac{1}{2}\sec^{-1} x^2 + c$

66. Question

Evaluate the following integrals:

$$\int \sqrt{e^x - 1}\, dx$$

Answer

Assume $e^x - 1 = t^2$

$d(e^x - 1) = d(t^2)$

$e^x.dx = 2t.dt$

$\Rightarrow dx = \frac{2t}{e^x} dt$

$e^x = t^2 + 1$

$\Rightarrow dx = \frac{2t}{t^2+1} dt$

Substituting t and dt

$\Rightarrow \int \sqrt{t^2} . \frac{2t}{t^2+1} dt$

$\Rightarrow \int t . \frac{2t}{t^2+1} dt$

$\Rightarrow \int \frac{2t^2}{t^2+1} dt$

$\Rightarrow 2\int \frac{t^2}{t^2+1} dt$

Add and subtract 1 in numerator

$\Rightarrow 2\int \frac{t^2+1-1}{t^2+1} dt$

$\Rightarrow 2\int \frac{t^2+1}{t^2+1}dt - 2\int \frac{1}{t^2+1}dt$

$\Rightarrow 2\int dt - 2\int \frac{1}{t^2+1}dt$

$\Rightarrow \int \frac{1}{t^2+1}dt = \tan^{-1}t + c$

$\Rightarrow 2t - 2\tan^{-1}(t) + c$

But $t = (e^x - 1)^{1/2}$

$\Rightarrow 2(e^x - 1)^{1/2} - 2\tan^{-1}(e^x - 1)^{1/2} + c$

67. Question

Evaluate the following integrals:

$$\int \frac{1}{(x+1)(x^2+2x+2)}dx$$

Answer

We can write $x^2 + 2x + 1 + 1 = (x + 1)^2 + 1$

$\Rightarrow \frac{1.dx}{(x+1)(x+1)^2+1}$

Assume $x + 1 = \tan t$

$\Rightarrow dx = \sec^2 t.dx$

$\Rightarrow \int \frac{\sec^2 t.dt}{\tan t.\tan^2 t+1}$

$\Rightarrow \tan^2 t + 1 = \sec^2 t.$

$\Rightarrow \int \frac{.dt}{\tan t.}$

$\Rightarrow \frac{\cos t}{\sin t}dt$

$\Rightarrow \log|\sin t| + c$

$\Rightarrow \sin t = \frac{\tan t}{\sec^2 t}$

But $\tan t = x + 1$

$\Rightarrow \sin t = \frac{x+1}{(1+x)^2+1}$

The final answer is

$\Rightarrow \log\sin\left|\frac{x+1}{x^2+2x+2}\right| + c$

68. Question

Evaluate the following integrals:

$$\int \frac{x^5}{\sqrt{1+x^3}}dx$$

Answer

Assume $x^3 + 1 = t^2$

$d(x^3 + 1) = d(t^2)$

$3x^2.dx = 2t.dt$

$\Rightarrow dx = \frac{2t}{3x^2} dt$

$x^3 + 1 = t^2$

$\Rightarrow dx = \frac{2t}{3x^2} dt$

Substituting t and dt

$\Rightarrow \int \frac{x^5}{\sqrt{t^2}} . \frac{2t}{3x^2} dt$

$\Rightarrow \int \frac{x^3}{t} . \frac{2t}{3} dt$

$\Rightarrow \int \frac{2x^3}{3} dt$

$\Rightarrow x^3 = t^2 - 1$

$\Rightarrow \frac{2}{3} \int (t^2 - 1).dt$

$\Rightarrow \frac{2}{3} \int t^2 dt - \frac{2}{3} \int dt$

$\Rightarrow \frac{2}{3} \times \frac{t^3}{3} - \frac{2}{3} t + c$

$\Rightarrow \frac{2}{9} (x^3 + 1)^{3\backslash 2} - \frac{2}{3} (x^3 + 1)^{1\backslash 2} + c$

69. Question

Evaluate the following integrals:

$\int 4x^3 \sqrt{5 - x^2}\, dx$

Answer

Assume $5 - x^2 = t^2$

$d(5 - x^2) = d(t^2)$

$- 2x.dx = 2t.dt$

$\Rightarrow x\, dx = - t.dx$

$\Rightarrow dx = \frac{-t}{x} dt$

Substituting t and dt

$\Rightarrow \int 4x^3 \sqrt{t^2} \frac{-t}{x} dt$

$\Rightarrow 4 \int x^2 t^2$

$\Rightarrow x^2 = 5 - t^2$

$\Rightarrow 4 \int (5 - t^2) t^2 .dt$

$\Rightarrow 20 \int t^2 dt - 4 \int t^4 dt$

$\Rightarrow 20 \times \frac{t^3}{3} - 4 \frac{t^5}{5} + c$

$\Rightarrow 20 (5 - x^2)^{3\backslash 2} - \frac{4}{5} (5 - x^2)^{5\backslash 2} + c$

70. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{x}+x} dx$$

Answer

$x = t^2$

d(x) = 2t.dt

dx = 2t.dt

Substituting t and dt we get

$\Rightarrow \int \frac{2t.dt}{t^2+t}$

$\Rightarrow 2\int \frac{t.dt}{t^2+t}$

$\Rightarrow 2\int \frac{1}{1+t} dt$

⇒ 2(ln|1 + t|)

But t = √x

⇒ 2(ln|1 + √x|) + c.

71. Question

Evaluate the following integrals:

$$\int \frac{1}{x^2\left(x^4+1\right)^{3/4}} dx$$

Answer

$I = \int \frac{1}{x^2(x^4+1)^{\frac{3}{4}}} dx$

$\Rightarrow \int \frac{1}{x^5\left(1+\frac{1}{x^4}\right)^{\frac{3}{4}}} dx$

Let $1 + \frac{1}{x^4} = t$

$\Rightarrow -\frac{4}{x^5} dx = dt$

$\Rightarrow \frac{1}{x^5} dx = \frac{-dt}{4}$

$I = \frac{-1}{4}\int \frac{1}{t^{\frac{3}{4}}} dt$

$\Rightarrow \frac{-1}{4}\left(\frac{t^{\frac{1}{4}}}{\frac{1}{4}}\right) + c$

$\Rightarrow -t^{\frac{1}{4}} + c$

But $t = 1 + \frac{1}{x^4}$

$\Rightarrow -\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}} + c$

72. Question

Evaluate the following integrals:

$$\int \frac{\sin^5 x}{\cos^4 x} dx$$

Answer

$\sin^5 x = \sin^4 x.\sin x$

Assume cos x = t

d(cosx) = dt

- sinx.dx = dt

$\Rightarrow dx = \frac{-dt}{\sin x}$

Substitute t and dt we get

$\Rightarrow \int \frac{\sin^4 x.\sin x}{\cos^4 x} \times \frac{-dt}{\sin x}$

$\Rightarrow \int \frac{-dt(1-\cos^2 x)^2}{\cos^4 x}$

$\Rightarrow \int \frac{-dt(1-t^2)^2}{t^4}$

$\Rightarrow -\int \frac{1+t^4-2t^2}{t^4} dt$

$\Rightarrow -\int \frac{1}{t^4} dt - \int \frac{t^4}{t^4} dt + 2\int \frac{t^2}{t^4} dt$

$\Rightarrow - \int t^{-4} dt - \int dt + 2\int t^{-2} dt$

$\Rightarrow \frac{t^{-3}}{3} - t - 2t^{-1} + c$

But t = cos x

$\Rightarrow \frac{\cos^{-3} x}{3} - \cos x - 2\cos^{-1} x + c$

Exercise 19.10

1. Question

Evaluate the followign integrals: $\int x^2 \sqrt{x+2}\, dx$

Answer

$\text{Let } I = \int x^2 \sqrt{x+2} dx$

Substituting, x + 2 = t ⇒dx = dt,

$I = \int (t-2)^2 \sqrt{t} dt$

$\Rightarrow I = \int (t^2 - 4t + 4) \sqrt{t} dt$

$\Rightarrow I = \int \left(t^{\frac{5}{2}} - 4t^{\frac{3}{2}} + 4t^{\frac{1}{2}}\right) dt$

$\Rightarrow \mathrm{I} = \frac{2}{7}t^{\frac{7}{2}} - \frac{8}{5}t^{\frac{5}{2}} + \frac{8}{2}t^{\frac{3}{2}} + c$

$\Rightarrow \mathrm{I} = \frac{2}{7}(x + 2)^{\frac{7}{2}} - \frac{8}{5}(x + 2)^{\frac{5}{2}} + \frac{8}{2}(x + 2)^{\frac{3}{2}} + c$

Therefore, $\int x^2\sqrt{x + 2}dx = \frac{2}{7}(x + 2)^{\frac{7}{2}} - \frac{8}{5}(x + 2)^{\frac{5}{2}} + \frac{8}{2}(x + 2)^{\frac{3}{2}} + c$

2. Question

Evaluate the following integrals: $\int \frac{x^2}{\sqrt{x-1}}dx$

Answer

Let $\mathrm{I} = \int \frac{x^2}{\sqrt{x - 1}}dx$

Substituting x - 1 = t ⇒ dx = dt,

$\Rightarrow \mathrm{I} = \int \frac{(t + 1)^2}{\sqrt{t}}dt$

$\Rightarrow \mathrm{I} = \int \frac{t^2 + 2t + 1}{\sqrt{t}}dt$

$\Rightarrow \mathrm{I} = \int \left(t^{\frac{3}{2}} + 2t^{\frac{1}{2}} + t^{-\frac{1}{2}}\right)dt$

$\Rightarrow \mathrm{I} = \frac{2}{5}t^{\frac{5}{2}} + 2t^{\frac{1}{2}} + \frac{4}{3}t^{\frac{3}{2}} + c$

$\Rightarrow \mathrm{I} = \frac{\left(6t^{\frac{5}{2}} + 30t^{\frac{1}{2}} + 20t^{\frac{3}{2}}\right)}{15} + c$

$\Rightarrow \mathrm{I} = \frac{2}{15}t^{\frac{1}{2}}(3t^2 + 15 + 10t) + c$

$\Rightarrow \mathrm{I} = \frac{2}{15}(x - 1)^{\frac{1}{2}}(3(x - 1)^2 + 15 + 10(x - 1)) + c$

$\Rightarrow \mathrm{I} = \frac{2}{15}(x - 1)^{\frac{1}{2}}(3(x^2 - 2x + 1)^2 + 15 + 10x - 10) + c$

$\Rightarrow \mathrm{I} = \frac{2}{15}(x - 1)^{\frac{1}{2}}(3x^2 + 4x + 8) + c$

Therefore, $\int \frac{x^2}{\sqrt{x - 1}}dx = \frac{2}{15}(x - 1)^{\frac{1}{2}}(3x^2 + 4x + 8) + c$

3. Question

Evaluate the following integrals: $\int \frac{x^2}{\sqrt{3x+4}}dx$

Answer

Let $\mathrm{I} = \int \frac{x^2}{\sqrt{3x + 4}}dx$

Substituting 3x + 4 = t ⇒ 3dx = dt,

$$\Rightarrow I = \int \frac{\left(\frac{t-4}{3}\right)^2}{3\sqrt{t}} dt$$

$$\Rightarrow I = \frac{1}{27}\int \frac{t^2 + 16 - 8t}{\sqrt{t}} dt$$

$$\Rightarrow I = \frac{1}{27}\int \left(t^{\frac{3}{2}} - 8t^{\frac{1}{2}} + 16t^{-\frac{1}{2}}\right) dt$$

$$\Rightarrow I = \frac{1}{27}\left[\frac{2}{5}t^{\frac{5}{2}} - \frac{16}{3}t^{\frac{3}{2}} + 32t^{\frac{1}{2}}\right] + c$$

$$\Rightarrow I = \frac{1}{27}\left[\frac{2}{5}(3x + 4)^{\frac{5}{2}} - \frac{16}{3}(3x + 4)^{\frac{3}{2}} + 32(3x + 4)^{\frac{1}{2}}\right] + c$$

$$\Rightarrow I = \frac{2}{135}(3x + 4)^{\frac{5}{2}} - \frac{16}{81}(3x + 4)^{\frac{3}{2}} + \frac{32}{27}(3x + 4)^{\frac{1}{2}} + c$$

Therefore, $\int \frac{x^2}{\sqrt{3x + 4}} dx = \frac{2}{135}(3x + 4)^{\frac{5}{2}} - \frac{16}{81}(3x + 4)^{\frac{3}{2}} + \frac{32}{27}(3x + 4)^{\frac{1}{2}} + c$

4. Question

Evaluate the following integrals: $\int \frac{2x - 1}{(x - 1)^2} dx$

Answer

Let $I = \int \frac{2x - 1}{(x - 1)^2} dx$

Substituting x - 1 = t ⇒dx = dt

$$\Rightarrow I = \int \frac{2(t + 1) - 1}{t^2} dt$$

$$\Rightarrow I = \int \frac{2t + 1}{t^2} dt$$

$$\Rightarrow I = \int \left(\frac{2}{t} + \frac{1}{t^2}\right) dt$$

$$\Rightarrow I = 2\log|t| + \frac{1}{t} + c$$

$$\Rightarrow I = 2\log|x - 1| + \frac{1}{x - 1} + c$$

Therefore, $\int \frac{2x - 1}{(x - 1)^2} dx = 2\log|x - 1| + \frac{1}{x - 1} + c$

5. Question

Evaluate the following integrals: $\int (2x^2 + 3)\sqrt{x + 2}\, dx$

Answer

Let $I = \int (2x^2 + 3)\sqrt{x + 2} dx$

Substituting x + 2 = t ⇒ dx = dt

$\Rightarrow I = \int [2(t-2)^2 + 3]\sqrt{t}dt$

$\Rightarrow I = \int [2t^2 - 8t + 8 + 3]\sqrt{t}dt$

$\Rightarrow I = \int \left[2t^{\frac{5}{2}} - 8t^{\frac{3}{2}} + 11^{\frac{1}{2}}\right]dt$

$\Rightarrow I = \frac{4}{7}t^{\frac{7}{2}} - \frac{16}{5}t^{\frac{5}{2}} + \frac{22}{3}t^{\frac{3}{2}} + c$

$\Rightarrow I = \frac{4}{7}(x+2)^{\frac{7}{2}} - \frac{16}{5}(x+2)^{\frac{5}{2}} + \frac{22}{3}(x+2)^{\frac{3}{2}} + c$

$\therefore \int (2x^2 + 3)\sqrt{x+2}dx = \frac{4}{7}(x+2)^{\frac{7}{2}} - \frac{16}{5}(x+2)^{\frac{5}{2}} + \frac{22}{3}(x+2)^{\frac{3}{2}} + c$

6. Question

Evaluate the following integrals: $\int \frac{x^2 + 3x + 1}{(x+1)^2}dx$

Answer

Let $I = \int \frac{x^2 + 3x + 1}{(x+1)^2}dx$

Substituting $x + 1 = t \Rightarrow dx = dt$

$\Rightarrow I = \int \frac{(t-1)^2 + 3(t-1) + 1}{t^2}dt$

$\Rightarrow I = \int \frac{t^2 - 2t + 1 + 3t - 3 + 1}{t^2}dt$

$\Rightarrow I = \int \frac{t^2 + t - 1}{t^2}dt$

$\Rightarrow I = \int \left(1 + \frac{1}{t} - \frac{1}{t^2}\right)dt$

$\Rightarrow I = t + \log|t| - \frac{1}{t} + c$

$\Rightarrow I = (x+1) + \log|x+1| + \frac{1}{x+1} + c$

Therefore, $\int \frac{x^2 + 3x + 1}{(x+1)^2}dx = (x+1) + \log|x+1| + \frac{1}{x+1} + c$

7. Question

Evaluate the following integrals: $\int \frac{x^2}{\sqrt{1-x}}dx$

Answer

Let $I = \int \frac{x^2}{\sqrt{1-x}}dx$

Substituting $1 - x = t \Rightarrow dx = -dt$,

$$\Rightarrow I = -\int \frac{(1-t)^2}{\sqrt{t}} dt$$

$$\Rightarrow I = -\int \frac{t^2 - 2t + 1}{\sqrt{t}} dt$$

$$\Rightarrow I = -\int \left(t^{\frac{3}{2}} - 2t^{\frac{1}{2}} + t^{-\frac{1}{2}}\right) dt$$

$$\Rightarrow I = -\left[\frac{2}{5}t^{\frac{5}{2}} + 2t^{\frac{1}{2}} - \frac{4}{3}t^{\frac{3}{2}}\right] + c$$

$$\Rightarrow I = \frac{-\left(6t^{\frac{5}{2}} + 30t^{\frac{1}{2}} - 20t^{\frac{3}{2}}\right)}{15} + c$$

$$\Rightarrow I = \frac{-2}{15}t^{\frac{1}{2}}(3t^2 + 15 - 10t) + c$$

$$\Rightarrow I = \frac{-2}{15}(1-x)^{\frac{1}{2}}(3(1-x)^2 + 15 - 10(1-x)) + c$$

$$\Rightarrow I = \frac{2}{15}(1-x)^{\frac{1}{2}}(3(x^2 - 2x + 1)^2 + 15 + 10x - 10) + c$$

$$\Rightarrow I = \frac{2}{15}(1-x)^{\frac{1}{2}}(3x^2 + 4x + 8) + c$$

Therefore, $\int \frac{x^2}{\sqrt{1-x}} dx = \frac{2}{15}(1-x)^{\frac{1}{2}}(3x^2 + 4x + 8) + c$

8. Question

Evaluate the following integrals:$\int x(1 - x)^{23}$ dx

Answer

Let $I = \int x(1-x)^{23} dx$

Substituting 1 - x = t ⇒ dx = - dt

$$\Rightarrow I = -\int (1-t)t^{23} dt$$

$$\Rightarrow I = -\int (t^{23} - t^{24}) dt$$

$$\Rightarrow I = -\left[\frac{t^{24}}{24} - \frac{t^{25}}{25}\right] + c$$

$$\Rightarrow I = \frac{t^{25}}{25} - \frac{t^{24}}{24} + c$$

$$\Rightarrow I = \frac{(1-x)^{25}}{25} - \frac{(1-x)^{24}}{24} + c$$

$$\Rightarrow I = \frac{1}{600}(1-x)^{24}[24(1-x) - 25]$$

$$\Rightarrow I = -\frac{1}{600}(1-x)^{24}[1 + 24x] + c$$

9. Question

Evaluate the following integrals: $\int \frac{1}{\sqrt{x}+\sqrt[4]{x}} dx$

Answer

Let $I = \int \frac{1}{\sqrt{x} + \sqrt[4]{x}} dx$

$\Rightarrow I = \int \frac{1}{\sqrt[4]{x}(\sqrt[4]{x} + 1)} dx$

Multiplying and dividing by $\sqrt{x}$

$\Rightarrow I = \int \frac{x^{\frac{1}{2}}}{x^{\frac{3}{4}}(\sqrt[4]{x} + 1)} dx$

Let, $\sqrt[4]{x} + 1 = t \Rightarrow \frac{1}{4}x^{-\frac{3}{4}}dx = dt$

So, $\Rightarrow I = 4\int \frac{(t-1)^2}{t} dt$

$\Rightarrow I = 4\int \frac{t^2 - 2t + 1}{t} dt$

$\Rightarrow I = 4\int \left(t - 2 + \frac{1}{t}\right) dt$

$\Rightarrow I = 4\left(\frac{t^2}{2} - 2t + \log|t|\right) + c$

$\Rightarrow I = 4\left(\frac{(\sqrt[4]{x} + 1)^2}{2} - 2(\sqrt[4]{x} + 1) + \log|(\sqrt[4]{x} + 1)|\right) + c$

Therefore, $\int \frac{1}{\sqrt{x} + \sqrt[4]{x}} dx$

$= 4\left(\frac{(\sqrt[4]{x} + 1)^2}{2} - 2(\sqrt[4]{x} + 1) + \log|(\sqrt[4]{x} + 1)|\right) + c$

10. Question

Evaluate the following integrals: $\int \frac{1}{x^{1/3}\left(x^{1/3}-1\right)} dx$

Answer

Let $I = \int \frac{1}{x^{\frac{1}{3}}\left(x^{\frac{1}{3}} - 1\right)} dx$

Multiplying and dividing by $x^{\frac{1}{3}}$

$\Rightarrow I = \int \frac{x^{\frac{1}{3}}}{x^{\frac{2}{3}}\left(x^{\frac{1}{3}} - 1\right)} dx$

Let, $x^{\frac{1}{3}} - 1 = t \Rightarrow \frac{1}{3}x^{-\frac{2}{3}}dx = dt$

$$\text{So,} \Rightarrow I = 3\int \frac{(t+1)}{t} dt$$

$$\Rightarrow I = 3\int \left(t + \frac{1}{t}\right) dt$$

$$\Rightarrow I = 3\left(\frac{t^2}{2} + \log|t|\right) + c$$

$$\Rightarrow I = 3\left(\frac{\left(x^{\frac{1}{3}} - 1\right)^2}{2} + \log\left|\left(x^{\frac{1}{3}} - 1\right)\right|\right) + c$$

$$\text{Therefore,} \int \frac{1}{\sqrt{x} + \sqrt[4]{x}} dx = 3\left(\frac{\left(x^{\frac{1}{3}} - 1\right)^2}{2} + \log\left|\left(x^{\frac{1}{3}} - 1\right)\right|\right) + c$$

Exercise 19.11

1. Question

Evaluate the following integrals:

$\int \tan^3 x \sec^2 x \, dx$

Answer

$$\text{Let } I = \int \tan^3 x \sec^2 x \, dx$$

Let tan x = t, then

$\Rightarrow \sec^2 x \, dx = dt$

$$\Rightarrow I = \int t^3 dt$$

$$\Rightarrow I = \frac{t^4}{4} + c$$

$$\Rightarrow I = \frac{\tan^4 x}{4} + c$$

$$\text{Therefore,} \int \tan^3 x \sec^2 x \, dx = \frac{\tan^4 x}{4} + c$$

2. Question

Evaluate the following integrals:

$\int \tan x \sec^4 x \, dx$

Answer

$$\text{Let } I = \int \tan x \sec^4 x \, dx$$

$$\Rightarrow I = \int \tan x \sec^2 x \sec^2 x \, dx$$

$$\Rightarrow I = \int \tan x (1 + \tan^2 x) \sec^2 x \, dx$$

$\Rightarrow I = \int (\tan x + \tan^3 x) \sec^2 x \, dx$

Let tan x = t, then

$\Rightarrow \sec^2 x \, dx = dt$

$\Rightarrow I = \int (t + t^3) dt$

$\Rightarrow I = \frac{t^2}{2} + \frac{t^4}{4} + c$

$\Rightarrow I = \frac{\tan^2 x}{2} + \frac{\tan^4 x}{4} + c$

Therefore, $\int \tan x \sec^4 x \, dx = \frac{\tan^2 x}{2} + \frac{\tan^4 x}{4} + c$

3. Question

Evaluate the following integrals:

$\int \tan^5 x \sec^4 x \, dx$

Answer

Let $I = \int \tan^5 x \sec^4 x \, dx$

$\Rightarrow I = \int \tan^5 x \sec^2 x \sec^2 x \, dx$

$\Rightarrow I = \int \tan^5 x (1 + \tan^2 x) \sec^2 x \, dx$

$\Rightarrow I = \int (\tan^5 x + \tan^7 x) \sec^2 x \, dx$

Let tan x = t, then

$\Rightarrow \sec^2 x \, dx = dt$

$\Rightarrow I = \int (t^5 + t^7) dt$

$\Rightarrow I = \frac{t^6}{6} + \frac{t^8}{8} + c$

$\Rightarrow I = \frac{\tan^6 x}{6} + \frac{\tan^8 x}{8} + c$

Therefore, $\int \tan^5 x \sec^4 x \, dx = \frac{\tan^6 x}{6} + \frac{\tan^8 x}{8} + c$

4. Question

Evaluate the following integrals:

$\int \sec^6 x \tan x \, dx$

Answer

Let $I = \int \sec^6 x \tan x \, dx$

$\Rightarrow I = \int \sec^5 x(\sec x \tan x)dx$

Substituting, sec x = t ⇒ sec x tan x dx = dt

$\Rightarrow I = \int t^5 dt$

$\Rightarrow I = \frac{t^6}{6} + c$

$\Rightarrow I = \frac{\sec^6 x}{6} + c$

Therefore, $\int \sec^5 x(\sec x \tan x)dx = \frac{\sec^6 x}{6} + c$

5. Question

Evaluate the following integrals:

$\int \tan^5 x\, dx$

Answer

Let $I = \int \tan^5 x\, dx$

$\Rightarrow I = \int \tan^2 x \tan^3 x\, dx$

$\Rightarrow I = \int (\sec^2 x - 1) \tan^3 x\, dx$

$\Rightarrow I = \int \tan^3 x \sec^2 x\, dx - \int \tan^3 x\, dx$

$\Rightarrow I = \int \tan^3 x \sec^2 x\, dx - \int (\sec^2 x - 1) \tan x\, dx$

$\Rightarrow I = \int \tan^3 x \sec^2 x\, dx - \int (\sec^2 x \tan x)dx + \int \tan x\, dx$

Let tan x = t, then

$\Rightarrow \sec^2 x\, dx = dt$

$\Rightarrow I = \int t^3 dt - \int t dt + \int \tan x\, dx$

$\Rightarrow I = \frac{t^4}{4} - \frac{t^2}{2} + \log|\sec x| + c$

$\Rightarrow I = \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \log|\sec x| + c$

Therefore, $\int \tan^5 x\, dx = \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \log|\sec x| + c$

6. Question

Evaluate the following integrals:

$\int \sqrt{\tan x} \sec^4 x\, dx$

Answer

Let $I = \int \sqrt{\tan x} \sec^4 x \, dx$

$\Rightarrow I = \int \sqrt{\tan x} \sec^2 x \sec^2 x \, dx$

$\Rightarrow I = \int \sqrt{\tan x} (1 + \tan^2 x) \sec^2 x \, dx$

$\Rightarrow I = \int (\tan^{\frac{1}{2}} x + \tan^{\frac{5}{2}} x) \sec^2 x \, dx$

Let tan x = t, then

$\Rightarrow \sec^2 x \, dx = dt$

$\Rightarrow I = \int \left(t^{\frac{1}{2}} + t^{\frac{5}{2}}\right) dt$

$\Rightarrow I = \frac{2}{3} t^{\frac{3}{2}} + \frac{2}{7} t^{\frac{7}{2}} + c$

$\Rightarrow I = \frac{2}{3} \tan^{\frac{3}{2}} x + \frac{2}{7} \tan^{\frac{7}{2}} x + c$

Therefore, $\int \sqrt{\tan x} \sec^4 x \, dx = \frac{2}{3} \tan^{\frac{3}{2}} x + \frac{2}{7} \tan^{\frac{7}{2}} x + c$

7. Question

Evaluate the following integrals:

$\int \sec^4 2x \, dx$

Answer

Let $I = \int \sec^4 2x \, dx$

$\Rightarrow I = \int \sec^2 2x \sec^2 2x \, dx$

$\Rightarrow I = \int (1 + \tan^2 2x) \sec^2 2x \, dx$

$\Rightarrow I = \int (\sec^2 2x + \tan^2 2x \sec^2 2x) dx$

Let tan 2x = t, then

$\Rightarrow 2\sec^2 2x \, dx = dt$

$\Rightarrow I = \frac{1}{2} \int (1 + t^2) dt$

$\Rightarrow I = \frac{1}{2} t + \frac{1}{2} \cdot \frac{1}{3} t^3 + c$

$\Rightarrow I = \frac{1}{2} \tan 2x + \frac{1}{6} \tan^3 2x + c$

Therefore, $\int \sec^4 2x \, dx = \frac{1}{2} \tan 2x + \frac{1}{6} \tan^3 2x + c$

8. Question

Evaluate the following integrals:

$\int \text{cosec}^4 3x\, dx$

Answer

$$\text{Let } I = \int \text{cosec}^4 3x\, dx$$

$$\Rightarrow I = \int \text{cosec}^2 3x\, \text{cosec}^2 3x\, dx$$

$$\Rightarrow I = \int (1 + \cot^2 3x)\, \text{cosec}^2 3x\, dx$$

$$\Rightarrow I = \int (\text{cosec}^2 3x + \cot^2 3x\, \text{cosec}^2 3x) dx$$

Let cot 3x = t, then

$\Rightarrow - 3\text{cosec}^2 3x\, dx = dt$

$$\Rightarrow I = -\frac{1}{3}\int (1 + t^2) dt$$

$$\Rightarrow I = -\frac{1}{3}t - \frac{1}{3}.\frac{1}{3}t^3 + c$$

$$\Rightarrow I = -\frac{1}{3}\cot 3x - \frac{1}{9}\cot^3 3x + c$$

$$\text{Therefore, } \int \text{cosec}^4 3x\, dx = -\frac{1}{3}\cot 3x - \frac{1}{9}\cot^3 3x + c$$

9. Question

Evaluate the following integrals:

$\int \cot^n x\, \text{cosec}^2 x\, dx,\ n \neq -1$

Answer

$$\text{Let } I = \int \cot^n x\, \text{cosec}^2 x dx$$

Let cot x = t ⇒ - $\text{cosec}^2 x\, dx = dt$

$$\Rightarrow I = -\int t^n dt$$

$$\Rightarrow I = -\frac{t^{n+1}}{n+1} + c$$

$$\Rightarrow I = -\frac{\cot^{n+1} x}{n+1} + c$$

$$\text{Therefore, } \int \cot^n x\, \text{cosec}^2 x dx = -\frac{\cot^{n+1} x}{n+1} + c$$

10. Question

Evaluate the following integrals:

$\int \cot^5 x\, \text{cosec}^4 x\, dx$

Answer

$$\text{Let } I = \int \cot^5 x\, \text{cosec}^4 x\, dx$$

$$\Rightarrow I = \int \cot^5 x \operatorname{cosec}^2 x \operatorname{cosec}^2 x \, dx$$

$$\Rightarrow I = \int \cot^5 x (1 + \cot^2 x) \operatorname{cosec}^2 x \, dx$$

$$\Rightarrow I = \int (\cot^5 x + \cot^7 x) \operatorname{cosec}^2 x \, dx$$

Let cot x = t, then

$\Rightarrow$ - cosec^2 x dx = dt

$$\Rightarrow I = -\int (t^5 + t^7) dt$$

$$\Rightarrow I = -\frac{t^6}{6} - \frac{t^8}{8} + c$$

$$\Rightarrow I = -\frac{\cot^6 x}{6} - \frac{\cot^8 x}{8} + c$$

Therefore, $\int \cot^5 x \operatorname{cosec}^4 x \, dx = -\frac{\cot^6 x}{6} - \frac{\cot^8 x}{8} + c$

11. Question

Evaluate the following integrals:

$\int \cot^5 x \, dx$

Answer

Let $I = \int \cot^5 x \, dx$

$$\Rightarrow I = \int \cot^2 x \cot^3 x \, dx$$

$$\Rightarrow I = \int (\operatorname{cosec}^2 x - 1) \cot^3 x \, dx$$

$$\Rightarrow I = \int \cot^3 x \operatorname{cosec}^2 x \, dx - \int \cot^3 x \, dx$$

$$\Rightarrow I = \int \cot^3 x \operatorname{cosec}^2 x \, dx - \int (\operatorname{cosec}^2 x - 1) \cot x \, dx$$

$$\Rightarrow I = \int \cot^3 x \operatorname{cosec}^2 x \, dx - \int (\operatorname{cosec}^2 x \cot x) dx + \int \cot x \, dx$$

Let cot x = t, then

$\Rightarrow$ - cosec^2 x dx = dt

$$\Rightarrow I = -\int t^3 dt + \int t dt + \int \cot x \, dx$$

$$\Rightarrow I = -\frac{t^4}{4} + \frac{t^2}{2} + \log|\sin x| + c$$

$$\Rightarrow I = -\frac{\cot^4 x}{4} + \frac{\cot^2 x}{2} + \log|\sin x| + c$$

Therefore, $\int \cot^5 x \, dx = -\frac{\cot^4 x}{4} + \frac{\cot^2 x}{2} + \log|\sin x| + c$

12. Question

Evaluate the following integrals:

$\int \cot^6 x \, dx$

Answer

Let $I = \int \cot^6 x \, dx$

$\Rightarrow I = \int \cot^2 x \cot^4 x \, dx$

$\Rightarrow I = \int (\operatorname{cosec}^2 x - 1) \cot^4 x \, dx$

$\Rightarrow I = \int \cot^4 x \operatorname{cosec}^2 x \, dx - \int \cot^4 x \, dx$

$\Rightarrow I = \int \cot^4 x \operatorname{cosec}^2 x \, dx - \int (\operatorname{cosec}^2 x - 1) \cot^2 x \, dx$

$\Rightarrow I = \int \cot^4 x \operatorname{cosec}^2 x \, dx - \int (\operatorname{cosec}^2 x \cot^2 x) dx + \int \cot^2 x \, dx$

$\Rightarrow I = \int \cot^4 x \operatorname{cosec}^2 x \, dx - \int (\operatorname{cosec}^2 x \cot^2 x) dx + \int (\operatorname{cosec}^2 x - 1) dx$

Let cot x = t, then

$\Rightarrow - \operatorname{cosec}^2 x \, dx = dt$

$\Rightarrow I = -\int t^4 dt + \int t^2 dt - \int dt - \int dx$

$\Rightarrow I = -\frac{t^5}{5} + \frac{t^3}{3} - t - x + c$

$\Rightarrow I = -\frac{\cot^5 x}{5} + \frac{\cot^3 x}{3} - \cot x - x + c$

Therefore, $\int \cot^6 x \, dx = \Rightarrow I = -\frac{\cot^5 x}{5} + \frac{\cot^3 x}{3} - \cot x - x + c$

Exercise 19.12

1. Question

Evaluate the following integrals:

$\int \sin^4 x \cos^3 x \, dx$

Answer

Let $\sin x = t$

We know the Differentiation of $\sin x = \cos x$

$dt = d(\sin x) = \cos x dx$

So, $dx = \frac{dt}{\cos x}$

substitute all in above equation,

$\int \sin^4 x \cos^3 x \, dx = \int t^4 \cos^3 x \frac{dt}{\cos x}$

$= \int t^4 \cos^2 x \, dt$

$= \int t^4(1 - \sin^2 x) \, dt$

$= \int t^4(1 - t^2) \, dt$

$= \int (t^4 - t^6) \, dt$

We know, basic integration formula, $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$ for any c≠-1

Hence, $\int (t^4 - t^6) dt = \frac{t^5}{5} - \frac{t^7}{7} + c$

Put back t = sin x

$\int \sin^4 x \cos^3 x \, dx = \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + c$

2. Question

Evaluate the following integrals:

$\int \sin^5 x \, dx$

Answer

$\int \sin^5 x \, dx = \int \sin^3 x \sin^2 x \, dx$

$= \int \sin^3 x(1 - \cos^2 x)dx$ { since $\sin^2 x + \cos^2 x = 1$}

$= \int (\sin^3 x - \sin^3 x \cos^2 x) \, dx$

$= \int(\sin x (\sin^2 x) - \sin^3 x \cos^2 x) \, dx$

$= \int(\sin x (1 - \cos^2 x) - \sin^3 x \cos^2 x) \, dx$ { since $\sin^2 x + \cos^2 x = 1$}

$= \int(\sin x - \sin x \cos^2 x - \sin^3 x \cos^2 x) dx$

$= \int \sin x \, dx - \int \sin x \cos^2 x \, dx - \int \sin^3 x \cos^2 x \, dx$ (separate the integrals)

We know , d(cos x) = -sin xdx

So put cos x = t and dt = -sin xdx in above integrals

$= \int \sin x \, dx - \int \sin x \cos^2 x \, dx - \int \sin^3 x \cos^2 x \, dx$

$= \int \sin x \, dx - \int t^2 (-dt) - \int (\sin^2 x \sin x) \, t^2 \, dx$

$= \int \sin x \, dx - \int t^2 (-dt) - \int (1 - \cos^2 x) t^2 \, (-dt)$

$= \int \sin x \, dx + \int t^2 dt) + \int (1 - t^2) t^2 \, dt$

$= \int \sin x \, dx + \int t^2 dt) + \int (t^2 - t^4) dt$

$= -\cos x + \frac{t^3}{3} + \frac{t^3}{3} - \frac{t^5}{5} + c$ (since $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$ for any $c \neq -1$)

Put back t = cos x

$= -\cos x + \frac{t^3}{3} + \frac{t^3}{3} - \frac{t^5}{5} + c$

$= -\cos x + \frac{\cos^3 x}{3} + \frac{\cos^3 x}{3} - \frac{\cos^5 x}{5} + c$

$= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c = -[\cos x - \frac{2}{3}\cos^3 x + \frac{1}{5}\cos^5 x] + c$

3. Question

Evaluate the following integrals:

$\int \cos^5 x\, dx$

Answer

$\int \cos^5 x\, dx = \int \cos^3 x \cos^2 x\, dx$

$= \int \cos^3 x(1 - \sin^2 x)dx$ { since $\sin^2 x + \cos^2 x = 1$}

$= \int (\cos^3 x - \cos^3 x \sin^2 x)\, dx$

$= \int(\cos x\, (\cos^2 x) - \cos^3 x \sin^2 x)\, dx$

$= \int(\cos x\, (1 - \sin^2 x) - \cos^3 x \sin^2 x)\, dx$ { since $\sin^2 x + \cos^2 x = 1$}

$= \int(\cos x\, - \cos x \sin^2 x - \cos^3 x \sin^2 x)dx$

$= \int \cos x\, dx - \int \cos x \sin^2 x\, dx - \int \cos^3 x \sin^2 x\, dx$ (separate the integrals)

We know , d(sin x) = cos xdx

So put sin x = t and dt = cos xdx in above integrals

$= \int \cos x\, dx - \int t^2 dt - \int \cos x \cos^2 x \sin^2 x\, dx$

$= \int \cos x\, dx - \int t^2 (dt) - \int (\cos^2 x \cos x)\, t^2\, dx$

$= \int \cos x\, dx - \int t^2 (dt) - \int (1 - \sin^2 x)t^2\, (dt)$

$= \int \cos x\, dx - \int t^2 dt) - \int (1 - t^2)t^2\, dt$

$= \int \cos x\, dx - \int t^2 dt) - \int (t^2 - t^4)dt$

$= \sin x - \frac{t^3}{3} - \frac{t^3}{3} + \frac{t^5}{5} + c$ (since $\int x^n\, dx = \frac{x^{n+1}}{n+1} + c$ for any $c \neq -1$)

Put back t = sin x

$= \sin x - \frac{\sin^3 x}{3} - \frac{\sin^3 x}{3} + \frac{\cos^5 x}{5} + c$

$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + c$

4. Question

Evaluate the following integrals:

$\int \sin^5 x \cos x\, dx$

Answer

Let sin x = t

Then d(sin x) = dt = cos xdx

Put t = sin x and dt = cos xdx in above equation

$\int \sin^5 x \cos x\, dx = \int t^5 dt$

$= \frac{t^6}{6} + c$ (since $\int x^n\, dx = \frac{x^{n+1}}{n+1} + c$ for any $c \neq -1$)

$= \frac{\sin^6 x}{6} + c$

5. Question

Evaluate the following integrals:

$\int \sin^3 x \cos^6 x\, dx$

Answer

Since power of sin is odd, put cos x = t

Then dt = -sin xdx

Substitute these in above equation,

$\int \sin^3 x \cos^6 x\, dx = \int \sin x \sin^2 x\, t^6\, dx$

$= \int (1-\cos^2 x)\, t^6 \sin x\, dx$

$= \int (1-t^2)\, t^6 dt$

$= \int (t^6 - t^8) dt$

$= \frac{t^7}{7} - \frac{t^9}{9} + c$ (since $\int x^n\, dx = \frac{x^{n+1}}{n+1} + c$ for any $c \neq -1$)

$= \frac{1}{7}\cos^7 x + \frac{1}{9}\cos^9 x + c$

6. Question

Evaluate the following integrals:

$\int \cos^7 x\, dx$

Answer

$\int \cos^7 x\, dx = \int \cos^6 x \cos x\, dx$

$= \int (\cos^2 x)^3 \cos x\, dx$

$= \int (1-\sin^2 x)^3 \cos x\, dx$ { since $\sin^2 x + \cos^2 x = 1$}

We know $(a-b)^3 = a^3 b^3 - 3a^2 b + 3ab^2$

Here, a = 1 and b = $\sin^2 x$

Hence, $\int (1-\sin^2 x)^3 \cos x\, dx = \int (1 - \sin^6 x - 3\sin^2 x + 3\sin^4 x) \cos x\, dx$

$= \int (\cos x\, dx - \sin^6 x \cos x\, dx - 3\sin^2 x \cos x\, dx + 3\sin^4 x \cos x\, dx)$ {take cos xdx inside brackets)

$= \int \cos x\, dx - \int \sin^6 x \cos x\, dx - 3\int \sin^2 x \cos x\, dx + 3\int \sin^4 x \cos x\, dx$ (separate the integrals)

Put sinx = t and cos xdx = dt

$= \int \cos x\, dx - \int t^6 dt - 3\int t^2 dt + 3\int t^4 dt$

$= \sin x - \frac{t^7}{7} - \frac{3t^3}{3} - \frac{3t^5}{5} + c$

$= \sin x - \frac{t^7}{7} - t^3 - \frac{3t^5}{5} + c$

Put back t = sin x

$= \sin x - \sin^3 x + \frac{3}{5}\sin^5 x - \frac{1}{7}\sin^7 x + c$

7. Question

Evaluate the following integrals:

$\int x \cos^3 x^2 \sin x^2\, dx$

Answer

Let $\cos x^2 = t$

Then $d(\cos x^2) = dt$

Since $d(x^n) = nx^{n-1}$ and $d(\cos x) = -\sin x\, dx$

$dt = 2x(-\sin x^2) = -2x \sin x^2\, dx$

$x \sin x^2 dx = -\frac{dt}{2}$

hence $\int x \cos^3 x^2 \sin x^2\, dx = \int t^3 \times -\frac{dt}{2}$

$= -\frac{1}{2}\int t^3 dt$

$= -\frac{1}{2} \times \frac{t^4}{4} + c$

$= -\frac{1}{8} \cos^4 x^2 + c$

8. Question

Evaluate the following integrals:

$\int \sin^7 x\, dx$

Answer

$\int \sin^7 x\, dx = \int \sin^6 x \sin x\, dx$

$= \int (\sin^2 x)^3 \sin x\, dx$

$= \int (1 - \cos^2 x)^3 \sin x\, dx$ { since $\sin^2 x + \cos^2 x = 1$}

We know $(a-b)^3 = a^3 - b^3 - 3a^2b + 3ab^2$

Here, $a = 1$ and $b = \cos^2 x$

Hence, $\int (1 - \cos^2 x)^3 \sin x\, dx = \int (1 - \cos^6 x - 3\cos^2 x + 3\cos^4 x) \sin x\, dx$

$= \int (\sin x\, dx - \cos^6 x \sin x\, dx - 3\cos^2 x \sin x\, dx + 3\cos^4 x \sin x\, dx)$ {take sin xdx inside brackets}

$= \int \sin x\, dx - \int \cos^6 x \sin x\, dx - 3\int \cos^2 x \sin x\, dx + 3\int \cos^4 x \sin x\, dx$ (separate the integrals)

Put $\cos x = t$ and $-\sin x\, dx = dt$

$= \int \sin x\, dx - \int t^6(-dt) - 3\int t^2(-dt) + 3\int t^4(-dt)$

$= -\cos x + \frac{t^7}{7} + \frac{3t^3}{3} - \frac{3t^5}{5} + c$

$= -\cos x + \frac{t^7}{7} + t^3 - \frac{3t^5}{5} + c$

Put back $t = \cos x$

$= -\cos x + \cos^3 x - \frac{3}{5}\cos^5 x + \frac{1}{7}\cos^7 x + c$

9. Question

Evaluate the following integrals:

$\int \sin^3 x \cos^5 x\, dx$

Answer

Let $\cos x = t$ then $dt = -\sin x dx$

$dx = -\frac{dt}{\sin x}$

Substitute all these in the above equation,

$\int \sin^3 x \cos^5 x \, dx = \int \sin^3 x \, t^5 (-\frac{dt}{\sin x})$

$= -\int \sin^2 x t^5 dt$

$= -\int (1 - \cos^2 x) t^5 dt$

$= -\int (1 - t^2) t^5 dt$

$= -\int t^5 dt - \int t^7 dt$

$= -\frac{t^6}{6} + \frac{t^8}{8} + c$ (since $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$ for any $c \neq -1$)

$= -\frac{\cos^6 x}{6} + \frac{\cos^8 x}{8} + c$

$= \frac{1}{8}\cos^8 x - \frac{1}{6}\cos^6 x + c$

10. Question

Evaluate the following integrals:

$$\int \frac{1}{\sin^4 x \cos^2 x} dx$$

Answer

$$\int \frac{1}{\sin^4 x \cos^2 x} dx = \int \sin^{-4} x \cos^{-2} x dx$$

Adding the powers : -4 + -2 = -6

Since all are even nos, we will divide each by $\cos^6 x$ to convert into positive power

So, $\int \frac{1}{\sin^4 x \cos^2 x} dx = \int \frac{\frac{1}{\cos^6 x}}{\frac{\sin^4 x \cos^2 x}{\cos^6 x}} dx$

$= \int \frac{\sec^6 x}{\frac{\sin^4 x}{coc^4 x}} dx = \int \frac{\sec^6 x}{\tan^4 x} dx$

$= \int \frac{\sec^4 x \sec^2 x}{\tan^4 x} dx = \int \frac{(\sec^2 x)^2 \sec^2 x}{\tan^4 x} dx$

$= \int \frac{(1 + \tan^2 x)^2 \sec^2 x}{\tan^4 x} dx$ { since $\sec^2 x = 1 + \tan^2 x$}

$= \int \frac{(1 + \tan^4 x + 2\tan^2 x)^2 \sec^2 x}{\tan^4 x} dx$ (apply $(a + b)^2 = a^2 + b^2 + 2ab$)

Let $\tan x = t$, so $dt = d(\tan x) = \sec^2 x dx$

So, $dx = \frac{dt}{\sec^2 x}$

Put t and dx in the above equation,

$\int \frac{(1 + \tan^4 x + 2\tan^2 x) \sec^2 x}{\tan^4 x} dx = \frac{\int (1 + t^4 + 2t^2)}{t^4} \sec^2 x * \frac{dt}{\sec^2 x}$

$= \frac{\int (1 + t^4 + 2t^2)}{t^4} dt$

$= \int (1 + t^{-4} + 2t^{-2})dt$

$= t - \frac{t^{-3}}{3} - 2t^{-1} + c$

$= t - \frac{2}{t} - \frac{1}{3t^3} + c$

$= \tan x - \frac{2}{\tan x} - \frac{1}{3\tan^3 x} + c$

$= \tan x - 2\cot x - \frac{1}{3}\cot^3 x + c$ {1/tanx = cotx)

11. Question

Evaluate the following integrals:

$$\int \frac{1}{\sin^3 x \cos^5 x} dx$$

Answer

$$\int \frac{1}{\sin^3 x \cos^5 x} dx = \int \sin^{-3} x \cos^{-5} x dx$$

Adding the powers , -3 + -5 = -8

Since it is an even number, we will divide numerator and denominator by $\cos^8 x$

$$\int \frac{1}{\sin^3 x \cos^5 x} dx = \int \frac{\frac{1}{\cos^8 x}}{\frac{\sin^3 x \cos^5 x}{\cos^8 x}} dx$$

$$= \int \frac{\sec^8 x}{\tan^3 x} dx = \int \frac{\sec^6 x \sec^2 x}{\tan^3 x} dx = \int \frac{(\sec^2 x)^3 \sec^2 x}{\tan^3 x} dx$$

$$= \int \frac{(1 + \tan^2 x)^3 \sec^2 x}{\tan^3 x} dx$$

We know, $(a + b)^3 = a^3 + b^3 + 3a^2b + 3ab^2$

Here, a = 1 and b = $\tan^2 x$

Hence, $\int \frac{(1 + \tan^2 x)^3 \sec^2 x}{\tan^3 x} dx = \int \frac{(1 + \tan^6 x + 3\tan^2 x + 3\tan^4 x) \sec^2 x}{\tan^3 x} dx$

Let tan x = t, then dt = d(tanx) = $\sec^2 x dx$

Put these values in above equation:

$$= \int \frac{1 + t^6 + 3t^2 + 3t^4}{t^3} dt = \int (t^{-3} + t^3 + 3t^{-1} + 3t)dt$$

$= -\frac{t^{-2}}{2} + \frac{t^4}{4} + 3\log t + \frac{3t^2}{2} + c$ (since $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ for any $c \neq -1$ and $\int t^{-1} dt = \log t$)

$$= -\frac{1}{2t^2} + \frac{1}{4}t^4 + 3\log t + \frac{3}{2}t^2 + c$$

$$= -\frac{1}{2\tan^2 x} + \frac{1}{4}\tan^4 x + 3\log(\tan x) + \frac{3}{2}\tan^2 x + c$$

12. Question

Evaluate the following integrals:

$$\int \frac{1}{\sin^3 x \cos x} dx$$

Answer

$$\int \frac{1}{\sin^3 x \cos x} dx = \int \sin^{-3}x\cos^{-1}x dx$$

Adding the powers , -3 + -1 = -4

Since it is an even number, we will divide numerator and denominator by cosx

$$\int \frac{1}{\sin^3 x \cos x} dx = \int \frac{\frac{1}{\cos^4 x}}{\frac{\sin^3 x\cos x}{\cos^4 x}} dx$$

$$= \int \frac{\sec^4 x}{\tan^3 x} dx = \int \frac{\sec^2 x\sec^2 x}{\tan^3 x} dx$$

$$= \int \frac{(1+\tan^2 x)\sec^2 x}{\tan^3 x} dx$$

Let tan x = t, then dt = d(tanx) = $\sec^2 x dx$

Put these values in the above equation:

$$= \int \frac{1+t^2}{t^3} dt = \int (t^{-3} + t^{-1}) dt$$

$= -\frac{t^{-2}}{2} + \log t + c$ (since $\int x^n\, dx = \frac{x^{n+1}}{n+1} + c$ for any $c \neq -1$ and $\int t^{-1}\, dt = \log t$)

$$= -\frac{1}{2t^2} + \log t + c$$

$$= -\frac{1}{2\tan^2 x} + \log(\tan x) + c$$

13. Question

Evaluate the following integrals:

$$\int \frac{1}{\sin x \cos^3 x} dx$$

Answer

We know, $\sin^2 x + \cos^2 x = 1$

Therefore $\frac{1}{\sin x\cos^3 x} = \frac{\sin^2 x + \cos^2 x}{\sin x\cos^3 x}$

Divide each term of numerator separately by $\sin x\cos^3 x$

$$= \frac{\sin^2 x}{\sin x\cos^3 x} + \frac{\cos^2 x}{\sin x\cos^3 x} = \frac{\sin x}{\cos^3 x} + \frac{1}{\sin x\cos x}$$

$= \frac{\sin x}{\cos x} * \left(\frac{1}{\cos^2 x}\right) + \frac{\frac{1}{\cos^2 x}}{\frac{\sin x\cos x}{\cos^2 x}}$ (divide second term each by $\cos^2 x$)

$$= \tan x \sec^2 x + \frac{\sec^2 x}{\tan x}$$

Therefore,

$$\int \frac{1}{\sin x \cos^3 x} dx = \int \left(\tan x \sec^2 x + \frac{\sec^2 x}{\tan x}\right) dx$$

$$= \int \tan x \sec^2 x\, dx + \int \frac{\sec^2 x}{\tan x} dx$$

Put tanx = t, dt = $\sec^2 x$ dx

$= \int \tan x \sec^2 x\, dx + \int \frac{\sec^2 x}{\tan x} dx = \int t dt + \int \frac{1}{t} dt$

$= \frac{t^2}{2} + \text{lot}\, t + c = \frac{1}{2}\tan^2 x + \log(\tan x) + c$

Exercise 19.13

1. Question

Evaluate the following integrals:

$$\int \frac{x^2}{\left(a^2 - x^2\right)^{3/2}} dx$$

Answer

$$\int \frac{x^2}{\left(a^2 - x^2\right)^{3/2}} dx$$

PUT x = a sinθ, so dx = a cosθ dθ and θ = sin⁻(x/a)

Above equation becomes,

$= \int \frac{a^2 \sin^2\theta}{(a^2 - a^2\sin^2\theta)^{3/2}} (a\cos\theta\, d\theta) = \int \frac{a^2 \sin^2\theta}{(a^2)(a^2 - a^2\sin^2\theta)^{3/2}} (a\cos\theta\, d\theta)$ {take a^2 outside)

$= \int \frac{a^2 \sin^2\theta}{(a^2)3/2(a^2 - a^2\sin^2\theta)^{3/2}} (a\cos\theta\, d\theta) = \int \sin^2\theta * \frac{\cos\theta}{\cos^3\theta} d\theta$

$= \int \frac{\sin^2\theta}{\cos^2\theta} d\theta = \int \tan^2\theta d\theta = \int (\sec^2\theta - 1)\, d\theta$ ($\sec^2\theta$-1 = $\tan^2\theta$)

$= \int \sec^2\theta\, d\theta - \int \theta\, d\theta = \tan\theta + c - \theta$

$= \tan\theta - \theta + c$

Put θ = sin⁻(x/a)

$= \tan\theta * \sin^{-}\left(\frac{x}{a}\right) - \sin^{-}\left(\frac{x}{a}\right) + c$

2. Question

Evaluate the following integrals:

$$\int \frac{x^7}{\left(a^2 - x^2\right)^5} dx$$

Answer

PUT x = a sinθ, so dx = a cosθ dθ and θ = sin⁻(x/a)

Above equation becomes,

$\int \frac{x^7}{\left(a^2 - x^2\right)^5} dx = = \int \frac{a^7 \sin^7\theta}{(a^2 - a^2\sin^2\theta)^5} (a\cos\theta\, d\theta) = \int \frac{a^7 \sin^7\theta}{(a^2)^5(1 - \sin^2\theta)^5} (a\cos\theta\, d\theta)$ {take a^2 outside)

$= \int \frac{a^7 \sin^7\theta}{(a^2)^5(1 - \sin^2\theta)^5} (a\cos\theta\, d\theta) = \int \frac{a^7 \sin^7\theta}{(a^{10}(1 - \sin^2\theta)^5} (a\cos\theta\, d\theta)$

$= \frac{1}{a^2} \int \frac{1}{\cos^2\theta} d\theta = \frac{1}{a^2} \int \sec^2\theta d\theta = \frac{1}{a^2}(\tan\theta + c)$

Put θ = sin⁻(x/a)

$= \frac{1}{a^2}\left(\tan \sin^{-}\left(\frac{x}{a}\right) + c\right)$

3. Question

Evaluate the following integrals:

$$\int \cos\left\{2\cot^{-1}\sqrt{\frac{1+x}{1-x}}\right\}dx$$

Answer

Let $x = \cos 2t$ and $t = \cos^{-} x\frac{x}{2}$

$= \sqrt{\frac{1+x}{1-x}} = \sqrt{\frac{1+\cos 2t}{1-\cos 2t}}$

We know 1 + cos 2t = 2cos^{2}t and 1-2cos2t = 2sin^{2}t

Hence, $\sqrt{\frac{1+\cos 2t}{1-\cos 2t}} = \sqrt{\frac{\cos^2 t}{\sin^2 t}} = \sqrt{\cot^2 t} = \cot t$

Therefore , $\int \cos\left\{2\cot^{-1}\sqrt{\frac{1+x}{1-x}}\right\}dx = \int \cos\theta\, dx$

Put $t = \cos^{-} x\frac{x}{2}$

$= \int \cos\theta\, dx = \int \cos\frac{\cos^{-} x}{2}dx = \int \frac{x}{2}\, dx = \frac{1}{2}\frac{x^2}{2} + c = \frac{x^2}{4} + c$

4. Question

Evaluate the following integrals:

$$\int \frac{\sqrt{1+x^2}}{x^4}dx$$

Answer

let x = tanθ , so dx = sec^{2}θ dθ and $\theta = \tan^{-} x$

Putting above values ,

$= \int \frac{\sqrt{1+x^2}}{x^4}dx = \int \frac{\sqrt{1+\tan^2\theta}}{\tan^4\theta}\sec^2\theta d\theta = \int \sec^2\theta/\tan^2\theta\, d\theta$

$= \int \mathrm{cosec}^2\theta d\theta = -\cot\theta + c$

Put $\theta = \tan^{-} x$

$= -\cot\theta + c = -\cot \tan^{-} x + c$

5. Question

Evaluate the following integrals:

$$\int \frac{1}{\left(x^2+2x+10\right)^2}dx$$

Answer

$= x^2 + 2x + 10 = x^2 + 2x + 1 - 1 + 10$ (add and substract 1)

$= (x^2 + 1)^2 - 1 + 10 = x^2 + 1)^2 + 9$

$= (x^2 + 1)^2 + 3^2$

Put x + 1 = t hence dx = dt and x = t-1

$\int \frac{1}{(x^2+2x+10)^2} dx = \int 1/((x^2 + 1)^2 + 3^2)\, dx$

$= \int \frac{1}{t^2+3^2} dt$

We have, $\int \frac{dt}{t^2+a^2} = \frac{1}{a}\log\left(\frac{t-a}{t+a}\right) + c$

Here a = 3

Therefore, $\int \frac{1}{t^2+3^2} dt = \frac{1}{3}\log\left(\frac{t-3}{t+3}\right) + c$

Put t = x + 1

$= \frac{1}{3}\log\left(\frac{t-3}{t+3}\right) + c = \frac{1}{3}\log\left(\frac{x+1-3}{x+1+3}\right) + c = \frac{1}{3}\log\left(\frac{x-2}{x+4}\right) + c$

Exercise 19.14

1. Question

Evaluate the following integrals:

$$\int \frac{1}{a^2 - b^2 x^2} dx$$

Answer

Taking out b^2, $\frac{1}{b^2}\int \frac{1}{\left(\frac{a^2}{b^2}\right) - x^2} dx$

$= \frac{1}{b^2}\int \frac{1}{\left(\frac{a^2}{b^2}\right) - x^2} dx = \frac{1}{b^2}\int \frac{1}{\left(\frac{a}{b}\right)^2 - x^2} dx$

$= \frac{1}{b^2} \times \frac{1}{2\left(\frac{a}{b}\right)}\log[\frac{\frac{a}{b}+x}{\frac{a}{b}-x}] + c$ { since $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a}\log\frac{x+a}{x-a} + c$ }

$= \frac{1}{2ab}\log\frac{a+bx}{a-bx} + c$

2. Question

Evaluate the following integrals:

$$\int \frac{1}{a^2 x^2 - b^2} dx$$

Answer

take out a^2

$= \frac{1}{a^2}\int \frac{1}{x^2 - \frac{b^2}{a^2}} dx$

$= \frac{1}{a^2}\int \frac{1}{x^2 - \left(\frac{b}{a}\right)^2} dx = \frac{1}{a^2} * \frac{1}{2\left(\frac{b}{a}\right)}\log[\frac{x-\left(\frac{b}{a}\right)}{x+\frac{b}{a}}] + c$ { since $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a}\log\frac{x+a}{x-a} + c$ }

$= \frac{1}{2ab}\log\frac{ax-b}{ax+b} + c$

3. Question

Evaluate the following integrals:

$$\int \frac{1}{a^2x^2+b^2}dx$$

Answer

take out a^2

$= \frac{1}{a^2}\int \frac{1}{x^2+\frac{b^2}{a^2}}\,dx$

$= \frac{1}{a^2}\int \frac{1}{x^2+(\frac{b}{a})^2}\,dx = \frac{1}{a^2} * \frac{1}{\left(\frac{b}{a}\right)}\tan^{-1}[\frac{x}{\frac{b}{a}}] + c$ { since $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{b}{a}\right) + c$}

$= \frac{1}{ab}\tan^{-1}\left(\frac{ax}{b}\right) + c$

4. Question

Evaluate the following integrals:

$$\int \frac{x^2-1}{x^2+4}dx$$

Answer

Add and subtract 4 in the numerator, we get

$= \int \frac{x^2+4-4-1}{x^2+4} = \int \frac{(x^2+4)-4-1}{x^2+4}dx$

$= \int \frac{(x^2+4)-5}{x^2+4}dx = \int \frac{(x^2+4)}{x^2+4}dx - \int \frac{5}{x^2+4}dx$ {separate the numerator terms)

$= \int dx - \int \frac{5}{x^2+4}dx = \int dx - 5\int \frac{1}{x^2+4}dx$

$= \int dx - 5\int \frac{1}{x^2+2^2}dx = x - 5 \times \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$ { since $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{b}{a}\right) + c$}

$= x - \frac{5}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$

5. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{1+4x^2}}dx$$

Answer

Let I = $\int \frac{1}{\sqrt{1+4x^2}}dx = \int \frac{1}{\sqrt{1+(2x)^2}}dx$

Let t = 2x, then dt = 2dx or dx = dt/2

Therefore, $\int \frac{1}{\sqrt{1+(2x)^2}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{1+t^2}}$

$= \frac{1}{2}\log[t + \sqrt{1+t^2}] + c$ {since $\int \frac{1}{\sqrt{(a^2+x^2)}}dx = \log[x + \sqrt{(a^2+x^2)} + c\}$}

$= \frac{1}{2}\log\left[2x + \sqrt{1 + 4x^2}\right] + c$

6. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{a^2 + b^2x^2}}dx$$

Answer

Let bx = t then dt = bdx or $dx = \frac{dt}{b}$

Hence, $\int \frac{1}{\sqrt{a^2 + b^2x^2}}dx = \frac{1}{b}\int \frac{1}{\sqrt{(a^2 + t^2)}}dt$

$= \frac{1}{b}\log\left[t + \sqrt{a^2 + t^2}\right] + c$ {since $\int \frac{1}{\sqrt{(a^2 + x^2)}}dx = \log[x + \sqrt{(a^2 + x^2)} + c\}$ }

Put t = bx

$= \frac{1}{b}\log\left[bx + \sqrt{a^2 + b^2x^2}\right] + c$

7. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{a^2 - b^2x^2}}dx$$

Answer

Let bx = t then dt = bdx or $dx = \frac{dt}{b}$

Hence, $\int \frac{1}{\sqrt{a^2 - b^2x^2}}dx = \frac{1}{b}\int \frac{1}{\sqrt{(a^2 - t^2)}}dt$

$= \frac{1}{b}\int \sin^{-1}\left(\frac{t}{a}\right) + c$ {since $\int \frac{1}{\sqrt{a^2 - x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + c\}$

Put t = bx

$= \frac{1}{b}\int \sin^{-1}\left(\frac{bx}{a}\right) + c$

8. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{(2 - x)^2 + 1}}dx$$

Answer

Let (2-x) = t , then dt = -dx , or dx = -dt

Hence, $\int \frac{1}{\sqrt{(2 - x)^2 + 1}}dx = \int \frac{1}{t^2 + 1}(-dt)$

$= -\int \frac{1}{t^2 + 1^2}dt = -\log\int\left(t + \sqrt{t^2 + 1}\right)) + c$ {since $\int \frac{1}{\sqrt{(a^2 + x^2)}}dx = \log[x + \sqrt{(a^2 + x^2)} + c\}$ }

Put t = 2-x

$= -\log \int \left((2-x) + \sqrt{(2-x)^2 + 1}\right) + c$

9. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{(2-x)^2 - 1}} dx$$

Answer

Let (2-x) = t , then dt = -dx , or dx = -dt

Hence, $\int \frac{1}{\sqrt{(2-x)^2 - 1}} dx = \int \frac{1}{t^2-1}(-dt)$

$= -\int \frac{1}{t^2-1^2} dt = -\log \int \left(t + \sqrt{t^2 - 1}\right) + c$ {since $\int \frac{1}{\sqrt{(x^2 + a^2)}} dx = \log[x + \sqrt{(x^2 - a^2)} + c\}$ }

Put t = 2-x

$= -\log \int \left((2-x) + \sqrt{(2-x)^2 - 1}\right) + c$

10. Question

Evaluate the following integrals:

$$\int \frac{x^4 + 1}{x^2 + 1} dx$$

Answer

We will use basic formula : $(a + b)^2 = a^2 + b^2 + 2ab$

Or, $a^2 + b^2 = (a + b)^2 - 2ab$

Here, $x^4 + 1 = x^4 + 1^4$

$= (x^2) + (1^2)^2$

Applying above formula, we get, $x^4 + 1 = (x^2 + 1)^2 - 2 \times 1 \times x^2$

$= (x^2 + 1)^2 - 2x^2$

Hence, $\int \frac{x^4 + 1}{x^2 + 1} dx = \int \frac{(x^2+1)^2 - 2x^2}{x^2+1} dx$

Separate the numerator terms,

$\int \frac{(x^2 + 1)^2 - 2x^2}{x^2 + 1} dx = \int \frac{(x^2 + 1)^2}{x^2 + 1} dx - \int \frac{2x^2}{x^2 + 1} dx$

$= \int (x^2 + 1) dx - \int \frac{2x^2 + 2 - 2}{x^2 + 1} dx$ { add and subtract 2 to the second term)

$= \int (x^2 + 1) dx - \int \frac{2(x^2 + 1)}{x^2 + 1} dx - 2\int 1/(x^2 + 1) dx \{ 2x^2 + 2 - 2 = 2(x^2 + 1) - 2\}$

$= \int (x^2 + 1) dx - \int 2 dx - 2\int 1/(x^2 + 1) dx$

$= \frac{x^3}{3} + x - 2x + 2\tan^{-1}x + c$ { since $\int \frac{1}{x^2 + 1} dx = \tan^{-1}(x) + c\}$

$= \frac{x^3}{3} - x + 2\tan^{-1}x + c$

Exercise 19.15

1. Question

Evaluate the following integrals:

$$\int \frac{1}{4x^2 + 12x + 5} dx$$

Answer

let $I = \int \frac{1}{4x^2+12x+5} dx$

$$= \frac{1}{4} \int \frac{1}{x^2 + 3x + \frac{5}{4}} dx$$

$$= \frac{1}{4} \int \frac{1}{x^2 + 2x \times \frac{3}{2} + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + \frac{5}{4}} dx$$

$$= \frac{1}{4} \int \frac{1}{\left(x + \frac{3}{2}\right)^2 - 1} dx$$

Let $\left(x + \frac{3}{2}\right) = t$ .....(i)

$\Rightarrow dx = dt$

so,

$$I = \frac{1}{4} \int \frac{1}{t^2 - (1)^2} dt$$

$$I = \frac{1}{4} \times \frac{1}{2 \times 1} \log \left|\frac{t-1}{t+1}\right| + c$$

$$[\text{since}, \int \frac{1}{x^2 - (a)^2} dx = \frac{1}{2 \times a} \log \left|\frac{x-a}{x+a}\right| + c]$$

$I = \frac{1}{8} \log \left|\frac{x - \frac{3}{2} - 1}{x + \frac{3}{2} + 1}\right| + c$ [using (i)]

$$I = \frac{1}{8} \log \left|\frac{2x-1}{2x+5}\right| + c$$

2. Question

Evaluate the following integrals:

$$\int \frac{1}{x^2 - 10x + 34} dx$$

Answer

let $I = \int \frac{1}{x^2-10x+34} dx$

$$I = \int \frac{1}{x^2 - 10x + 34} dx$$

$$= \int \frac{1}{x^2 + 2x \times 5 + (5)^2 - (5)^2 + 34} dx$$

$$= \int \frac{1}{(x-5)^2 - 9} dx$$

Let $(x-5) = t$ .....(i)

⇒ dx = dt

so,

$$I = \int \frac{1}{t^2 + (3)^2} dt$$

$$I = \frac{1}{3} \tan^{-1}\left(\frac{t}{3}\right) + c$$

$$\left[\text{since}, \int \frac{1}{x^2 + (a)^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c\right]$$

$I = \frac{1}{3} \tan^{-1}\left(\frac{x-5}{3}\right) + c$ [using (i)]

$$I = \frac{1}{3} \tan^{-1}\left(\frac{x-5}{3}\right) + c$$

3. Question

Evaluate the following integrals:

$$\int \frac{1}{1 + x - x^2} dx$$

Answer

: let $I = \int \frac{1}{1+x-x^2} dx = \int \frac{1}{-(x^2-x-1)} dx$

$$= \int \frac{1}{-(x^2 - x - 1)} dx$$

$$= \int \frac{1}{-(x^2 - x - \frac{1}{4} - 1 + \frac{1}{4})} dx$$

$$= \int \frac{1}{-\left(\left(x - \frac{1}{2}\right)^2 - \frac{5}{4}\right)} dx$$

$$= \int \frac{1}{\left(\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2\right)} dx$$

$$I = \frac{1}{2 \times \frac{\sqrt{5}}{2}} \log \left| \frac{\frac{\sqrt{5}}{2} + (x - \frac{1}{2})}{\frac{\sqrt{5}}{2} - (x - \frac{1}{2})} \right| + c$$

$$\left[\text{since}, \int \frac{1}{x^2 - (a)^2} dx = \frac{1}{2 \times a} \log \left|\frac{x-a}{x+a}\right| + c\right]$$

$$I = \frac{1}{\sqrt{5}} \log \left| \frac{\sqrt{5} + 2x - 1}{\sqrt{5} - 2x + 1} \right| + c$$

$$I = \frac{1}{\sqrt{5}} \log \left| \frac{\sqrt{5} - 1 + 2x}{\sqrt{5} + 1 - 2x} \right| + c$$

4. Question

Evaluate the following integrals:

$$\int \frac{1}{2x^2 - x - 1} dx$$

Answer

let $I = \int \frac{1}{2x^2 - x - 1} dx$

$$= \frac{1}{2} \int \frac{1}{x^2 - \frac{x}{2} - \frac{1}{2}} dx$$

$$= \frac{1}{2} \int \frac{1}{x^2 + 2x \times \frac{1}{4} + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 - \frac{1}{2}} dx$$

$$= \frac{1}{2} \int \frac{1}{\left(x - \frac{1}{4}\right)^2 - \frac{9}{16}} dx$$

Let $\left(x - \frac{1}{4}\right) = t$ ......(i)

⇒ dx = dt

so,

$$I = \frac{1}{2} \int \frac{1}{t^2 - \left(\frac{3}{4}\right)^2} dt$$

$$I = \frac{1}{2} \times \frac{1}{2 \times \frac{3}{4}} \log \left| \frac{t - \frac{3}{4}}{t + \frac{3}{4}} \right| + c$$

$$\left[\text{since}, \int \frac{1}{x^2 - (a)^2} dx = \frac{1}{2 \times a} \log \left| \frac{x - a}{x + a} \right| + c\right]$$

$I = \frac{1}{3} \log \left| \frac{x - \frac{1}{4} - \frac{3}{4}}{x - \frac{1}{4} + \frac{3}{4}} \right| + c$ [using (i)]

$$I = \frac{1}{3} \log \left| \frac{x - 1}{2x + 1} \right| + c$$

5. Question

Evaluate the following integrals:

$$\int \frac{1}{x^2 + 6x + 13} dx$$

Answer

We have,

$x^2 + 6x + 13 = x^2 + 6x + 3^2 - 3^2 + 13$

$= (x + 3)^2 + 4$

Sol, $\int \frac{1}{x^2 + 6x + 13} dx = \int \frac{1}{(x+3)^2 + 2^2} dx$

Let x+3 =t

Then dx = dt

$$\int \frac{1}{(t)^2 + 2^2} dt = \frac{1}{2} \tan^{-1} \frac{t}{2} + c$$

$$[\text{since}, \int \frac{1}{x^2 + (a)^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a}\right) + c]$$

$$\frac{1}{2} \tan^{-1} \frac{x+3}{2} + c$$

Exercise 19.16

1. Question

Evaluate the following integrals:

$$\int \frac{\sec^2 x}{1 - \tan^2 x} dx$$

Answer

let $I = \int \frac{\sec^2 x}{1 - \tan^2 x} dx$

Let tan x = t .....(i)

⇒ $\sec^2 x$ dx = dt

so,

$$I = \int \frac{dt}{(1)^2 - t^2}$$

$$I = \frac{1}{2 \times 1} \log \left|\frac{1+t}{1-t}\right| + c \ [\text{since}, \int \frac{1}{a^2 - (x)^2} dx = \frac{1}{2 \times a} \log \left|\frac{a+x}{a-x}\right| + c]$$

$I = \frac{1}{2} \log \left|\frac{1 + \tan x}{1 - \tan x}\right| + c$ [using (i)]

2. Question

Evaluate the following integrals:

$$\int \frac{e^x}{1 + e^{2x}} dx$$

Answer

: let $I = \int \frac{e^x}{1 + e^{2x}} dx$

Let e^x = t .....(i)

⇒ e^x dx = dt

so,

$$I = \int \frac{dt}{(1)^2 + t^2}$$

$$I = \tan^{-1} t + c$$

$$[\text{since}, \int \frac{1}{1 + (x)^2} dx = \tan^{-1} x + c]$$

$I = \tan^{-1}(e^x) + c$ [using(i)]

3. Question

Evaluate the following integrals:

$$\int \frac{\cos x}{\sin^2 x + 4\sin x + 5} dx$$

Answer

Let $I = \int \frac{\cos x}{\sin^2 x + 4\sin x + 5} dx$

Let sin x = t .....(i)

⇒ cos x dx = dt

So, $I = \int \frac{dt}{t^2 + 4t + 5}$

$$= \int \frac{dt}{t^2 + (2t)(2) + 2^2 - 2^2 + 5}$$

$$\int \frac{dt}{(t+2)^2 + 1}$$

Again, let t + 2 = u .....(ii)

⇒ dt = du

$$I = \int \frac{du}{u^2 + 1}$$

$$= \tan^{-1} u + c$$

$$\left[\text{since}, \int \frac{1}{1 + (x)^2} dx = \tan^{-1} x + c\right]$$

$= \tan^{-1}(\sin x + 2) + c$ [using(i),(ii)]

4. Question

Evaluate the following integrals:

$$\int \frac{e^x}{e^{2x} + 5e^x + 6} dx$$

Answer

let $I = \int \frac{e^x}{e^{2x} + 5e^x + 6} dx$

Let e^x = t .....(i)

⇒ e^x dx = dt

$$= \int \frac{1}{t^2 + 5t + 6} dt$$

$$= \int \frac{1}{t^2 + 2t \times \frac{5}{2} + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 6} dt$$

$$= \int \frac{1}{\left(t + \frac{5}{2}\right)^2 - \frac{1}{4}} dt$$

Let $t + \frac{5}{2} = u$ .....(i)

$\Rightarrow$ dt = du

so,

$$I=\int \frac{1}{u^2-\left(\frac{1}{2}\right)^2} du$$

$$I=\frac{1}{2\times\frac{1}{2}}\log\left|\frac{u-\frac{1}{2}}{u+\frac{1}{2}}\right|+c$$

$$\left[\text{since, } \int \frac{1}{x^2-(a)^2}dx=\frac{1}{2\times a}\log\left|\frac{x-a}{x+a}\right|+c\right]$$

$$I=\log\left|\frac{2u-1}{2u+1}\right|+c$$

$I=\log\left|\frac{2\left(t+\frac{5}{2}\right)-1}{2\left(t+\frac{5}{2}\right)+1}\right|+c$ [using (i)]

$I=\log\left|\frac{e^x+2}{e^x+3}\right|+c$ [using (ii)]

5. Question

Evaluate the following integrals:

$$\int \frac{e^{3x}}{4e^{6x}-9}dx$$

Answer

let $I=\int \frac{e^{3x}}{4e^{6x}-9}dx$

Let $e^{3x}=t$ .....(i)

$\Rightarrow 3e^{3x}\,dx = dt$

$$I=\frac{1}{3}\int \frac{1}{4t^2-9}dt$$

$$=\frac{1}{12}\int \frac{1}{t^2-\frac{9}{4}}dt$$

$$I=\frac{1}{12}\int \frac{1}{t^2-\left(\frac{3}{2}\right)^2}dt$$

$$I=\frac{1}{36}\log\left|\frac{t-\frac{3}{2}}{t+\frac{3}{2}}\right|+c$$

$$\left[\text{since, } \int \frac{1}{x^2-(a)^2}dx=\frac{1}{2\times a}\log\left|\frac{x-a}{x+a}\right|+c\right]$$

$$I=\log\left|\frac{2t-3}{2t+3}\right|+c$$

$I=\log\left|\frac{2e^{3x}-3}{2e^{3x}+3}\right|+c$ [using (i)]

6. Question

Evaluate the following integrals:

$$\int \frac{1}{e^{x}+e^{-x}} dx$$

Answer

let $I = \int \frac{1}{e^{x}+e^{-x}} dx$

$$= \int \frac{1}{e^{x}+\frac{1}{e^{x}}} dx$$

$$= \int \frac{e^{x}}{(e^{x})^2+1} dx$$

Let e^{x} = t .....(i)

⇒ e^{x} dx = dt

$$I = \int \frac{1}{(t)^2+1} dt$$

$$I = \tan^{-1} t + c$$

$$[\text{since}, \int \frac{1}{1+(x)^2} dx = \tan^{-1} x + c]$$

$I = \tan^{-1}(e^{x}) + c$ [using (i)]

7. Question

Evaluate the following integrals:

$$\int \frac{x}{x^4+2x^2+3} dx$$

Answer

Let $I = \int \frac{x}{x^4+2x^2+3} dx$

Let x^2 = t .....-----(i)

⇒ 2x dx = dt

$$I = \frac{1}{2} \int \frac{1}{t^2+2t+3} dt$$

$$= \frac{1}{2} \int \frac{1}{t^2+2t+1-1+3} dt$$

$$= \frac{1}{2} \int \frac{1}{(t+1)^2+2} dt$$

Put t + 1 = u .....-----(ii)

⇒ dt = du

$$I = \frac{1}{2} \int \frac{1}{(u)^2+\left(\sqrt{2}\right)^2} du$$

$$I = \frac{1}{2\sqrt{2}} \tan^{-1} \frac{u}{\sqrt{2}} + c$$

[since, $\int \frac{1}{x^2 + (a)^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$]

$I = \frac{1}{2\sqrt{2}} \tan^{-1} \frac{t+1}{\sqrt{2}} + c$ [using (i)]

$I = \frac{1}{2\sqrt{2}} \tan^{-1} \frac{x^2+1}{\sqrt{2}} + c$ [using (ii)]

8. Question

Evaluate the following integrals:

$$\int \frac{3x^5}{1+x^{12}} dx$$

Answer

let $I = \int \frac{3x^5}{1+x^{12}} dx$

$= \int \frac{3x^5}{1 + (x^6)^2} dx$

Let $x^6 = t$ .....(i)

$\Rightarrow 6x^5$ dx = dt

$I = \frac{3}{6} \int \frac{1}{(t)^2 + 1} dt$

$I = \frac{1}{2} \tan^{-1} t + c$

[since, $\int \frac{1}{1 + (x)^2} dx = \tan^{-1} x + c$]

$I = \frac{1}{2} \tan^{-1}(x^6) + c$ [using (i)]

9. Question

Evaluate the following integrals:

$$\int \frac{x^2}{x^6 - a^6} dx$$

Answer

let $I = \int \frac{x^2}{x^6 - a^6} dx$

$= \int \frac{x^2}{(x^3)^2 - (a^3)^2} dx$

Let $x^3 = t$ .....(i)

$\Rightarrow 3x^2$ dx = dt

$I = \frac{1}{3} \int \frac{1}{t^2 - (a^3)^2} dt$

$I = \frac{1}{3} \times \frac{1}{2 \times a^3} \log\left|\frac{t - a^3}{t + a^3}\right| + c$

[since, $\int \frac{1}{x^2-(a)^2}dx = \frac{1}{2\times a}\log\left|\frac{x-a}{x+a}\right| + c$]

$I = \frac{1}{6a^3}\log\left|\frac{x^3-a^3}{x^3+a^3}\right| + c$ [using (i)]

10. Question

Evaluate the following integrals:

$$\int \frac{x^2}{x^6+a^6}dx$$

Answer

let $I = \int \frac{x^2}{x^6+a^6}dx$

$$= \int \frac{x^2}{(x^3)^2+(a^3)^2}dx$$

Let $x^3 = t$ .....(i)

$\Rightarrow 3x^2$ dx = dt

$$I = \frac{1}{3}\int \frac{1}{t^2+(a^3)^2}dt$$

$$I = \frac{1}{3a^3}\tan^{-1}\frac{t}{a^3} + c$$

[since, $\int \frac{1}{x^2+(a)^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$]

$I = \frac{1}{3a^3}\tan^{-1}\frac{x^3}{a^3} + c$ [using (i)]

11. Question

Evaluate the following integrals:

$$\int \frac{1}{x(x^6+1)}dx$$

Answer

let $I = \int \frac{1}{x(x^6+1)}dx$

$$= \int \frac{x^5}{x^6(x^6+1)}dx$$

Let $x^6 = t$ .....(i)

$\Rightarrow 6x^5$ dx = dt

$$I = \frac{1}{6}\int \frac{1}{t(t+1)}dt$$

$$I = \frac{1}{6}\int \left(\frac{1}{t} - \frac{1}{t+1}\right)dt$$

$$I = \frac{1}{6}\left(\int \frac{1}{t}dt - \int \frac{1}{(t+1)}dt\right)$$

$I = \frac{1}{6}(\log t - \log(t+1)) + c$

$I = \frac{1}{6}(\log x^6 - \log(x^6+1)) + c$ [using (i)]

$I = \frac{1}{6}\log\frac{x^6}{x^6+1} + c$ [log m – log n= $\log\frac{m}{n}$]

12. Question

Evaluate the following integrals:

$$\int \frac{x}{x^4 - x^2 + 1} dx$$

Answer

Let $I = \int \frac{x}{x^4-x^2+1} dx$

Let x^2 = t .....-----(i)

⇒ 2x dx = dt

$$I = \frac{1}{2}\int \frac{1}{t^2 - t + 1} dt$$

$$= \frac{1}{2}\int \frac{1}{t^2 - 2t(\frac{1}{2}) + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1} dt$$

$$= \frac{1}{2}\int \frac{1}{(t - \frac{1}{2})^2 + \frac{3}{4}} dt$$

Put t – 1/2 = u .....-----(ii)

⇒ dt = du

$$I = \frac{1}{2}\int \frac{1}{(u)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} du$$

$$I = \frac{1}{2\frac{\sqrt{3}}{2}}\tan^{-1}\frac{u}{\frac{\sqrt{3}}{2}} + c$$

$$[\text{since}, \int \frac{1}{x^2 + (a)^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c]$$

$I = \frac{1}{2\frac{\sqrt{3}}{2}}\tan^{-1}\frac{t-\frac{1}{2}}{\frac{\sqrt{3}}{2}} + c$ [using (i)]

$I = \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x^2-1}{\sqrt{3}} + c$ [using (ii)]

13. Question

Evaluate the following integrals:

$$\int \frac{x}{3x^4 - 18x^2 + 11} dx$$

Answer

Let $I = \int \frac{x}{3x^4-18x^2+11} dx$

Let x^2 = t .....(i)

⇒ 2x dx = dt

$$I=\frac{1}{6}\int\frac{1}{t^2-6t+\frac{11}{3}}dt$$

$$=\frac{1}{6}\int\frac{1}{t^2-2t(3)+(3)^2-(3)^2+11}dt$$

$$=\frac{1}{6}\int\frac{1}{(t-3)^2-\frac{16}{3}}dt$$

Put t - 3 = u .....(ii)

⇒ dt = du

$$I=\frac{1}{6}\int\frac{1}{(u)^2-\left(\frac{4}{\sqrt{3}}\right)^2}du$$

$$I=\frac{1}{6}\times\frac{1}{2\times\frac{4}{\sqrt{3}}}\log\left|\frac{u-\frac{4}{\sqrt{3}}}{u+\frac{4}{\sqrt{3}}}\right|+c$$

$$\left[\text{since, }\int\frac{1}{x^2-(a)^2}dx=\frac{1}{2\times a}\log\left|\frac{x-a}{x+a}\right|+c\right]$$

$I=\frac{\sqrt{3}}{48}\log\left|\frac{t-3-\frac{4}{\sqrt{3}}}{t-3+\frac{4}{\sqrt{3}}}\right|+c$ [using (ii)]

$I=\frac{\sqrt{3}}{48}\log\left|\frac{x^2-3-\frac{4}{\sqrt{3}}}{x^2-3+\frac{4}{\sqrt{3}}}\right|+c$ [using (i)]

14. Question

Evaluate the following integrals:

$$\int\frac{e^x}{\left(1+e^x\right)\left(2+e^x\right)}dx$$

Answer

To evaluate the following integral following steps:

Let e^x = t .....(i)

⇒ e^x dx = dt

Now

$$\int\frac{e^x}{(1+e^x)(2+e^x)}dx=\int\frac{1}{(1+t)(2+t)}dt$$

$$=\int\frac{1}{(1+t)}dt-\int\frac{1}{(2+t)}dt$$

= log |(1+t)| - log |(2+t)| + c

$=\log\left|\frac{1+t}{2+t}\right|+c$ [log m – log n= $\log\frac{m}{n}$]

$= \log\left|\frac{1+e^x}{2+e^x}\right| + c$ [using(i)]

15. Question

Evaluate the following integrals:

$$\int \frac{1}{\cos x + \cos ecx} dx$$

Answer

let $I = \frac{1}{\cos x + \operatorname{cosec} x} dx$

Multiply and divide by sinx

$$I = \frac{\frac{1}{\sin x}}{\frac{\cos x}{\sin x} + \frac{\operatorname{cosec} x}{\sin x}} dx$$

$$= \frac{\operatorname{cosec} x}{\cot x + \operatorname{cosec}^2 x} dx$$

$$= \frac{\operatorname{cosec} x}{\cot x + 1 + \cot^2 x} dx$$

$$= \frac{\operatorname{cosec} x}{\cot^2 x + \cot x + 1} dx$$

Let cot x = t

-cosec x dx =dt

So, $I = -\int \frac{dt}{t^2 + t + 1}$

$$= \int \frac{dt}{t^2 + 2t.\frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1}$$

$$= \int \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1} \frac{t + \frac{1}{2}}{\frac{\sqrt{3}}{2}} + c$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \frac{2t + 1}{\sqrt{3}} + c$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \frac{2\cot x + 1}{\sqrt{3}} + c$$

Exercise 19.17

1. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{2x - x^2}} dx$$

Answer

let $I = \int \frac{1}{\sqrt{2x-x^2}} dx$

$= \int \frac{1}{\sqrt{-(x^2-2x)}} dx$

$= \int \frac{1}{\sqrt{-[x^2-2x(1)+1^2-1^2]}} dx$

$= \int \frac{1}{\sqrt{-[(x-1)^2-1]}} dx$

$= \int \frac{1}{\sqrt{1-(x-1)^2}} dx$

let (x-1)=t

dx=dt

so, $I = \int \frac{1}{\sqrt{1-t^2}} dt$

$= \sin^{-1} t + c$ [since $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$]

$I = \sin^{-1}(x-1) + c$

2. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{8+3x-x^2}} dx$$

Answer

8+3x-x2 can be written as 8-$\left(x^2 - 3x + \frac{9}{4} - \frac{9}{4}\right)$

Therefore

$8 - \left(x^2 - 3x + \frac{9}{4} - \frac{9}{4}\right)$

$= \frac{41}{4} - \left(x - \frac{3}{2}\right)^2$

$\int \frac{1}{\sqrt{8+3x-x^2}} dx = \int \frac{1}{\sqrt{\frac{41}{4} - \left(x-\frac{3}{2}\right)^2}} dx$

Let x-3/2=t

dx=dt

$\int \frac{1}{\sqrt{\frac{41}{4} - \left(x-\frac{3}{2}\right)^2}} dx = \int \frac{1}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2 - t^2}} dt$

$= \sin^{-1}\left(\frac{t}{\frac{\sqrt{41}}{2}}\right) + c$

[since $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$]

$$= \sin^{-1}\left(\frac{x - \frac{3}{2}}{\frac{\sqrt{41}}{2}}\right) + c$$

$$= \sin^{-1}\left(\frac{2x - 3}{\sqrt{41}}\right) + c$$

3. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{5 - 4x - 2x^2}} dx$$

Answer

Let I = $\int \frac{1}{\sqrt{5-4x-2x^2}} dx$

$$= \int \frac{1}{\sqrt{-2\left[x^2 + 2x - \frac{5}{2}\right]}} dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{-\left[x^2 + 2x + (1)^2 - (1)^2 - \frac{5}{2}\right]}} dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{-\left[(x+1)^2 - \frac{7}{2}\right]}} dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\frac{7}{2} - (x+1)^2}} dx$$

Let (x + 1) = t

Differentiating both sides, we get,

dx = dt

So, $I = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\left(\sqrt{\left(\frac{7}{2}\right)}\right)^2 - t^2}} dt$

$$= \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{t}{\sqrt{\frac{7}{2}}}\right) + c$$

[since $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$]

$$I = \frac{1}{\sqrt{2}} \sin^{-1}\left(\sqrt{\frac{2}{7}} \times (x+1)\right) + c$$

4. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{3x^2+5x+7}} dx$$

Answer

let $I = \int \frac{1}{\sqrt{3x^2+5x+7}} dx$

$$= \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{x^2+\frac{5}{3}x+\frac{7}{3}}} dx$$

$$= \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{x^2+2x\left(\frac{5}{6}\right)+\left(\frac{5}{6}\right)^2-\left(\frac{5}{6}\right)^2+\frac{7}{3}}} dx$$

$$= \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{\left(x+\frac{5}{6}\right)^2-\frac{59}{36}}} dx$$

let $\left(x+\frac{5}{6}\right) = t$

dx=dt

$$I = \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{t^2-\left(\frac{\sqrt{59}}{6}\right)^2}} dt$$

$$= \frac{1}{\sqrt{3}} \log\left|t+\sqrt{t^2-\left(\frac{\sqrt{59}}{6}\right)}\right| + c \;[\text{since } \int \frac{1}{\sqrt{x^2-a^2}} dx = \log\left|x+\sqrt{x^2-a^2}\right| + c$$

$$I = \frac{1}{\sqrt{3}} \log\left|x+\frac{5}{6}+\sqrt{\left(x+\frac{5}{6}\right)^2-\left(\frac{\sqrt{59}}{6}\right)^2}\right| + c$$

$$I = \frac{1}{\sqrt{3}} \log\left|x+\frac{5}{6}+\sqrt{x^2+\frac{5x}{3}+\frac{7}{3}}\right| + c$$

5. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{(x-\alpha)(\beta-x)}} dx, (\beta > \alpha)$$

Answer

let $I = \int \frac{1}{\sqrt{(x-\alpha)(\beta-x)}} dx, (\text{as } \beta > \alpha)$

$$= \int \frac{1}{\sqrt{-x^2-x(\alpha+\beta)-\alpha\beta}} dx$$

$$= \int \frac{1}{\sqrt{-\left[x^2-2x\left(\frac{\alpha+\beta}{2}\right)+\left(\frac{\alpha+\beta}{2}\right)^2-\left(\frac{\alpha+\beta}{2}\right)^2+\alpha\beta\right]}} dx$$

$$= \int \frac{1}{\sqrt{-\left[\left(x - \frac{\alpha+\beta}{2}\right)^2 - \left(\frac{\alpha+\beta}{2}\right)^2\right]}} dx$$

$$= \int \frac{1}{\sqrt{\left[\left(\frac{\beta-\alpha}{2}\right)^2 - \left(x - \frac{\alpha+\beta}{2}\right)^2\right]}} dx \quad [\beta > \alpha]$$

Let (x-(α+β)/2)=t

dx=dt

$$I = \int \frac{1}{\sqrt{\left(\frac{\beta-\alpha}{2}\right)^2 - t^2}} dt$$

$$= \sin^{-1}\left(\frac{t}{\frac{\beta-\alpha}{2}}\right) + c$$

$$I = \sin^{-1}\left(2\frac{x - \frac{\alpha+\beta}{2}}{\beta-\alpha}\right) + c$$

$$I = \sin^{-1}\left(\frac{2x-\alpha-\beta}{\beta-\alpha}\right)$$

6. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{7-3x-2x^2}} dx$$

Answer

let $I = \int \frac{1}{\sqrt{7-3x-2x^2}} dx$

$$= \int \frac{1}{\sqrt{-2\left[x^2 + \frac{3}{2}x - \frac{7}{2}\right]}} dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{-\left[x^2 + 2x\left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^2 - \frac{7}{2}\right]}} dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{-\left[\left(x - \frac{3}{4}\right)^2 - \frac{65}{16}\right]}} dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\left(\frac{\sqrt{65}}{4}\right)^2 - \left(x + \frac{3}{4}\right)^2}} dx$$

let $\left(x + \frac{3}{4}\right) = t$

dx=dt

$$I=\frac{1}{\sqrt{2}}\int\frac{1}{\sqrt{\left(\frac{\sqrt{65}}{4}\right)^2-(t)^2}}dt$$

$$=\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{t}{\frac{\sqrt{65}}{4}}\right)+c$$

$$I=\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{4\left(x+\frac{3}{4}\right)}{\sqrt{65}}\right)+c$$

$$I=\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{4x+3}{\sqrt{65}}\right)+c$$

7. Question

Evaluate the following integrals:

$$\int\frac{1}{\sqrt{16-6x-x^2}}dx$$

Answer

let $I=\int\frac{1}{\sqrt{16-6x-x^2}}dx$

$$=\int\frac{1}{\sqrt{-[x^2+6x-16]}}dx$$

$$=\int\frac{1}{\sqrt{-[x^2+2x(3)+(3)^2-(3)^2-16]}}dx$$

$$=\int\frac{1}{\sqrt{-[(x-3)^2-25]}}dx$$

$$=\int\frac{1}{\sqrt{25-(x+3)^2}}dx$$

let $(x+3)=t$

dx=dt

$$I=\int\frac{1}{\sqrt{5^2-t^2}}dt$$

$$=\sin^{-1}\left(\frac{t}{5}\right)+c$$

$$I=\sin^{-1}\left(\frac{x+3}{5}\right)+c$$

8. Question

Evaluate the following integrals:

$$\int\frac{1}{\sqrt{7-6x-x^2}}dx$$

Answer

$7-6x-x^2$ can be written as $7-(x^2+6x+9-9)$

Therefore

$7-(x^2+6x+9-9)$

$= 16-(x^2+6x+9)$

$= 16-(x+3)^2$

$= (4)^2-(x+3)^2$

$$\int \frac{1}{\sqrt{7-6x-x^2}}dx = \int \frac{1}{\sqrt{(4)^2-(x+3)^2}}dx$$

Let x+3=t

dx=dt

$$\int \frac{1}{\sqrt{(4)^2-(x+3)^2}}dx = \int \frac{1}{\sqrt{(4)^2-(t)^2}}dt$$

$$= \sin^{-1}\left(\frac{t}{4}\right) + c$$

$$= \sin^{-1}\left(\frac{x+3}{4}\right) + c$$

9. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{5x^2-2x}}dx$$

Answer

we have $\int \frac{dx}{\sqrt{5x^2-2x}} = \int \frac{dx}{\sqrt{5\left(x^2-\frac{2x}{5}\right)}}$

$= \frac{1}{\sqrt{5}}\int \frac{dx}{\sqrt{\left(x-\frac{1}{5}\right)^2-\left(\frac{1}{5}\right)^2}}$ completing the square

Put x-1/5=t then dx =dt

Therefore $\int \frac{dx}{\sqrt{5x^2-2x}} = \frac{1}{\sqrt{5}}\int \frac{dx}{\sqrt{(t)^2-\left(\frac{1}{5}\right)^2}}$

$$= \frac{1}{\sqrt{5}}\log\left|t+\sqrt{t^2-\left(\frac{1}{5}\right)^2}\right| + c$$

$$= \frac{1}{\sqrt{5}}\log\left|x-\frac{1}{5}+\sqrt{x^2-\frac{2x}{5}}\right| + c$$

Exercise 19.18

1. Question

Evaluate the following integrals:

$$\int \frac{x}{\sqrt{x^4 + a^4}} dx$$

Answer

$$\int \frac{x}{\sqrt{x^4 + a^4}} dx = \int \frac{x}{\sqrt{(x^2)^2 + (a^2)^2}} dx$$

Let $x^2 = t$, so $2x\ dx = dt$

Or, $x\ dx = dt/2$

Hence, $\int \frac{x}{\sqrt{(x^2)^2 + (a^2)^2}} dx = \int \frac{1}{\sqrt{t^2 + (a^2)^2}} \frac{dt}{2} = \frac{1}{2} \int \frac{1}{\sqrt{t^2 + (a^2)^2}} dt$

Since, $\int \frac{1}{\sqrt{(x^2 + a^2)}} dx = \log[x + \sqrt{(x^2 + a^2)}) + c$

Hence, $\frac{1}{2} \int \frac{1}{\sqrt{t^2 + (a^2)^2}} dt = \frac{1}{2} \log(t + \sqrt{t^2 + (a^2)^2} + c$

Put $t = x^2$

$= \frac{1}{2} \log(x^2 + \sqrt{(x^2)^2 + (a^2)^2} + c$

$= \frac{1}{2} \log[x^2 + \sqrt{x^4 + a^4}] + c$

2. Question

Evaluate the following integrals:

$$\int \frac{\sec^2 x}{\sqrt{4 + \tan^2 x}} dx$$

Answer

Let tan x = t

Then $dt = \sec^2 x\ dx$

Therefore, $\int \frac{\sec^2 x}{\sqrt{4 + \tan^2 x}} dx = \int \frac{dt}{\sqrt{2^2 + t^2}}$

Since, $\int \frac{1}{\sqrt{(x^2 + a^2)}} dx = \log[x + \sqrt{(x^2 + a^2)}) + c$

Hence, $\int \frac{dt}{\sqrt{2^2 + t^2}} = \log[t + \sqrt{t^2 + 2^2}] + c$

$= \log[\tan x + \sqrt{\tan^2 x + 4}] + c$

3. Question

Evaluate the following integrals:

$$\int \frac{e^x}{\sqrt{16 - e^{2x}}} dx$$

Answer

Let $e^x = t$

Then we have, $e^x\ dx = dt$

Therefore, $\int \frac{e^x}{\sqrt{16 - e^{2x}}} dx = \int \frac{dt}{\sqrt{4^2 - t^2}}$

Since we have, $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

Hence, $\int \frac{dt}{\sqrt{4^2-t^2}} = \sin^{-1}\left(\frac{e^x}{a}\right) + c$

4. Question

Evaluate the following integrals:

$$\int \frac{\cos x}{\sqrt{4+\sin^2 x}}dx$$

Answer

Let $\sin x = t$

Then $dt = \cos x\, dx$

Hence, $\int \frac{\cos x}{\sqrt{4+\sin^2 x}}dx = \int \frac{dt}{\sqrt{2^2+t^2}}$

Since we have, $\int \frac{1}{\sqrt{(x^2+a^2)}}dx = \log[x + \sqrt{(x^2 + a^2)}] + c$

Therefore, $\int \frac{dt}{\sqrt{2^2+t^2}} = \log\left[t + \sqrt{t^2 + 2^2}\right] + c$

$= \log\left[t + \sqrt{t^2 + 2^2}\right] + c = \log\left[\sin x + \sqrt{\sin^2 x + 4}\right] + c$

5. Question

Evaluate the following integrals:

$$\int \frac{\sin x}{\sqrt{4\cos^2 x - 1}}dx$$

Answer

Let $2\cos x = t$

Then $dt = -2\sin x\, dx$

Or, $\sin x\, dx = -\frac{dt}{2}$

Therefore, $\int \frac{\sin x}{\sqrt{4\cos^2 x - 1}}dx = \int -\frac{dt}{2\sqrt{(t^2-1^2)}}$

Since, $\int \frac{1}{\sqrt{(x^2-a^2)}}dx = \log[x + \sqrt{(x^2 - a^2)}] + c$

Therefore, $\int -\frac{dt}{2\sqrt{(t^2-1^2)}} = -\frac{1}{2}\, \text{lod}\left[t + \sqrt{t^2 - 1}\right] + c$

$= -\frac{1}{2}\log\left[2\cos x + \sqrt{4\cos^2 x - 1}\right] + c$

6. Question

Evaluate the following integrals:

$$\int \frac{x}{\sqrt{4 - x^4}}dx$$

Answer

Let $x^2 = t$

2x dx = dt or x dx = dt/2

Hence, $\int \frac{x}{\sqrt{4-x^4}} = \int \frac{dt}{2\left(\sqrt{2^2-t^2}\right)}$

Since we have, $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

So, $\int \frac{dt}{2\left(\sqrt{2^2-t^2}\right)} = \frac{1}{2}\sin^{-1}\left(\frac{t}{2}\right) + c$

Put t = x^2

$= \frac{1}{2}\sin^{-1}\left(\frac{t}{2}\right) + c = \frac{1}{2}\sin^{-1}\left(\frac{x^2}{2}\right) + c$

7. Question

Evaluate the following integrals:

$$\int \frac{1}{x\sqrt{4-9(\log x)^2}} dx$$

Answer

Put 3logx = t

We have $d(\log x) = 1/x$

Hence, $d(3\log x) = dt = 3/x\, dx$

Or $1/x\, dx = dt/3$

Hence, $\int \frac{1}{x\sqrt{4-9(\log x)^2}} dx = \int \frac{1}{3}\frac{dt}{\sqrt{2^2-t^2}}$

Since we have, $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

Hence, $\int \frac{1}{3}\frac{dt}{\sqrt{2^2-t^2}} = \frac{1}{3}\sin^{-1}\left(\frac{t}{2}\right) + c$

Put t = 3logx

$= \frac{1}{3}\sin^{-1}\left(\frac{t}{2}\right) + c = \frac{1}{3}\sin^{-1}\left(\frac{3\log x}{2}\right) + c$

8. Question

Evaluate the following integrals:

$$\int \frac{\sin 8x}{\sqrt{9+\sin^4 4x}} dx$$

Answer

Let t = $\sin^2 4x$

dt = 2sin4x cos4x × 4 dx

we know sin2x = 2sins2xcos2x

therefore, dt = 4 sin8x dx

or, sin8x dx = dt/4

$$\int \frac{\sin 8x}{\sqrt{9+\sin^4 x}} dx = \frac{1}{4}\int \frac{dt}{\sqrt{3^2 + t^2}}$$

Since we have, $\int \frac{1}{\sqrt{(x^2+a^2)}}dx = \log[x + \sqrt{(x^2 + a^2)}] + c$

$= \frac{1}{4}\int \frac{dt}{\sqrt{3^2+t^2}} = \frac{1}{4}\log[t + \sqrt{t^2 + 3^2} + c$

$= \frac{1}{4}\log[\sin^2 4x + \sqrt{9 + \sin^4 4x} + c$

9. Question

Evaluate the following integrals:

$$\int \frac{\cos 2x}{\sqrt{\sin^2 2x + 8}}dx$$

Answer

Let = sin2x

dt = 2cos2x dx

Cos2x dx = dt/2

$$\int \frac{\cos 2x}{\sqrt{\sin^2 2x + 8}}dx = \frac{1}{2}\int dt/\sqrt{(t^2 + (2\sqrt{2})^2}$$

Since we have, $\int \frac{1}{\sqrt{(x^2+a^2)}}dx = \log[x + \sqrt{(x^2 + a^2)}] + c$

$$= \frac{1}{2}\int dt/\sqrt{(t^2 + (2\sqrt{2})^2} = \frac{1}{2}\log\left[t + \sqrt{t^2 + 8}\right] + c$$

$$= \frac{1}{2}\log[t + \sqrt{t^2 + 8}] + c = \frac{1}{2}\log\left[\sin 2x + \sqrt{\sin^2 2x + 8}\right] + c$$

10. Question

Evaluate the following integrals:

$$\int \frac{\sin 2x}{\sqrt{\sin^4 x + 4\sin^2 x - 2}}dx$$

Answer

Let t = sin^{2}x

dt = 2sinx cosx dx

we know sin2x = 2sins2xcos2x

therefore, dt = sin2x dx

$$\int \frac{\sin 2x}{\sqrt{\sin^4 x + 4\sin^2 x - 2}}dx = \int \frac{dt}{\sqrt{t^2 + 4t - 2}}$$

Add and subtract 2^2 in denominator

$$= \int \frac{dt}{\sqrt{t^2 + 4t - 2}} = \int \frac{dt}{\sqrt{t^2 + 2 \times 2t + 2^2 - 2^2 - 2}}$$

Let t + 2 = u

dt = du

$$= \int dt/\sqrt{((t + 2)^2 - 6)} = \int dt/\sqrt{(u^2 - 6)}$$

Since, $\int \frac{1}{\sqrt{(x^2-a^2)}}dx = \log[x + \sqrt{(x^2 - a^2)}] + c$

$= \int dt/\sqrt{(u^2 - 6)} = \log[u + \sqrt{u^2 - 6} + c$

$= \log[t + 2 + \sqrt{(t + 2)^2 - 6} + c$

$= \log[t + 2 + \sqrt{(t + 2)^2 - 6} + c = \log[\sin^2 x + 2 + \sqrt{(\sin^2 x + 2)^2 - 6} + c$

11. Question

Evaluate the following integrals:

$$\int \frac{\sin 2x}{\sqrt{\cos^4 x - \sin^2 x + 2}} dx$$

Answer

Let t = $\cos^2 x$

dt = 2cosx sinx dx = - sin2x dx

therefore, $\int \frac{\sin 2x}{\sqrt{\cos^4 x - \sin^2 x + 2}} dx = \int -\frac{dt}{\sqrt{t^2 - (1-t^2) + 2}}$

since, [$\sin^2 x = 1 - \cos^2 x$]

$\int -\frac{dt}{\sqrt{t^2 - (1-t^2) + 2}} = \int -\frac{dt}{\sqrt{t^2 + t + 1}} = \int -\frac{dt}{\sqrt{t^2 + t + \frac{1}{4} + \frac{3}{4}}}$

$$= \int -\frac{dt}{\sqrt{(t + \frac{1}{2})^2 + \frac{3}{4}}}$$

Since, $\int \frac{1}{\sqrt{(x^2-a^2)}}dx = \log[x + \sqrt{(x^2 - a^2)}] + c$

$$= \int -\frac{dt}{\sqrt{(t + \frac{1}{2})^2 + \frac{3}{4}}} = \log[t + \frac{1}{2} + \sqrt{(t + \frac{1}{2})^2 - (\frac{\sqrt{3}}{2})^2} + c$$

$= \log[t + \frac{1}{2} + \sqrt{t^2 + t + 1} + c = \log[\cos^2 x + \frac{1}{2} + \sqrt{\cos^4 x + \cos^2 x + 1} + c$

12. Question

Evaluate the following integrals:

$$\int \frac{\cos x}{\sqrt{4 - \sin^2 x}} dx$$

Answer

Let sinx = t

dt = cosxdx

therefore, $\int \frac{\cos x}{\sqrt{4-\sin^2 x}} dx = \int \frac{dt}{\sqrt{2^2 - t^2}}$

Since we have, $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

$= \int \frac{dt}{\sqrt{2^2 - t^2}} = \sin^{-1}\left(\frac{t}{2}\right) + c = \sin^{-1}\left(\frac{\sin x}{2}\right) + c$

13. Question

Evaluate the following integrals:

$$\int \frac{1}{x^{\frac{2}{3}}\sqrt{x^{\frac{2}{3}} - 4}} dx$$

Answer

Let $x^{\frac{1}{3}} = t$

So, $dt = 1/3\, x^{\frac{1}{3}-1} dx$

$= dt = \frac{1}{3} x^{\frac{1}{3}-1} dx = \frac{1}{3} x^{-\frac{2}{3}}$

Or, $\frac{dx}{x^{\frac{2}{3}}} = 3\, dt$

$\int \frac{1}{x^{\frac{2}{3}}\sqrt{x^{\frac{2}{3}}-4}} dx = 3\int \frac{dt}{\sqrt{t^2 - 2^2}}$

Since, $\int \frac{1}{\sqrt{(x^2 - a^2)}} dx = \log[x + \sqrt{(x^2 - a^2)}] + c$

$= 3\int \frac{dt}{\sqrt{t^2 - 2^2}} = 3\log\left[t + \sqrt{t^2 - 4}\right] + c$

$= 3\log\left[x^{\frac{1}{3}} + \sqrt{(x^{\frac{1}{3}})^2 - 4}\right] + c = 3\log\left[x^{\frac{1}{3}} + \sqrt{x^{\frac{2}{3}} - 4}\right] + c$

14. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{(1 - x^2)\left\{9 + \left(\sin^{-1} x\right)^2\right\}}} dx$$

Answer

Let $\sin^{-1} x = t$

$dt = \frac{1}{\sqrt{1 - x^2}} dx$

Therefore, $\int \frac{1}{\sqrt{(1 - x^2)\left\{9 + \left(\sin^{-1} x\right)^2\right\}}} dx = \int \frac{1}{\sqrt{3^2 - t^2}} dt$

Since we have, $\int \frac{1}{\sqrt{(x^2 + a^2)}} dx = \log[x + \sqrt{(x^2 + a^2)}] + c$

$= \int \frac{1}{\sqrt{3^2 - t^2}} dt = \log\left[t + \sqrt{9 + t^2}\right] + c$

$= \log\left[t + \sqrt{9 + t^2}\right] + c = \log\left[\sin^{-1} x + \sqrt{9 + (\sin^{-1} x)^2}\right] + c$

15. Question

Evaluate the following integrals:

$$\int \frac{\cos x}{\sqrt{\sin^2 x - 2\sin x - 3}} dx$$

Answer

Let sinx = t

Cosx dx = dt

$$\int \frac{\cos x}{\sqrt{\sin^2 x - 2\sin x - 3}} dx = \int \frac{dt}{\sqrt{t^2 - 2t - 3}}$$

Add and subtract 1^2 in denominator

$$= \int \frac{dt}{\sqrt{t^2 - 2t - 3}} = \int \frac{dt}{\sqrt{t^2 - 2t + 1^2 - 1^2 - 3}} = \int \frac{dt}{\sqrt{((t-1)^2 - 2^2)}}$$

Let t - 1 = u

dt = du

$$= \int \frac{dt}{\sqrt{((t-1)^2 - 2^2)}} = \int \frac{dt}{\sqrt{(u^2 - 2^2)}}$$

Since, $\int \frac{1}{\sqrt{(x^2 - a^2)}} dx = \log[x + \sqrt{(x^2 - a^2)}] + c$

$$= \int \frac{dt}{\sqrt{(u^2 - 2^2)}} = \log\left[u + \sqrt{u^2 - 4}\right] + c$$

Put u = t - 1

$$= \log\left[t - 1 + \sqrt{(t-1)^2 - 4}\right] + c$$

Put t = sinx

$$= \log\left[t - 1 + \sqrt{(t-1)^2 - 4}\right] + c$$

$$= \log\left[\sin x - 1 + \sqrt{(\sin x - 1)^2 - 4}\right] + c$$

$$= \log[\sin x - 1 + \sqrt{\sin^2 x - 2\sin x - 3}] + c$$

16. Question

Evaluate the following integrals:

$$\int \sqrt{\operatorname{cosec} x - 1}\, dx$$

Answer

$$\int \sqrt{\operatorname{cosec} x - 1}\, dx$$

Since cosec x = 1/ sinx

$$\int \sqrt{\operatorname{cosec} x - 1}\, dx = \int \sqrt{\frac{1}{\sin x} - 1}\, dx = \int \sqrt{\frac{1 - \sin x}{\sin x}}\, dx$$

Multiply with (1 + sinx) both numerator and denominator

$$= \int \sqrt{\frac{1 - \sin x}{\sin x}}\, dx = \int \sqrt{\frac{1 - \sin x * (1 + \sin x)}{\sin x * (1 + \sin x)}}\, dx$$

Since (a + b) × (a - b) = $a^2 - b^2$,

$$= \int \sqrt{\frac{1 - \sin x \times (1 + \sin x)}{\sin x \times (1 + \sin x)}}\, dx = \int \sqrt{\frac{1 - \sin^2 x}{\sin x + \sin^2 x)}}\, dx$$

$$= \int \sqrt{\frac{\cos^2 x}{\sin x + \sin^2 x)}}\, dx$$

$$= \int \frac{\cos x}{\sqrt{\sin x + \sin^2 x}}\, dx$$

Let sinx = t

dt = cosx dx

therefore, $\int \frac{\cos x}{\sqrt{\sin x + \sin^2 x}}\, dx = \int \frac{dt}{\sqrt{t^2 - t}}$

multiply and divide by 2 and add and subtract $(1/2)^2$ in denominator,

$$= \int \frac{dt}{\sqrt{t^2 - 2t\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} = \frac{\int dt}{\sqrt{\left(t + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}$$

Let t + 1/2 = u

dt = du

$$= \frac{\int dt}{\sqrt{\left(t + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2)}} = \int \frac{dt}{\sqrt{(u^2 - \left(\frac{1}{2}\right)^2}}$$

Since, $\int \frac{1}{\sqrt{(x^2 - a^2)}} dx = \log[x + \sqrt{(x^2 - a^2)}] + c$

$$= \int \frac{dt}{\sqrt{(u^2 - \left(\frac{1}{2}\right)^2}} = \log\left[u + \sqrt{\left(\left(u^2 - \left(\frac{1}{2}\right)^2\right)\right)}\right] + c$$

$$= \log\left[t + \frac{1}{2} + \sqrt{\left(\left(\left(t + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right)\right)}\right] + c$$

$$= \log\left[\sin x + \frac{1}{2} + \sqrt{\sin^2 x + \sin x}\right] + c$$

17. Question

Evaluate the following integrals:

$$\int \frac{\sin x - \cos x}{\sqrt{\sin 2x}} dx$$

Answer

$\int \frac{\sin x - \cos x}{\sqrt{\sin 2x}} dx = \int (\sin x - \cos x)/\sqrt{((\sin x + \cos x)^2 - 1)}\, dx$

Let sinx + cosx = t

(Cosx - sinx) = dt

Therefore, $\int \frac{\sin x - \cos x}{\sqrt{(\sin x + \cos x)^2 - 1}}\, dx = \int -\frac{dt}{\sqrt{t^2 - 1}}$

Since, $\int \frac{1}{\sqrt{(x^2-a^2)}}dx = \log[x + \sqrt{(x^2-a^2)}] + c$

$$= \int -\frac{dt}{\sqrt{t^2-1}} = -\log\left[t + \sqrt{t^2-1}\right] + c$$

$$= -\log\left[t + \sqrt{t^2-1}\right] + c = -\log\left[\sin x + \cos x + \sqrt{\sin 2x}\right] + c$$

18. Question

Evaluate the following integrals:

$$\int \frac{\cos x - \sin x}{\sqrt{8 - \sin 2x}} dx$$

Answer

$$= \int \frac{\cos x - \sin x}{\sqrt{8 - \sin 2x}} dx = \int \frac{\sin x - \cos x}{\sqrt{8 - (\sin x + \cos x)^2 + 1}} dx$$

Let sinx + cosx = t

(Cosx - sinx) = dt

Therefore, $\int \frac{\sin x - \cos x}{\sqrt{8-(\sin x + \cos x)^2 + 1}} dx = \int \frac{dt}{\sqrt{9-t^2}}$

Since we have, $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

$$= \int \frac{dt}{\sqrt{9-t^2}} = \int \frac{dt}{\sqrt{3^2-t^2}} = \sin^{-1}\left(\frac{t}{3}\right) + c$$

$$= \sin^{-1}\left(\frac{\sin x + \cos x}{3}\right) + c = \sin^{-1}\left(\frac{\sin x}{3} + \frac{\cos x}{3}\right) + c = \sin^{-1}\left(\frac{\sin x}{3}\right) + \sin^{-1}\left(\frac{\cos x}{3}\right) + c$$

$$= \frac{x}{3} + \sin^{-1}\left(\frac{\sin x}{3}\right) + c$$

Exercise 19.19

1. Question

Evaluate the integral:

$$\int \frac{x}{x^2 + 3x + 2} dx$$

Answer

$I = \int \frac{x}{x^2+3x+2} dx$

As we can see that there is a term of x in numerator and derivative of x^2 is also 2x. So there is a chance that we can make substitution for $x^2 + 3x + 2$ and I can be reduced to a fundamental integration.

As, $\frac{d}{dx}(x^2 + 3x + 2) = 2x + 3$

∴ Let, x = A(2x + 3) + B

⇒ x = 2Ax + 3A + B

On comparing both sides –

We have,

2A = 1 ⇒ A = 1/2

3A + B = 0 ⇒ B = –3A = –3/2

Hence,

$I = \int \frac{\frac{1}{2}(2x+3)-\frac{3}{2}}{x^2+3x+2}dx$

$\therefore I = \frac{1}{2}\int \frac{2x+3}{x^2+3x+2}dx - \frac{3}{2}\int \frac{1}{x^2+3x+2}dx$

Let, $I_1 = \frac{1}{2}\int \frac{2x+3}{x^2+3x+2}dx$ and $I_2 = \frac{3}{2}\int \frac{1}{x^2+3x+2}dx$

Now, $I = I_1 - I_2$ ....eqn 1

We will solve I_1 and I_2 individually.

As, $I_1 = \frac{1}{2}\int \frac{2x+3}{x^2+3x+2}dx$

Let $u = x^2 + 3x + 2 \Rightarrow du = (2x + 3)dx$

$\therefore I_1$ reduces to $\frac{1}{2}\int \frac{du}{u}$

Hence,

$I_1 = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\log|u| + C$ $\{\because \int \frac{dx}{x} = \log|x| + C\}$

On substituting value of u, we have:

$I_1 = \frac{1}{2}\log|x^2 + 3x + 2| + C$ ....eqn 2

As, $I_2 = \frac{3}{2}\int \frac{1}{x^2+3x+2}dx$ and we don't have any derivative of function present in denominator. ∴ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

i) $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$ ii) $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$\therefore I_2 = \frac{3}{2}\int \frac{1}{x^2+3x+2}dx$

$\Rightarrow I_2 = \frac{3}{2}\int \frac{1}{\{x^2+2\left(\frac{3}{2}\right)x+\left(\frac{3}{2}\right)^2\}+2-\left(\frac{3}{2}\right)^2}dx$

Using: $a^2 + 2ab + b^2 = (a + b)^2$

We have:

$I_2 = \frac{3}{2}\int \frac{1}{\left(x+\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}dx$

I_2 matches with $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

$\therefore I_2 = \frac{3}{2}\left\{\frac{1}{2\left(\frac{1}{2}\right)}\log\left|\frac{\left(x+\frac{3}{2}\right)-\frac{1}{2}}{\left(x+\frac{3}{2}\right)+\frac{1}{2}}\right| + C\right\}$

$\Rightarrow I_2 = \frac{3}{2}\log\left|\frac{2x+3-1}{2x+3+1}\right| + C$

$\Rightarrow I_2 = \frac{3}{2}\log\left|\frac{2x+2}{2x+4}\right| + C = \frac{3}{2}\log\left|\frac{x+1}{x+2}\right| + C$ ...eqn 3

From eqn 1:

$I = I_1 - I_2$

Using eqn 2 and eqn 3:

$I = \frac{1}{2}\log|x^2 + 3x + 2| + \frac{3}{2}\log\left|\frac{x+1}{x+2}\right| + C$

2. Question

Evaluate the integral:

$$\int \frac{x+1}{x^2+x+3}dx$$

Answer

$I = \int \frac{x+1}{x^2+x+3}dx$

As we can see that there is a term of x in numerator and derivative of x^2 is also 2x. So there is a chance that we can make substitution for $x^2 + x + 3$ and I can be reduced to a fundamental integration.

As, $\frac{d}{dx}(x^2 + x + 1) = 2x + 1$

∴ Let, x = A(2x + 1) + B

⇒ x = 2Ax + A + B

On comparing both sides –

We have,

2A = 1 ⇒ A = 1/2

A + B = 0 ⇒ B = –A = –1/2

Hence,

$I = \int \frac{\frac{1}{2}(2x+1)-\frac{1}{2}}{x^2+x+3}dx$

$\therefore I = \frac{1}{2}\int \frac{2x+1}{x^2+x+3}dx - \frac{1}{2}\int \frac{1}{x^2+x+3}dx$

Let, $I_1 = \frac{1}{2}\int \frac{2x+1}{x^2+x+3}dx$ and $I_2 = \frac{1}{2}\int \frac{1}{x^2+x+3}dx$

Now, $I = I_1 - I_2$ ....eqn 1

We will solve I_1 and I_2 individually.

As $I_1 = \frac{1}{2}\int \frac{2x+1}{x^2+x+3}dx$

Let $u = x^2 + x + 3 \Rightarrow du = (2x + 1)dx$

$\therefore I_1$ reduces to $\frac{1}{2}\int \frac{du}{u}$

Hence,

$I_1 = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\log|u| + C$ $\{\because \int \frac{dx}{x} = \log|x| + C\}$

On substituting the value of u, we have:

$I_1 = \frac{1}{2}\log|x^2 + x + 3| + C$ ....eqn 2

As, $I_2 = \frac{1}{2}\int \frac{1}{x^2+x+3}dx$ and we don't have any derivative of function present in denominator. ∴ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

i) $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$ ii) $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$\therefore I_2 = \frac{1}{2}\int \frac{1}{x^2+x+3}dx$

$\Rightarrow I_2 = \frac{1}{2}\int \frac{1}{\{x^2+2\left(\frac{1}{2}\right)x+\left(\frac{1}{2}\right)^2\}+3-\left(\frac{1}{2}\right)^2}\,dx$

Using: $a^2 + 2ab + b^2 = (a + b)^2$

We have:

$I_2 = \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{11}}{2}\right)^2}\,dx$

I_2 matches with $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

$\therefore I_2 = \frac{1}{2}\left\{\frac{1}{\left(\frac{\sqrt{11}}{2}\right)}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{11}}{2}}\right) + C\right\}$

$\Rightarrow I_2 = \frac{1}{\sqrt{11}}\tan^{-1}\left(\frac{2x+1}{\sqrt{11}}\right) + C$ ...eqn 3

From eqn 1:

$I = I_1 - I_2$

Using eqn 2 and eqn 3:

$I = \frac{1}{2}\log|x^2+x+3| + \frac{1}{\sqrt{11}}\tan^{-1}\left(\frac{2x+1}{\sqrt{11}}\right) + C$

3. Question

Evaluate the integral:

$$\int \frac{x-3}{x^2+2x-4}dx$$

Answer

$I = \int \frac{x-3}{x^2+2x-4}dx$

As we can see that there is a term of x in numerator and derivative of x^2 is also 2x. So there is a chance that we can make substitution for $x^2 + 2x - 4$ and I can be reduced to a fundamental integration.

As, $\frac{d}{dx}(x^2+2x-4) = 2x+2$

∴ Let, $x - 3 = A(2x + 2) + B$

$\Rightarrow x - 3 = 2Ax + 2A + B$

On comparing both sides –

We have,

$2A = 1 \Rightarrow A = 1/2$

$2A + B = -3 \Rightarrow B = -3-2A = -4$

Hence,

$I = \int \frac{\frac{1}{2}(2x+2)-4}{x^2+2x-4} dx$

$\therefore I = \frac{1}{2}\int \frac{2x+2}{x^2+2x-4} dx - 4\int \frac{1}{x^2+2x-4} dx$

Let, $I_1 = \frac{1}{2}\int \frac{2x+2}{x^2+2x-4} dx$ and $I_2 = \int \frac{1}{x^2+2x-4} dx$

Now, $I = I_1 - 4I_2$ ....eqn 1

We will solve I_1 and I_2 individually.

As, $I_1 = \frac{1}{2}\int \frac{2x+2}{x^2+2x-4} dx$

Let $u = x^2 + 2x - 4 \Rightarrow du = (2x + 2)dx$

$\therefore I_1$ reduces to $\frac{1}{2}\int \frac{du}{u}$

Hence,

$I_1 = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\log|u| + C \; \{\because \int \frac{dx}{x} = \log|x| + C\}$

On substituting value of u, we have:

$I_1 = \frac{1}{2}\log|x^2 + 2x - 4| + C$ ....eqn 2

As, $I_2 = \int \frac{1}{x^2+2x-4} dx$ and we don't have any derivative of function present in denominator. $\therefore$ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

i) $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$ ii) $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$\therefore I_2 = \int \frac{1}{x^2+2x-4} dx$

$\Rightarrow I_2 = \int \frac{1}{\{x^2+2(1)x+(1)^2\}-4-(1)^2} dx$

Using: $a^2 + 2ab + b^2 = (a + b)^2$

We have:

$I_2 = \int \frac{1}{(x+1)^2-(\sqrt{5})^2} dx$

I_2 matches with $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

$\therefore I_2 = \frac{1}{2\sqrt{5}}\log\left|\frac{x+1-\sqrt{5}}{x+1+\sqrt{5}}\right| + C$ ...eqn 3

From eqn 1:

$I = I_1 - 4I_2$

Using eqn 2 and eqn 3:

$I = \frac{1}{2}\log|x^2 + 2x - 4| - 4(\frac{1}{2\sqrt{5}}\log\left|\frac{x+1-\sqrt{5}}{x+1+\sqrt{5}}\right|) + C$

$I = \frac{1}{2}\log|x^2 + 2x - 4| - \frac{2}{\sqrt{5}}\log\left|\frac{x+1-\sqrt{5}}{x+1+\sqrt{5}}\right| + C$

4. Question

Evaluate the integral:

$$\int \frac{2x-3}{x^2+6x+13}dx$$

Answer

$I = \int \frac{2x-3}{x^2+6x+13}dx$

As we can see that there is a term of x in numerator and derivative of x^2 is also 2x. So there is a chance that we can make a substitution for $x^2 + 6x + 13$ and I can be reduced to a fundamental integration.

As $\frac{d}{dx}(x^2 + 6x + 13) = 2x + 6$

∴ Let, 2x – 3 = A(2x + 6) + B

⇒ 2x – 3 = 2Ax + 6A + B

On comparing both sides –

We have,

2A = 2 ⇒ A = 1

6A + B = –3 ⇒ B = –3–6A = –9

Hence,

$I = \int \frac{(2x+6)-9}{x^2+6x+13}dx$

$\therefore I = \int \frac{2x+6}{x^2+6x+13}dx - 9\int \frac{1}{x^2+6x+13}dx$

Let, $I_1 = \int \frac{2x+6}{x^2+6x+13}dx$ and $I_2 = \int \frac{1}{x^2+6x+13}dx$

Now, $I = I_1 - 9I_2$ ....eqn 1

We will solve I_1 and I_2 individually.

As, $I_1 = \int \frac{2x+6}{x^2+6x+13}dx$

Let $u = x^2 + 6x + 13 \Rightarrow du = (2x + 6)dx$

∴ I_1 reduces to $\int \frac{du}{u}$

Hence,

$I_1 = \int \frac{du}{u} = \log|u| + C$ $\{\because \int \frac{dx}{x} = \log|x| + C\}$

On substituting value of u, we have:

$I_1 = \log|x^2 + 6x + 13| + C$ ....eqn 2

As, $I_2 = \int \frac{1}{x^2+6x+13}dx$ and we don't have any derivative of function present in denominator. ∴ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

i) $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + C$ ii) $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$\therefore I_2 = \int \frac{1}{x^2+6x+13} dx$

$\Rightarrow I_2 = \int \frac{1}{\{x^2+2(3)x+(3)^2\}+13-(3)^2} dx$

Using: $a^2 + 2ab + b^2 = (a + b)^2$

We have:

$I_2 = \int \frac{1}{(x+3)^2+(2)^2} dx$

I_2 matches with $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$

$\therefore I_2 = \frac{1}{2} \tan^{-1}\left(\frac{x+3}{2}\right) + C$ ...eqn 3

From eqn 1:

$I = I_1 - 9I_2$

Using eqn 2 and eqn 3:

$I = \log|x^2 + 6x + 13| - 9\,\frac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right) + C$

$I = \log|x^2 + 6x + 13| - \frac{9}{2}\tan^{-1}\left(\frac{x+3}{2}\right) + C$

5. Question

Evaluate the integral:

$$\int \frac{x-1}{3x^2 - 4x + 3} dx$$

Answer

$I = \int \frac{x-1}{3x^2-4x+3} dx$

As we can see that there is a term of x in numerator and derivative of x^2 is also 2x. So there is a chance that we can make substitution for $3x^2$ –4x + 3 and I can be reduced to a fundamental integration.

As, $\frac{d}{dx}(3x^2 - 4x + 3) = 6x - 4$

∴ Let, x – 1 = A(6x – 4) + B

⇒ x – 1 = 6Ax – 4A + B

On comparing both sides –

We have,

6A = 1 ⇒ A = 1/6

–4A + B = –1 ⇒ B = –1+4A = –2/6 = –1/3

Hence,

$I = \int \frac{\frac{1}{6}(6x-4)-\frac{1}{3}}{3x^2-4x+3} dx$

$\therefore I = \frac{1}{6}\int \frac{6x-4}{3x^2-4x+3} dx - \frac{1}{3}\int \frac{1}{3x^2-4x+3} dx$

Let, $I_1 = \frac{1}{6}\int \frac{6x-4}{3x^2-4x+3} dx$ and $I_2 = \frac{1}{3}\int \frac{1}{3x^2-4x+3} dx$

Now, $I = I_1 - I_2$ ....eqn 1

We will solve I_1 and I_2 individually.

As, $I_1 = \frac{1}{6}\int \frac{6x-4}{3x^2-4x+3} dx$

Let $u = 3x^2 - 4x + 3 \Rightarrow du = (6x - 4)dx$

$\therefore I_1$ reduces to $\frac{1}{6}\int \frac{du}{u}$

Hence,

$I_1 = \frac{1}{6}\int \frac{du}{u} = \frac{1}{6}\log|u| + C$ $\{\because \int \frac{dx}{x} = \log|x| + C\}$

On substituting value of u, we have:

$I_1 = \frac{1}{6}\log|3x^2 - 4x + 3| + C$ ....eqn 2

As, $I_2 = \frac{1}{3}\int \frac{1}{3x^2-4x+3} dx$ and we don't have any derivative of function present in denominator. $\therefore$ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

i) $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$ ii) $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in the denominator.

$\therefore I_2 = \frac{1}{9}\int \frac{1}{x^2-\frac{4}{3}x+1} dx$ {on taking 3 common from denominator}

$\Rightarrow I_2 = \frac{1}{9}\int \frac{1}{\{x^2-2\left(\frac{2}{3}\right)x+\left(\frac{2}{3}\right)^2\}+1-\left(\frac{2}{3}\right)^2} dx$

Using: $a^2 + 2ab + b^2 = (a + b)^2$

We have:

$I_2 = \frac{1}{9}\int \frac{1}{\left(x-\frac{2}{3}\right)^2+\left(\frac{\sqrt{5}}{3}\right)^2} dx$

I_2 matches with $\int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

$\therefore I_2 = \frac{1}{9}\frac{1}{\frac{\sqrt{5}}{3}}\tan^{-1}\left(\frac{x-\frac{2}{3}}{\frac{\sqrt{5}}{3}}\right) + C$

$\therefore I_2 = \frac{3}{9\sqrt{5}}\tan^{-1}\left(\frac{3x-2}{\sqrt{5}}\right) + C = \frac{1}{3\sqrt{5}}\tan^{-1}\left(\frac{3x-2}{\sqrt{5}}\right) + C$ ...eqn 3

From eqn 1:

$I = I_1 - I_2$

Using eqn 2 and eqn 3:

$I = \frac{1}{6}\log|3x^2 - 4x + 3| - \frac{1}{3\sqrt{5}}\tan^{-1}\left(\frac{3x-2}{\sqrt{5}}\right) + C$

6. Question

Evaluate the integral:

$$\int \frac{2x}{2+x-x^2}dx$$

Answer

$I = \int \frac{2x}{2+x-x^2}dx$

As we can see that there is a term of x in numerator and derivative of x^2 is also 2x. So there is a chance that we can make substitution for $-x^2 + x + 2$ and I can be reduced to a fundamental integration.

As, $\frac{d}{dx}(-x^2 + x + 2) = -2x + 1$

∴ Let, 2x = A(–2x + 1) + B

⇒ 2x = –2Ax + A + B

On comparing both sides –

We have,

–2A = 2 ⇒ A = –1

A + B = 0 ⇒ B = –A = 1

Hence,

$I = \int \frac{-(-2x+1)+1}{2+x-x^2}dx$

$\therefore I = -\int \frac{(-2x+1)}{2+x-x^2}dx + \int \frac{1}{2+x-x^2}dx$

Let, $I_1 = -\int \frac{(-2x+1)}{2+x-x^2}dx$ and $I_2 = \int \frac{1}{2+x-x^2}dx$

Now, $I = I_1 + I_2$ ....eqn 1

We will solve I_1 and I_2 individually.

As, $I_1 = -\int \frac{(-2x+1)}{2+x-x^2}dx$

Let $u = 2 + x - x^2 \Rightarrow du = (-2x + 1)dx$

$\therefore I_1$ reduces to $-\int \frac{du}{u}$

Hence,

$I_1 = -\int \frac{du}{u} = -\log|u| + C$ $\{\because \int \frac{dx}{x} = \log|x| + C\}$

On substituting value of u, we have:

$I_1 = -\log|2 + x - x^2| + C$ ....eqn 2

As, $I_2 = \int \frac{1}{2+x-x^2}dx$ and we don't have any derivative of function present in denominator. ∴ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

i) $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left|\frac{x - a}{x + a}\right| + C$ ii) $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$\therefore I_2 = -\int \frac{1}{x^2 - x - 2} dx$

$\Rightarrow I_2 = -\int \frac{1}{\{x^2 - 2\left(\frac{1}{2}\right)x + \left(\frac{1}{2}\right)^2\} - 2 - \left(\frac{1}{2}\right)^2} dx$

Using: $a^2 + 2ab + b^2 = (a + b)^2$

We have:

$I_2 = -\int \frac{1}{\left(x - \frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2} dx$

I_2 matches with $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left|\frac{x - a}{x + a}\right| + C$

$\therefore I_2 = -\frac{1}{2\left(\frac{3}{2}\right)} \log \left|\frac{\left(x - \frac{1}{2}\right) - \frac{3}{2}}{\left(x - \frac{1}{2}\right) + \frac{3}{2}}\right| + C$

$\therefore I_2 = -\frac{1}{3} \log \left|\frac{2x - 1 - 3}{2x - 1 + 3}\right| + C = -\frac{1}{3} \log \left|\frac{2x - 4}{2x + 2}\right| + C = -\frac{1}{3} \log \left|\frac{x - 2}{x + 1}\right| + C$ ...eqn 3

From eqn 1:

$I = I_1 + I_2$

Using eqn 2 and eqn 3:

$\therefore I = -\log|2 + x - x^2| - \frac{1}{3} \log \left|\frac{x - 2}{x + 1}\right| + C$

7. Question

Evaluate the integral:

$$\int \frac{1 - 3x}{3x^2 + 4x + 2} dx$$

Answer

$I = \int \frac{1 - 3x}{3x^2 + 4x + 2} dx$

As we can see that there is a term of x in numerator and derivative of x^2 is also 2x. So there is a chance that we can make substitution for $3x^2 + 4x + 2$ and I can be reduced to a fundamental integration.

As, $\frac{d}{dx}(3x^2 + 4x + 2) = 6x + 4$

$\therefore$ Let, $1 - 3x = A(6x + 4) + B$

$\Rightarrow 1 - 3x = 6Ax + 4A + B$

On comparing both sides –

We have,

$6A = -3 \Rightarrow A = -1/2$

$4A + B = 1 \Rightarrow B = -4A + 1 = 3$

Hence,

$I = \int \frac{-\frac{1}{2}(6x+4)+3}{3x^2+4x+2} dx$

$\therefore I = -\frac{1}{2}\int \frac{6x+4}{3x^2+4x+2} dx + \int \frac{3}{3x^2+4x+2} dx$

Let, $I_1 = -\frac{1}{2}\int \frac{6x+4}{3x^2+4x+2} dx$ and $I_2 = \int \frac{3}{3x^2+4x+2} dx$

Now, $I = I_1 + I_2$ ....eqn 1

We will solve I_1 and I_2 individually.

As $I_1 = -\frac{1}{2}\int \frac{6x+4}{3x^2+4x+2} dx$

Let $u = 3x^2 + 4x + 2 \Rightarrow du = (6x + 4)dx$

$\therefore I_1$ reduces to $-\frac{1}{2}\int \frac{du}{u}$

Hence,

$I_1 = -\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\log|u| + C$ $\{\because \int \frac{dx}{x} = \log|x| + C\}$

On substituting the value of u, we have:

$I_1 = -\frac{1}{2}\log|3x^2 + 4x + 2| + C$ ....eqn 2

As, $I_2 = \int \frac{3}{3x^2+4x+2} dx$ and we don't have any derivative of function present in denominator. $\therefore$ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

i) $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$ ii) $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$\therefore I_2 = \int \frac{3}{3(x^2+\frac{4}{3}x+\frac{2}{3})} dx = \int \frac{1}{x^2+\frac{4}{3}x+\frac{2}{3}} dx$

$\Rightarrow I_2 = \int \frac{1}{\{x^2+2(\frac{2}{3})x+(\frac{2}{3})^2\}+\frac{2}{3}-(\frac{2}{3})^2} dx$

Using: $a^2 + 2ab + b^2 = (a + b)^2$

We have:

$I_2 = \int \frac{1}{\left(x+\frac{2}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2} dx$

I_2 matches with $\int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

$\therefore I_2 = \frac{1}{\frac{\sqrt{2}}{3}}\tan^{-1}\left(\frac{x+\frac{2}{3}}{\frac{\sqrt{2}}{3}}\right) + C$

$\therefore I_2 = \frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{3x+2}{\sqrt{2}}\right) + C$ ...eqn 3

From eqn 1:

$I = I_1 + I_2$

Using eqn 2 and eqn 3:

$\therefore I = -\frac{1}{2}\log|3x^2 + 4x + 2| + \frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{3x+2}{\sqrt{2}}\right) + C$

8. Question

Evaluate the integral:

$$\int \frac{2x+5}{x^2 - x - 2}dx$$

Answer

$I = \int \frac{2x+5}{x^2-x-2}dx$

As we can see that there is a term of x in numerator and derivative of x^2 is also 2x. So there is a chance that we can make substitution for $x^2 - x - 2$ and I can be reduced to a fundamental integration.

As, $\frac{d}{dx}(x^2 - x - 2) = 2x - 1$

$\therefore$ Let, $2x + 5 = A(2x - 1) + B$

$\Rightarrow 2x + 5 = 2Ax - A + B$

On comparing both sides –

We have,

$2A = 2 \Rightarrow A = 1$

$-A + B = 5 \Rightarrow B = A + 5 = 6$

Hence,

$I = \int \frac{(2x-1)+6}{x^2-x-2}dx$

$\therefore I = \int \frac{(2x-1)}{x^2-x-2}dx + \int \frac{6}{x^2-x-2}dx$

Let, $I_1 = \int \frac{(2x-1)}{x^2-x-2}dx$ and $I_2 = \int \frac{6}{x^2-x-2}dx$

Now, $I = I_1 + I_2$ ....eqn 1

We will solve I_1 and I_2 individually.

As, $I_1 = \int \frac{(2x-1)}{x^2-x-2}dx$

Let $u = x^2 - x - 2 \Rightarrow du = (2x - 1)dx$

$\therefore I_1$ reduces to $\int \frac{du}{u}$

Hence,

$I_1 = \int \frac{du}{u} = \log|u| + C \ \{\because \int \frac{dx}{x} = \log|x| + C\}$

On substituting value of u, we have:

$I_1 = \log|x^2 - x - 2| + C$ ....eqn 2

As, $I_2 = \int \frac{6}{x^2-x-2}dx$ and we don't have any derivative of function present in denominator. $\therefore$ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

i) $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$ ii) $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$\therefore I_2 = \int \frac{6}{x^2-x-2}dx$

$\Rightarrow I_2 = \int \frac{6}{\{x^2-2\left(\frac{1}{2}\right)x+\left(\frac{1}{2}\right)^2\}-2-\left(\frac{1}{2}\right)^2} dx$

Using: $a^2 - 2ab + b^2 = (a - b)^2$

We have:

$I_2 = 6\int \frac{1}{\left(x-\frac{1}{2}\right)^2-\left(\frac{3}{2}\right)^2} dx$

I_2 matches with $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

$\therefore I_2 = \frac{6}{2\left(\frac{3}{2}\right)}\log\left|\frac{\left(x-\frac{1}{2}\right)-\frac{3}{2}}{\left(x-\frac{1}{2}\right)+\frac{3}{2}}\right| + C$

$\therefore I_2 = \frac{6}{3}\log\left|\frac{2x-1-3}{2x-1+3}\right| + C = 2\log\left|\frac{2x-4}{2x+2}\right| + C = 2\log\left|\frac{x-2}{x+1}\right| + C$ ...eqn 3

From eqn 1, we have:

$I = I_1 + I_2$

Using eqn 2 and 3, we get –

$I = \log|x^2 - x - 2| + 2\log\left|\frac{x-2}{x+1}\right| + C$ .....ans

9. Question

Evaluate the integral:

$$\int \frac{ax^3+bx}{x^4+c^2}dx$$

Answer

$I = \int \frac{ax^3+bx}{x^4+c^2}dx$

As we can see that there is a term of x^3 in numerator and derivative of x^4 is also $4x^3$. So there is a chance that we can make substitution for $x^4 + c^2$ and I can be reduced to a fundamental integration but there is also a x term present. So it is better to break this integration.

$I = \int \frac{ax^3}{x^4+c^2}dx + \int \frac{bx}{x^4+c^2}dx = I_1 + I_2$ ...eqn 1

$I_1 = \int \frac{ax^3}{x^4+c^2}dx = \frac{a}{4}\int \frac{4x^3}{x^4+c^2}dx$

As, $\frac{d}{dx}(x^4+c^2) = 4x^3$

To make the substitution, I_1 can be rewritten as

$I_1 = \frac{a}{4}\int \frac{4x^3}{x^4+c^2}dx$

$\therefore$ Let, $x^4 + c^2 = u$

$\Rightarrow du = 4x^3\, dx$

I_1 is reduced to simple integration after substituting u and du as:

$$I_1 = \frac{a}{4}\int \frac{du}{u} = \frac{a}{4}\log|u| + C$$

$\therefore I_1 = \frac{a}{4}\log|x^4 + c^2| + C$ ...eqn 2

As,

$I_2 = \int \frac{bx}{x^4+c^2}dx$

$\because$ we have derivative of x^2 in numerator and term of x^2 in denominator. So we can apply method of substitution here also.

As, $I_2 = \int \frac{bx}{(x^2)^2+c^2}dx$

Let, $x^2 = v$

$\Rightarrow dv = 2x\, dx$

$\because I_2 = \frac{b}{2}\int \frac{2x}{(x^2)^2+c^2}dx = \frac{b}{2}\int \frac{dv}{(v)^2+c^2}$

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

i) $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$ ii) $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

I_2 matches with $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

$\therefore I_2 = \frac{b}{2}\frac{1}{c}\tan^{-1}\left(\frac{v}{c}\right) + K = \frac{b}{2c}\tan^{-1}\left(\frac{v}{c}\right) + K$

$\Rightarrow I_2 = \frac{b}{2c}\tan^{-1}\left(\frac{x^2}{c}\right) + K$ ...eqn 3

From eqn 1, we have:

$I = I_1 + I_2$

Using eqn 2 and 3, we get –

$I = \frac{a}{4}\log|x^4 + c^2| + \frac{b}{2c}\tan^{-1}\left(\frac{x^2}{c}\right) + K$ .....ans

10. Question

Evaluate the integral:

$$\int \frac{(3\sin x - 2)\cos x}{5 - \cos^2 x - 4\sin x}dx$$

Answer

$I = \int \frac{(3\sin x-2)\cos x}{5-\cos^2 x-4\sin x}dx = \int \frac{(3\sin x-2)\cos x}{5-(1-\sin^2 x)-4\sin x}dx$

$\Rightarrow I = \int \frac{(3\sin x-2)\cos x}{4+\sin^2 x-4\sin x}dx$

Let, $\sin x = t \Rightarrow \cos x\, dx = dt$

$\therefore I = \int \frac{(3t-2)}{t^2-4t+4}dt$

As we can see that there is a term of t in numerator and derivative of t^2 is also 2t. So there is a chance that we can make substitution for $t^2 - 4t + 4$ and I can be reduced to a fundamental integration.

As, $\frac{d}{dt}(t^2 - 4t - 4) = 2t - 4$

∴ Let, $3t - 2 = A(2t - 4) + B$

⇒ $3t - 2 = 2At - 4A + B$

On comparing both sides –

We have,

$2A = 3 \Rightarrow A = 3/2$

$-4A + B = -2 \Rightarrow B = 4A - 2 = 4$

Hence,

$I = \int \frac{(3t-2)}{t^2-4t+4} dt$

$\therefore I = \int \frac{\frac{3}{2}(2t-4)}{t^2-4t+4} dt + \int \frac{4}{t^2-4t+4} dt$

Let, $I_1 = \frac{3}{2} \int \frac{(2t-4)}{t^2-4t+4} dt$ and $I_2 = \int \frac{4}{t^2-4t+4} dt$

Now, $I = I_1 + I_2$ ....eqn 1

We will solve I_1 and I_2 individually.

As, $I_1 = \frac{3}{2} \int \frac{(2t-4)}{t^2-4t+4} dt$

Let $u = t^2 - 4t + 4 \Rightarrow du = (2t - 4)dx$

∴ I_1 reduces to $\frac{3}{2} \int \frac{du}{u}$

Hence,

$I_1 = \frac{3}{2} \int \frac{du}{u} = \log|u| + C$ $\{\because \int \frac{dx}{x} = \log|x| + C\}$

On substituting value of u, we have:

$I_1 = \frac{3}{2} \log|t^2 - 4t + 4| + C$

$I_1 = \frac{3}{2} \log|t - 2|^2 + C = 3 \log|t - 2| + C$ ....eqn 2

$\because I_2 = \int \frac{4}{t^2-4t+4} dt$

$\Rightarrow I_2 = \int \frac{4}{\{t^2-2(2)t+2^2\}} dx$

Using: $a^2 - 2ab + b^2 = (a - b)^2$

We have:

$I_2 = 4 \int \frac{1}{(t-2)^2} dx$

As, $\int \frac{1}{x^2} dx = -\frac{1}{x}$

$\therefore I_2 = \frac{-4}{t-2} = \frac{4}{2-t} + C$ ...eqn 3

From eqn 1, we have:

I = $I_1 + I_2$

Using eqn 2 and 3, we get –

I = $3\log|t-2| + \frac{4}{2-t} + C$

Putting value of t in I:

I = $3\log|\sin x - 2| + \frac{4}{2-\sin x} + C$ .....ans

11. Question

Evaluate the integral:

$$\int \frac{x+2}{2x^2+6x+5}dx$$

Answer

I $= \int \frac{x+2}{2x^2+6x+5}dx$

As we can see that there is a term of x in numerator and derivative of x^2 is also 2x. So there is a chance that we can make substitution for $2x^2 +6x +5$ and I can be reduced to a fundamental integration.

As, $\frac{d}{dx}(2x^2+6x+5) = 4x+6$

∴ Let, x + 2 = A(4x + 6) + B

⇒ x + 2 = 4Ax + 6A + B

On comparing both sides –

We have,

4A = 1 ⇒ A = 1/4

6A + B = 2 ⇒ B = –6A + 2 = 1/2

Hence,

I $= \int \frac{\frac{1}{4}(4x+6)+\frac{1}{2}}{2x^2+6x+5}dx$

∴ I = $\int \frac{\frac{1}{4}(4x+6)}{2x^2+6x+5}dx + \int \frac{\frac{1}{2}}{2x^2+6x+5}dx$

Let, $I_1 = \frac{1}{4}\int \frac{(4x+6)}{2x^2+6x+5}dx$ and $I_2 = \frac{1}{2}\int \frac{1}{2x^2+6x+5}dx$

Now, I = $I_1 + I_2$ ....eqn 1

We will solve I_1 and I_2 individually.

As, $I_1 = \frac{1}{4}\int \frac{(4x+6)}{2x^2+6x+5}dx$

Let u = $2x^2 + 6x + 5$ ⇒ du = (4x + 6)dx

∴ I_1 reduces to $\frac{1}{4}\int \frac{du}{u}$

Hence,

$I_1 = \frac{1}{4}\int \frac{du}{u} = \frac{1}{4}\log|u| + C$ {∵ $\int \frac{dx}{x} = \log|x| + C$ }

On substituting value of u, we have:

$I_1 = \frac{1}{4}\log|2x^2 + 6x + 5| + C$ ....eqn 2

As, $I_2 = \frac{1}{2}\int \frac{1}{2x^2+6x+5} dx$ and we don't have any derivative of function present in denominator. ∴ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

i) $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$ ii) $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$\therefore I_2 = \frac{1}{2}\int \frac{1}{2x^2+6x+5} dx = \frac{1}{2}\int \frac{1}{2(x^2+3x+\frac{5}{2})} dx = \frac{1}{4}\int \frac{1}{x^2+3x+\frac{5}{2}} dx$

$\Rightarrow I_2 = \frac{1}{4}\int \frac{6}{\{x^2+2(\frac{3}{2})x+(\frac{3}{2})^2\}+\frac{5}{2}-(\frac{3}{2})^2} dx$

Using: $a^2 + 2ab + b^2 = (a + b)^2$

We have:

$I_2 = \frac{1}{4}\int \frac{1}{\left(x+\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} dx$

I_2 matches with 1x2+a2dx=1112 tan−1x +32112+ CI_2 matches with the form $\int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

$\therefore I_2 = \frac{1}{4}\frac{1}{\frac{1}{2}}\tan^{-1}\left(\frac{x+\frac{3}{2}}{\frac{1}{2}}\right) + C$

$\therefore I_2 = \frac{1}{2}\tan^{-1}(2x+3) + C$ ...eqn 3

From eqn 1, we have:

$I = I_1 + I_2$

Using eqn 2 and 3, we get –

$I = \frac{1}{4}\log|2x^2 + 6x + 5| + C + \frac{1}{2}\tan^{-1}(2x+3) + C$ .....ans

12. Question

Evaluate the integral:

$$\int \frac{5x - 2}{1 + 2x + 3x^2} dx$$

Answer

$I = \int \frac{5x-2}{3x^2+2x+1} dx$

As we can see that there is a term of x in numerator and derivative of x^2 is also 2x. So there is a chance that we can make substitution for $3x^2$ +2x +1 and I can be reduced to a fundamental integration.

As, $\frac{d}{dx}(3x^2 + 2x + 1) = 6x + 2$

∴ Let, 5x – 2 = A(6x + 2) + B

⇒ 5x – 2 = 6Ax + 2A + B

On comparing both sides –

We have,

$6A = 5 \Rightarrow A = 5/6$

$2A + B = -2 \Rightarrow B = -2A - 2 = -11/3$

Hence,

$I = \int \frac{\frac{5}{6}(6x+2) - \frac{11}{3}}{3x^2+2x+1} dx$

$\therefore I = \int \frac{\frac{5}{6}(6x+2)}{3x^2+2x+1} dx + \int \frac{-\frac{11}{3}}{3x^2+2x+1} dx$

Let, $I_1 = \frac{5}{6} \int \frac{(6x+2)}{3x^2+2x+1} dx$ and $I_2 = -\frac{11}{3} \int \frac{1}{3x^2+2x+1} dx$

Now, $I = I_1 + I_2$ ....eqn 1

We will solve I_1 and I_2 individually.

As, $I_1 = \frac{5}{6} \int \frac{(6x+2)}{3x^2+2x+1}$

Let $u = 3x^2 + 2x + 1 \Rightarrow du = (6x + 2)dx$

$\therefore I_1$ reduces to $\frac{5}{6} \int \frac{du}{u}$

Hence,

$I_1 = \frac{5}{6} \int \frac{du}{u} = \frac{5}{6} \log|u| + C \{\because \int \frac{dx}{x} = \log|x| + C \}$

On substituting value of u, we have:

$I_1 = \frac{5}{6} \log|3x^2 + 2x + 1| + C$ ....eqn 2

As, $I_2 = -\frac{11}{3} \int \frac{1}{3x^2+2x+1} dx$ and we don't have any derivative of function present in denominator. $\therefore$ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

i) $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + C$ ii) $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$\therefore I_2 = -\frac{11}{3} \int \frac{1}{3x^2+2x+1} dx = \frac{-11}{3} \int \frac{1}{3(x^2+\frac{2}{3}x+\frac{1}{3})} dx = -\frac{11}{9} \int \frac{1}{x^2+\frac{2}{3}x+\frac{1}{3}} dx$

$\Rightarrow I_2 = -\frac{11}{9} \int \frac{6}{\{x^2+2(\frac{1}{3})x+(\frac{1}{3})^2\}+\frac{1}{3}-(\frac{1}{3})^2} dx$

Using: $a^2 + 2ab + b^2 = (a + b)^2$

We have:

$I_2 = -\frac{11}{9} \int \frac{1}{(x+\frac{1}{3})^2 + (\frac{\sqrt{2}}{3})^2} dx$

I_2 matches with the form $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$

$\therefore I_2 = -\frac{11}{9} \frac{1}{\frac{\sqrt{2}}{3}} \tan^{-1}\left(\frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}}\right) + C$

$\therefore I_2 = -\frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C$ ...eqn 3

From eqn 1, we have:

$I = I_1 + I_2$

Using eqn 2 and 3, we get –

$I = \frac{5}{6}\log|3x^2 + 2x + 1| - \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C$

13. Question

Evaluate the integral:

$$\int \frac{x+5}{3x^2+13x-10}dx$$

Answer

$I = \int \frac{x+5}{3x^2+13x-10}dx$

As we can see that there is a term of x in numerator and derivative of x^2 is also 2x. So there is a chance that we can make substitution for $3x^2 + 13x – 10$ and I can be reduced to a fundamental integration.

As, $\frac{d}{dx}(3x^2 + 13x - 10) = 6x + 13$

$\therefore$ Let, $x + 5 = A(6x + 13) + B$

$\Rightarrow x + 5 = 6Ax + 13A + B$

On comparing both sides –

We have,

$6A = 1 \Rightarrow A = 1/6$

$13A + B = 5 \Rightarrow B = -13A + 5 = 17/6$

Hence,

$I = \int \frac{\frac{1}{6}(6x+13)+\frac{17}{6}}{3x^2+13x-10}dx$

$\therefore I = \int \frac{\frac{1}{6}(6x+13)}{3x^2+13x-10}dx + \int \frac{\frac{17}{6}}{3x^2+13x-10}dx$

Let, $I_1 = \frac{1}{6}\int \frac{(6x+13)}{3x^2+13x-10}dx$ and $I_2 = \frac{17}{6}\int \frac{1}{3x^2+13x-10}dx$

Now, $I = I_1 + I_2$ ....eqn 1

We will solve I_1 and I_2 individually.

As, $I_1 = \frac{1}{6}\int \frac{(6x+13)}{3x^2+13x-10}dx$

Let $u = 3x^2 + 13x - 10 \Rightarrow du = (6x + 13)dx$

$\therefore I_1$ reduces to $\frac{1}{6}\int \frac{du}{u}$

Hence,

$I_1 = \frac{1}{6}\int \frac{du}{u} = \frac{1}{6}\log|u| + C$ $\{\because \int \frac{dx}{x} = \log|x| + C\}$

On substituting value of u, we have:

$I_1 = \frac{1}{6}\log|3x^2 + 13x - 10| + C$ ....eqn 2

As, $I_2 = \frac{17}{6}\int \frac{1}{3x^2+13x-10}dx$ and we don't have any derivative of function present in denominator. ∴ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

i) $\int \frac{1}{x^2 - a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$ ii) $\int \frac{1}{x^2 + a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$\therefore I_2 = \frac{17}{6}\int \frac{1}{3x^2+13x-10}dx = \frac{17}{6}\int \frac{1}{3(x^2+\frac{13}{3}x-\frac{10}{3})}dx = \frac{17}{18}\int \frac{1}{x^2+\frac{13}{3}x-\frac{10}{3}}dx$

$\Rightarrow I_2 = \frac{17}{18}\int \frac{6}{\{x^2+2(\frac{13}{6})x+(\frac{13}{6})^2\}-\frac{10}{3}-(\frac{13}{6})^2}dx$

Using: $a^2 + 2ab + b^2 = (a + b)^2$

We have:

$I_2 = \frac{17}{18}\int \frac{1}{\left(x+\frac{13}{6}\right)^2-\left(\frac{17}{6}\right)^2}dx$

I_2 matches with the form $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

$\therefore I_2 = \frac{17}{18} \times \frac{1}{2\times\frac{17}{6}}\log\left|\frac{\left(x+\frac{13}{6}\right)-\frac{17}{6}}{\left(x+\frac{13}{6}\right)+\frac{17}{6}}\right| + C$

$\therefore I_2 = \frac{1}{6}\log\left|\frac{6x+13-17}{6x+13+17}\right| + C = \frac{1}{6}\log\left|\frac{6x-4}{6x+30}\right| + C$ ...eqn 3

From eqn 1, we have:

$I = I_1 + I_2$

Using eqn 2 and 3, we get –

$I = \frac{1}{6}\log|3x^2 + 13x - 10| + \frac{1}{6}\log\left|\frac{6x-4}{6x+30}\right| + C$

4. Question

Evaluate the integral:

$$\int \frac{(3\sin x - 2)\cos x}{13 - \cos^2 x - 7\sin x}dx$$

Answer

$I = \int \frac{(3\sin x-2)\cos x}{13-\cos^2 x-7\sin x}dx = \int \frac{(3\sin x-2)\cos x}{13-(1-\sin^2 x)-7\sin x}dx$

$\Rightarrow I = \int \frac{(3\sin x-2)\cos x}{12+\sin^2 x-7\sin x}dx$

Let, $\sin x = t \Rightarrow \cos x\, dx = dt$

$\therefore I = \int \frac{(3t-2)}{t^2-7t+12}dt$

As we can see that there is a term of t in numerator and derivative of t^2 is also 2t. So there is a chance that we can make substitution for $t^2 - 7t + 12$ and I can be reduced to a fundamental integration.

As, $\frac{d}{dt}(t^2 - 7t + 12) = 2t - 7$

∴ Let, 3t – 2 = A(2t – 7) + B

⇒ 3t – 2 = 2At – 7A + B

On comparing both sides –

We have,

2A = 3 ⇒ A = 3/2

–7A + B = –2 ⇒ B = 7A – 2 = 17/2

Hence,

$I = \int \frac{(3t-2)}{t^2-7t+12} dt$

$\therefore I = \int \frac{\frac{3}{2}(2t-7)}{t^2-7t+12} dt + \int \frac{\frac{17}{2}}{t^2-7t+12} dt$

Let, $I_1 = \frac{3}{2}\int \frac{(2t-7)}{t^2-7t+12} dt$ and $I_2 = \frac{17}{2}\int \frac{1}{t^2-7t+12} dt$

Now, $I = I_1 + I_2$ ....eqn 1

We will solve I_1 and I_2 individually.

As, $I_1 = \frac{3}{2}\int \frac{(2t-7)}{t^2-7t+12} dt$

Let $u = t^2 - 7t + 12 \Rightarrow du = (2t - 7)dx$

∴ I_1 reduces to $\frac{3}{2}\int \frac{du}{u}$

Hence,

$I_1 = \frac{3}{2}\int \frac{du}{u} = \log|u| + C \ \{\because \int \frac{dx}{x} = \log|x| + C\ \}$

On substituting value of u, we have:

$I_1 = \frac{3}{2}\log|t^2 - 7t + 12| + C$ ....eqn 2

As, $I_2 = \frac{17}{2}\int \frac{1}{t^2-7t+12} dt$and we don't have any derivative of function present in denominator. ∴ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

i) $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$ ii) $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

$\because I_2 = \frac{17}{2}\int \frac{1}{t^2-7t+12} dt$

$\Rightarrow I_2 = \frac{17}{2}\int \frac{4}{\{t^2-2\left(\frac{7}{2}\right)t+\left(\frac{7}{2}\right)^2\}+12-\left(\frac{7}{2}\right)^2} dx$

Using: $a^2 - 2ab + b^2 = (a - b)^2$

We have:

$I_2 = \frac{17}{2}\int \frac{1}{\left(t-\frac{7}{2}\right)^2-\left(\frac{1}{2}\right)^2} dx$

I_2 matches with the form $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

$\therefore I_2 = \frac{17}{2}\frac{1}{2\left(\frac{1}{2}\right)}\log\left|\frac{\left(t-\frac{7}{2}\right)-\frac{1}{2}}{\left(t-\frac{7}{2}\right)+\frac{1}{2}}\right| + C$

$I_2 = \frac{17}{2}\log\left|\frac{2t-7-1}{2t-7+1}\right| + C = \frac{17}{2}\log\left|\frac{2t-8}{2t-6}\right| + C$

$I_2 = \frac{17}{2}\log\left|\frac{t-4}{t-3}\right| + C$ ...eqn 3

From eqn 1, we have:

$I = I_1 + I_2$

Using eqn 2 and 3, we get –

$I = \frac{3}{2}\log|t^2 - 7t + 12| + \frac{17}{2}\log\left|\frac{t-4}{t-3}\right| + C$

Putting value of t in I:

$I = \frac{3}{2}\log|\sin^2 x - 7\sin x + 12| + \frac{17}{2}\log\left|\frac{4-\sin x}{3-\sin x}\right| + C$ .....ans

5. Question

Evaluate the integral:

$$\int \frac{x+7}{3x^2 + 25x + 28}\,dx$$

Answer

$I = \int \frac{x+7}{3x^2+25x+28}dx$

As we can see that there is a term of x in numerator and derivative of x^2 is also 2x. So there is a chance that we can make substitution for $3x^2 + 13x - 10$ and I can be reduced to a fundamental integration.

As, $\frac{d}{dx}(3x^2 + 25x + 28) = 6x + 25$

$\therefore$ Let, $x + 7 = A(6x + 25) + B$

$\Rightarrow x + 7 = 6Ax + 25A + B$

On comparing both sides –

We have,

$6A = 1 \Rightarrow A = 1/6$

$25A + B = 5 \Rightarrow B = -25A + 5 = 5/6$

Hence,

$I = \int \frac{\frac{1}{6}(6x+25)+\frac{5}{6}}{3x^2+25x+28}dx$

$\therefore I = \int \frac{\frac{1}{6}(6x+25)}{3x^2+25x+28}dx + \int \frac{\frac{5}{6}}{3x^2+25x+28}dx$

Let, $I_1 = \frac{1}{6}\int \frac{(6x+25)}{3x^2+25x+28}dx$ and $I_2 = \frac{5}{6}\int \frac{1}{3x^2+25x+28}dx$

Now, $I = I_1 + I_2$ ....eqn 1

We will solve I_1 and I_2 individually.

As, $I_1 = \frac{1}{6}\int \frac{(6x+25)}{3x^2+25x+28}dx$

Let $u = 3x^2 + 25x + 28 \Rightarrow du = (6x + 25)dx$

$\therefore I_1$ reduces to $\frac{1}{6}\int \frac{du}{u}$

Hence,

$I_1 = \frac{1}{6}\int \frac{du}{u} = \frac{1}{6}\log|u| + C \{\because \int \frac{dx}{x} = \log|x| + C\}$

On substituting value of u, we have:

$I_1 = \frac{1}{6}\log|3x^2 + 25x + 28| + C$ ....eqn 2

As, $I_2 = \frac{5}{6}\int \frac{1}{3x^2+25x+28}dx$ and we don't have any derivative of function present in denominator. $\therefore$ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

i) $\int \frac{1}{x^2 - a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$ ii) $\int \frac{1}{x^2 + a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$\therefore I_2 = \frac{5}{6}\int \frac{1}{3x^2+25x+28}dx = \frac{5}{6}\int \frac{1}{3(x^2+\frac{25}{3}x+\frac{28}{3})}dx = \frac{5}{18}\int \frac{1}{x^2+\frac{25}{3}x+\frac{28}{3}}dx$

$\Rightarrow I_2 = \frac{5}{18}\int \frac{1}{\{x^2+2(\frac{25}{6})x+(\frac{25}{6})^2\}+\frac{28}{3}-(\frac{25}{6})^2}dx$

Using: $a^2 + 2ab + b^2 = (a + b)^2$

We have:

$I_2 = \frac{5}{18}\int \frac{1}{(x+\frac{25}{6})^2-(\frac{17}{6})^2}dx$

I_2 matches with the form $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

$\therefore I_2 = \frac{5}{18} \times \frac{1}{2\times\frac{17}{6}}\log\left|\frac{(x+\frac{25}{6})-\frac{17}{6}}{(x+\frac{25}{6})+\frac{17}{6}}\right| + C$

$\therefore I_2 = \frac{5}{102}\log\left|\frac{6x+25-17}{6x+25+17}\right| + C = \frac{5}{102}\log\left|\frac{6x-8}{6x+42}\right| + C$ ...eqn 3

From eqn 1, we have:

$I = I_1 + I_2$

Using eqn 2 and 3, we get –

$I = \frac{1}{6}\log|3x^2 + 25x + 28| + \frac{5}{102}\log\left|\frac{6x-8}{6x+42}\right| + C$

16. Question

Evaluate the integral:

$$\int \frac{x^3}{x^4 + x^2 + 1}dx$$

Answer

Let, $I = \int \frac{x^3}{x^4+x^2+1}dx$

$\mathrm{I} = \int \frac{x^2\, x}{(x^2)^2+x^2+1} dx$

If we assume x^2 to be an another variable, we can simplify the integral as derivative of x^2 i.e. x is present in numerator.

Let, $x^2 = u$

$\Rightarrow 2x\, dx = du$

$\Rightarrow x\, dx = 1/2\, du$

$\therefore \mathrm{I} = \frac{1}{2}\int \frac{u}{u^2+u+1} du$

As, $\frac{d}{du}(u^2+u+1) = 2u+1$

$\therefore$ Let, $u = A(2u + 1) + B$

$\Rightarrow u = 2Au + A + B$

On comparing both sides –

We have,

$2A = 1 \Rightarrow A = 1/2$

$A + B = 0 \Rightarrow B = -A = -1/2$

Hence,

$\mathrm{I} = \frac{1}{2}\int \frac{\frac{1}{2}(2u+1)-\frac{1}{2}}{u^2+u+1} du$

$\therefore \mathrm{I} = \frac{1}{4}\int \frac{(2u+1)}{u^2+u+1} du + \frac{1}{2}\int \frac{-\frac{1}{2}}{u^2+u+1} du$

Let, $I_1 = \frac{1}{4}\int \frac{(2u+1)}{u^2+u+1} du$ and $I_2 = -\frac{1}{4}\int \frac{1}{u^2+u+1} du$

Now, $I = I_1 + I_2$ ....eqn 1

We will solve I_1 and I_2 individually.

As, $I_1 = \frac{1}{4}\int \frac{(2u+1)}{u^2+u+1} du$

Let $v = u^2 + u + 1 \Rightarrow dv = (2u + 1)du$

$\therefore I_1$ reduces to $\frac{1}{4}\int \frac{dv}{v}$

Hence,

$I_1 = \frac{1}{4}\int \frac{dv}{v} = \frac{1}{4}\log|v| + C$ $\{\because \int \frac{dx}{x} = \log|x| + C\}$

On substituting value of u, we have:

$I_1 = \frac{1}{4}\log|u^2+u+1| + C$ ....eqn 2

As, $I_2 = -\frac{1}{4}\int \frac{1}{u^2+u+1} du$ and we don't have any derivative of function present in denominator. $\therefore$ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

i) $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$ ii) $\int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$\therefore I_2 = -\frac{1}{4}\int \frac{1}{u^2+u+1} du$

$\Rightarrow I_2 = -\frac{1}{4}\int \frac{1}{\{u^2+2\left(\frac{1}{2}\right)u+\left(\frac{1}{2}\right)^2\}+1-\left(\frac{1}{2}\right)^2} du$

Using: $a^2 + 2ab + b^2 = (a + b)^2$

We have:

$I_2 = -\frac{1}{4}\int \frac{1}{\left(u+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} du$

$\therefore I_2 = -\frac{1}{4}\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{u+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$

$\therefore I_2 = -\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{2u+1}{\sqrt{3}}\right) + C$ ...eqn 3

From eqn 1, we have:

$I = I_1 + I_2$

Using eqn 2 and 3, we get –

$I = \frac{1}{4}\log|u^2 + u + 1| - \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{2u+1}{\sqrt{3}}\right) + C$

Putting value of u in I:

$I = \frac{1}{4}\log\left|x^{2^2} + x^2 + 1\right| - \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right) + C$

$I = \frac{1}{4}\log|x^4 + x^2 + 1| - \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right) + C$

17. Question

Evaluate the integral:

$$\int \frac{x^3 - 3x}{x^4 + 2x^2 - 4}$$

Answer

Let, $I = \int \frac{x^3-3x}{x^4+2x^2-4} dx$

$I = \int \frac{(x^2-3)x}{(x^2)^2+2x^2-4} dx$

If we assume x^2 to be an another variable, we can simplify the integral as derivative of x^2 i.e. x is present in numerator.

Let, $x^2 = u$

$\Rightarrow 2x\,dx = du$

$\Rightarrow x\,dx = 1/2\,du$

$\therefore I = \frac{1}{2}\int \frac{u-3}{u^2+2u-4} du$

As, $\frac{d}{du}(u^2 + 2u - 4) = 2u + 2$

$\therefore$ Let, $u - 3 = A(2u + 2) + B$

$\Rightarrow u - 3 = 2Au + 2A + B$

On comparing both sides –

We have,

$2A = 1 \Rightarrow A = 1/2$

$2A + B = -3 \Rightarrow B = -3-2A = -4$

Hence,

$I = \int \frac{\frac{1}{2}(2u+2)-4}{u^2+2u-4} du$

$\therefore I = \frac{1}{2}\int \frac{2u+2}{u^2+2u-4} du - 4\int \frac{1}{u^2+2u-4} du$

Let, $I_1 = \frac{1}{2}\int \frac{2u+2}{u^2+2u-4} du$ and $I_2 = \int \frac{1}{u^2+2u-4} du$

Now, $I = I_1 - 4I_2$ ....eqn 1

We will solve I_1 and I_2 individually.

As, $I_1 = \frac{1}{2}\int \frac{2u+2}{u^2+2u-4} du$

Let $v = u^2 + 2u - 4 \Rightarrow dv = (2u + 2)du$

$\therefore I_1$ reduces to $\frac{1}{2}\int \frac{dv}{v}$

Hence,

$I_1 = \frac{1}{2}\int \frac{dv}{v} = \log|u| + C \; \{\because \int \frac{dx}{x} = \log|x| + C\}$

On substituting value of u, we have:

$I_1 = \frac{1}{2}\log|u^2 + 2u - 4| + C$ ....eqn 2

As, $I_2 = \int \frac{1}{u^2+2u-4} du$ and we don't have any derivative of function present in denominator.

$\therefore$ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

i) $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$ ii) $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$\therefore I_2 = \int \frac{1}{u^2+2u-4} du$

$\Rightarrow I_2 = \int \frac{1}{\{u^2+2(1)u+(1)^2\} - 4 - (1)^2} du$

Using: $a^2 + 2ab + b^2 = (a + b)^2$

We have:

$I_2 = \int \frac{1}{(u+1)^2 - (\sqrt{5})^2} du$

I_2 matches with $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

$\therefore I_2 = \frac{1}{2\sqrt{5}}\log\left|\frac{u+1-\sqrt{5}}{u+1+\sqrt{5}}\right| + C$ ...eqn 3

From eqn 1:

$I = I_1 - 4I_2$

Using eqn 2 and eqn 3:

$I = \frac{1}{2}\log|u^2 + 2u - 4| - 4\left(\frac{1}{2\sqrt{5}}\log\left|\frac{u+1-\sqrt{5}}{u+1+\sqrt{5}}\right|\right) + C$

$I = \frac{1}{2}\log|u^2 + 2u - 4| - \frac{2}{\sqrt{5}}\log\left|\frac{u+1-\sqrt{5}}{u+1+\sqrt{5}}\right| + C$

Putting value of u in I:

$I = \frac{1}{2}\log|x^4 + 2x^2 - 4| - \frac{2}{\sqrt{5}}\log\left|\frac{x^2+1-\sqrt{5}}{x^2+1+\sqrt{5}}\right| + C$

Exercise 19.20

1. Question

Evaluate the following integrals:

$$\int \frac{x^2 + x + 1}{x^2 - x}\, dx$$

Answer

Given $I = \int \frac{x^2+x+1}{x^2-x} dx$

Expressing the integral $\int \frac{P(x)}{ax^2+bx+c} dx = \int Q(x)\, dx + \int \frac{R(x)}{ax^2+bx+c} dx$

$$\Rightarrow \int \frac{x^2 + x + 1}{(x - 1)x} dx$$

$$\Rightarrow \int \left(\frac{2x + 1}{(x - 1)x} + 1\right) dx$$

$$\Rightarrow \int \frac{2x + 1}{(x - 1)x} dx + \int 1\ dx$$

Consider $\int \frac{2x+1}{(x-1)x} dx$

By partial fraction decomposition,

$$\Rightarrow \frac{2x + 1}{(x - 1)x} = \frac{A}{x - 1} + \frac{B}{x}$$

⇒ 2x + 1 = Ax + B(x – 1)

⇒ 2x + 1 = Ax + Bx – B

⇒ 2x + 1 = (A + B)x – B

∴ B = -1 and A + B = 2

∴ A = 2 + 1 = 3

Thus, $\Rightarrow \frac{2x+1}{(x-1)x} = \frac{3}{x-1} - \frac{1}{x}$

$$\Rightarrow \int \left(\frac{3}{x - 1} - \frac{1}{x}\right) dx$$

$$\Rightarrow 3\int \frac{1}{x - 1} dx - \int \frac{1}{x}\, dx$$

Consider $\int \frac{1}{x-1} dx$

Substitute u = x - 1 → dx = du.

$$\Rightarrow \int \frac{1}{x-1} dx = \int \frac{1}{u} du$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\therefore \int \frac{1}{u} du = \log|u| = \log|x-1|$$

Then,

$$\Rightarrow 3\int \frac{1}{x-1} dx - \int \frac{1}{x} dx = 3(\log|x-1|) - \int \frac{1}{x} dx$$

$$= 3(\log|x-1|) - \log|x|$$

$$\therefore \int \frac{2x+1}{(x-1)x} dx = 3(\log|x-1|) - \log|x|$$

Then,

$$\Rightarrow \int \frac{2x+1}{(x-1)x} dx + \int 1\ dx = 3(\log|x-1|) - \log|x| + \int 1\ dx$$

We know that $\int 1\ dx = x + c$

$$\Rightarrow \int \frac{2x+1}{(x-1)x} dx + \int 1\ dx = 3(\log|x-1|) - \log|x| + x + c$$

$$\therefore I = \int \frac{x^2+x+1}{x^2-x} dx = -\log|x| + x + 3(\log|x-1|) + c$$

2. Question

Evaluate the following integrals:

$$\int \frac{x^2+x-1}{x^2+x-6} dx$$

Answer

Consider $I = \int \frac{x^2+x-1}{x^2+x-6} dx$

Expressing the integral $\int \frac{P(x)}{ax^2+bx+c} dx = \int Q(x)\,dx + \int \frac{R(x)}{ax^2+bx+c} dx$

Let $x^2 + x - 1 = x^2 + x - 6 + 5$

$$\Rightarrow \int \frac{x^2+x-1}{x^2+x-6} dx = \int \left(\frac{x^2+x-6}{x^2+x-6} + \frac{5}{x^2+x-6}\right) dx$$

$$= \int \left(\frac{5}{x^2+x-6} + 1\right) dx$$

$$= 5\int \left(\frac{1}{x^2+x-6}\right) dx + \int 1\, dx$$

Consider $\int \frac{1}{x^2+x-6} dx$

Factorizing the denominator,

$\Rightarrow \int \frac{1}{x^2+x-6} dx = \int \frac{1}{(x-2)(x+3)} dx$

By partial fraction decomposition,

$\Rightarrow \frac{1}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3}$

⇒ 1 = A(x + 3) + B(x – 2)

⇒ 1 = Ax + 3A + Bx – 2B

⇒ 1 = (A + B) x + (3A – 2B)

⇒ Then A + B = 0 … (1)

And 3A – 2B = 1 … (2)

Solving (1) and (2),

2 × (1) → 2A + 2B = 0

1 × (2) → 3A – 2B = 1

5A = 1

∴ A = 1/5

Substituting A value in (1),

⇒ A + B = 0

⇒ 1/5 + B = 0

∴ B = -1/5

Thus, $\frac{1}{(x-2)(x+3)} = \frac{1}{5(x-2)} - \frac{1}{5(x+3)}$

$= \frac{1}{5}\int \frac{1}{x-2} dx - \frac{1}{5}\int \frac{1}{x+3} dx$

Let x – 2 = u → dx = du

And x + 3 = v → dx = dv.

$\Rightarrow \frac{1}{5}\int \frac{1}{u} du - \frac{1}{5}\int \frac{1}{v} dv$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$\Rightarrow \frac{1}{5}\log|u| - \frac{1}{5}\log|v|$

$\Rightarrow \frac{1}{5}\log|x-2| - \frac{1}{5}\log|x+3|$

$\Rightarrow \frac{1}{5}(\log|x-2| - \log|x+3|)$

Then,

$\Rightarrow 5\int \left(\frac{1}{x^2+x-6}\right) dx + \int 1\, dx = 5\left(\frac{1}{5}(\log|x-2| - \log|x+3|)\right) + \int 1\, dx$

We know that $\int 1 \;\; dx = x + c$

$\Rightarrow (\log|x-2| - \log|x+3|) + x + c$

$\therefore I = \int \frac{x^2+x-1}{x^2+x-6}dx = -\log|x+3| + x + \log|x-2| + c$

3. Question

Evaluate the following integrals:

$$\int \frac{\left(1-x^2\right)}{x\left(1-2x\right)}dx$$

Answer

Given $I = \int \frac{1-x^2}{(1-2x)x}dx$

Rewriting, we get $\int \frac{x^2-1}{x(2x-1)}dx$

Expressing the integral $\int \frac{P(x)}{ax^2+bx+c}dx = \int Q(x)\,dx + \int \frac{R(x)}{ax^2+bx+c}dx$

$$\Rightarrow \int \frac{x^2-1}{x(2x-1)}dx = \int \left(\frac{x-2}{2x(2x-1)} + \frac{1}{2}\right)dx$$

$$= \frac{1}{2}\int \frac{x-2}{x(2x-1)}dx + \frac{1}{2}\int 1dx$$

Consider $\int \frac{x-2}{x(2x-1)}dx$

By partial fraction decomposition,

$$\Rightarrow \frac{x-2}{x(2x-1)} = \frac{A}{x} + \frac{B}{2x-1}$$

$\Rightarrow x - 2 = A(2x - 1) + Bx$

$\Rightarrow x - 2 = 2Ax - A + Bx$

$\Rightarrow x - 2 = (2A + B)x - A$

$\therefore A = 2$ and $2A + B = 1$

$\therefore B = 1 - 4 = -3$

Thus, $\Rightarrow \frac{x-2}{x(2x-1)} = \frac{2}{x} - \frac{3}{2x-1}$

$$\Rightarrow \int (\frac{2}{x} - \frac{3}{2x-1})dx$$

$$\Rightarrow 2\int \frac{1}{x}dx - 3\int \frac{1}{2x-1}dx$$

Consider $\int \frac{1}{x}dx$

We know that $\int \frac{1}{x}dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{x}dx = \log|x|$$

And consider $\int \frac{1}{2x-1}dx$

Let $u = 2x - 1 \rightarrow dx = 1/2\, du$

$$\Rightarrow \int \frac{1}{2x-1}dx = \frac{1}{2}\int \frac{1}{u}du$$

We know that $\int \frac{1}{x}dx = \log|x| + c$

$$\Rightarrow \frac{1}{2}\int \frac{1}{u}du = \frac{\log|u|}{2} = \frac{\log|2x-1|}{2}$$

Then,

$$\Rightarrow \int \frac{x-2}{x(2x-1)}dx = 2\int \frac{1}{x}dx - 3\int \frac{1}{2x-1}\, dx$$

$$= 2(\log|x|) - 3\left(\frac{\log|2x-1|}{2}\right)$$

Then,

$$\Rightarrow \int \frac{x^2-1}{x(2x-1)}dx = \frac{1}{2}\int \frac{x-2}{x(2x-1)}dx + \frac{1}{2}\int 1dx$$

$$= \frac{1}{2}\left(2(\log|x|) - 3\left(\frac{\log|2x-1|}{2}\right)\right) + \frac{1}{2}\int 1\, dx$$

We know that $\int 1\ dx = x + c$

$$\Rightarrow \log|x| - \frac{3\log|2x-1|}{4} + \frac{x}{2} + c$$

$$\therefore I = \int \frac{1-x^2}{(1-2x)x}dx = -\frac{3\log|2x-1|}{4} + \log|x| + \frac{x}{2} + c$$

4. Question

Evaluate the following integrals:

$$\int \frac{x^2+1}{x^2-5x+6}dx$$

Answer

Consider $I = \int \frac{x^2+1}{x^2-5x+6}dx$

Expressing the integral $\int \frac{P(x)}{ax^2+bx+c}dx = \int Q(x)\, dx + \int \frac{R(x)}{ax^2+bx+c}dx$

$$\Rightarrow \int \frac{x^2+1}{x^2-5x+6}dx = \int \left(\frac{5x-5}{x^2-5x+6} + 1\right) dx$$

$$= 5\int \frac{x-1}{x^2-5x+6}dx + \int 1\, dx$$

Consider $\int \frac{x-1}{x^2-5x+6}dx$

Let $x - 1 = \frac{1}{2}(2x-5) + \frac{3}{2}$ and split,

$$\Rightarrow \int \left(\frac{2x-5}{2(x^2-5x+6)} + \frac{3}{2(x^2-5x+6)}\right) dx$$

$$\Rightarrow \frac{1}{2}\int \frac{2x-5}{(x^2-5x+6)}dx + \frac{3}{2}\int \frac{1}{x^2-5x+6}dx$$

Consider $\int \frac{2x-5}{(x^2-5x+6)}dx$

Let $u = x^2 - 5x + 6 \rightarrow dx = \frac{1}{2x-5}du$

$$\Rightarrow \int \frac{2x-5}{(x^2-5x+6)}dx = \int \frac{2x-5}{u}\frac{1}{2x-5}du$$

$$= \int \frac{1}{u}du$$

We know that $\int \frac{1}{x}dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{u}du = \log|u| = \log|x^2 - 5x + 6|$$

Now consider $\int \frac{1}{x^2-5x+6}dx$

$$\Rightarrow \int \frac{1}{x^2-5x+6}dx = \int \frac{1}{(x-3)(x-2)}dx$$

By partial fraction decomposition,

$$\Rightarrow \frac{1}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}$$

⇒ 1 = A (x – 2) + B (x – 3)

⇒ 1 = Ax – 2A + Bx – 3B

⇒ 1 = (A + B) x – (2A + 3B)

⇒ A + B = 0 and 2A + 3B = -1

Solving the two equations,

⇒ 2A + 2B = 0

2A + 3B = -1

-B = 1

∴ B = -1 and A = 1

$$\Rightarrow \int \frac{1}{(x-3)(x-2)}dx = \int \left(\frac{1}{x-3} - \frac{1}{x-2}\right)dx$$

$$= \int \frac{1}{x-3}dx - \int \frac{1}{x-2}dx$$

Consider $\int \frac{1}{x-3}dx$

Let u = x – 3 → dx = du

$$\Rightarrow \int \frac{1}{x-3}dx = \int \frac{1}{u}du$$

We know that $\int \frac{1}{x}dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{u}du = \log|u| = \log|x-3|$$

Similarly $\int \frac{1}{x-2}dx$

Let u = x – 2 → dx = du

$$\Rightarrow \int \frac{1}{x-2}dx = \int \frac{1}{u}du$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{u} du = \log|u| = \log|x-2|$$

Then,

$$\Rightarrow \int \frac{1}{x^2-5x+6} dx = \int \frac{1}{(x-3)(x-2)} dx = \int \frac{1}{x-3} dx - \int \frac{1}{x-2} dx$$

$$= \log|x-3| - \log|x-2|$$

Then,

$$\Rightarrow \int \frac{x-1}{x^2-5x+6} dx = \frac{1}{2}\int \frac{2x-5}{(x^2-5x+6)} dx + \frac{3}{2}\int \frac{1}{x^2-5x+6} dx$$

$$= \frac{1}{2}(\log|x^2-5x+6|) + \frac{3}{2}(\log|x-3| - \log|x-2|)$$

$$= \frac{\log|x^2-5x+6|}{2} + \frac{3\log|x-3|}{2} - \frac{3\log|x-2|}{2}$$

Then,

$$\Rightarrow \int \frac{x^2+1}{x^2-5x+6} dx = 5\int \frac{x-1}{x^2-5x+6} dx + \int 1\, dx$$

We know that $\int 1\ dx = x + c$

$$\Rightarrow 5\int \frac{x-1}{x^2-5x+6} dx + \int 1\, dx$$
$$= \frac{5\log|x^2-5x+6|}{2} + \frac{15\log|x-3|}{2} - \frac{15\log|x-2|}{2} + x + c$$

$$= \frac{5\log|x-2|\log|x-3|}{2} + \frac{15\log|x-3|}{2} - \frac{15\log|x-2|}{2} + x + c$$

$$= x - 5\log|x-2| + 10\log|x-3| + c$$

$$\therefore I = \int \frac{x^2+1}{x^2-5x+6} dx = x - 5\log|x-2| + 10\log|x-3| + c$$

5. Question

Evaluate the following integrals:

$$\int \frac{x^2}{x^2+7x+10} dx$$

Answer

Given $I = \int \frac{x^2}{x^2+7x+10} dx$

Expressing the integral $\int \frac{P(x)}{ax^2+bx+c} dx = \int Q(x)\, dx + \int \frac{R(x)}{ax^2+bx+c} dx$

$$\Rightarrow \int \frac{x^2}{x^2+7x+10} dx = \int \left(\frac{-7x-10}{x^2+7x+10} + 1\right) dx$$

$$= -\int \frac{7x+10}{x^2+7x+10} dx + \int 1\, dx$$

Consider $\int \frac{7x+10}{x^2+7x+10} dx$

Let $7x + 10 = \frac{7}{2}(2x + 7) - \frac{29}{2}$ and split,

$$\Rightarrow \int \frac{7x + 10}{x^2 + 7x + 10} dx = \int \left(\frac{7(2x + 7)}{2(x^2 + 7x + 10)} - \frac{29}{2(x^2 + 7x + 10)}\right) dx$$

$$= \frac{7}{2} \int \frac{2x + 7}{x^2 + 7x + 10} dx - \frac{29}{2} \int \frac{1}{x^2 + 7x + 10} dx$$

Consider $\int \frac{2x+7}{x^2+7x+10} dx$

Let $u = x^2 + 7x + 10 \rightarrow dx = \frac{1}{2x+7} du$

$$\Rightarrow \int \frac{2x + 7}{(x^2 + 7x + 10)} dx = \int \frac{2x + 7}{u} \frac{1}{2x + 7} du$$

$$= \int \frac{1}{u} du$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{u} du = \log|u| = \log|x^2 + 7x + 10|$$

Now consider $\int \frac{1}{x^2+7x+10} dx$

$$\Rightarrow \int \frac{1}{x^2 + 7x + 10} dx = \int \frac{1}{(x + 2)(x + 5)} dx$$

By partial fraction decomposition,

$$\Rightarrow \frac{1}{(x + 2)(x + 5)} = \frac{A}{x + 2} + \frac{B}{x + 5}$$

$\Rightarrow 1 = A(x + 2) + B(x + 5)$

$\Rightarrow 1 = Ax + 2A + Bx + 5B$

$\Rightarrow 1 = (A + B)x + (2A + 5B)$

$\Rightarrow A + B = 0$ and $2A + 5B = 1$

Solving the two equations,

$\Rightarrow 2A + 2B = 0$

$2A + 5B = 1$

$-3B = -1$

$\therefore B = 1/3$ and $A = -1/3$

$$\Rightarrow \int \frac{1}{(x + 2)(x + 5)} dx = \int \left(\frac{-1}{3(x + 2)} + \frac{1}{3(x + 5)}\right) dx$$

$$= -\frac{1}{3} \int \frac{1}{x + 2} dx + \frac{1}{3} \int \frac{1}{x + 5} dx$$

Consider $\int \frac{1}{x+2} dx$

Let $u = x + 2 \rightarrow dx = du$

$$\Rightarrow \int \frac{1}{x + 2} dx = \int \frac{1}{u} du$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{u} du = \log|u| = \log|x+2|$$

Similarly $\int \frac{1}{x+5} dx$

Let u = x + 5 → dx = du

$$\Rightarrow \int \frac{1}{x+5} dx = \int \frac{1}{u} du$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{u} du = \log|u| = \log|x+5|$$

Then,

$$\Rightarrow \int \frac{1}{x^2+7x+10} dx = \int \frac{1}{(x+2)(x+5)} dx = -\frac{1}{3}\int \frac{1}{x+2} dx + \frac{1}{3}\int \frac{1}{x+5} dx$$

$$= \frac{-\log|x+2|}{3} + \frac{\log|x+5|}{3}$$

Then,

$$\Rightarrow \int \frac{7x+10}{x^2+7x+10} dx = \frac{7}{2}\int \frac{2x+7}{x^2+7x+10} dx - \frac{29}{2}\int \frac{1}{x^2+7x+10} dx$$

$$= \frac{7}{2}(\log|x^2+7x+10|) - \frac{29}{2}\left(\frac{-\log|x+2|}{3} + \frac{\log|x+5|}{3}\right)$$

$$= \frac{7\log|x^2+7x+10|}{2} + \frac{29\log|x+2|}{6} - \frac{29\log|x+5|}{6}$$

Then,

$$\Rightarrow \int \frac{x^2}{x^2+7x+10} dx = -\int \frac{7x+10}{x^2+7x+10} dx + \int 1\, dx$$

We know that $\int 1 \ dx = x + c$

$$\Rightarrow -\int \frac{7x+10}{x^2+7x+10} dx + \int 1\, dx$$
$$= \frac{-7\log|x^2+7x+10|}{2} - \frac{29\log|x+2|}{6} + \frac{29\log|x+5|}{6} + x + c$$

$$= \frac{-7\log|x+2|\log|x+5|}{2} - \frac{29\log|x+2|}{6} + \frac{29\log|x+5|}{6} + x + c$$

$$= -\frac{25\log|x+2|}{3} + \frac{4\log|x+5|}{3} + x + c$$

$$\therefore I = \int \frac{x^2}{x^2+7x+10} dx = -\frac{25\log|x+2|}{3} + \frac{4\log|x+5|}{3} + x + c$$

6. Question

Evaluate the following integrals:

$$\int \frac{x^2+x+1}{x^2-x+1} dx$$

Answer

Given $I = \int \frac{x^2+x+1}{x^2-x+1} dx$

Expressing the integral $\int \frac{P(x)}{ax^2+bx+c} dx = \int Q(x)\, dx + \int \frac{R(x)}{ax^2+bx+c} dx$

$$\Rightarrow \int \frac{x^2+x+1}{x^2-x+1} dx = \int \left(\frac{2x}{x^2-x+1} + 1\right) dx$$

$$= 2\int \left(\frac{x}{x^2-x+1}\right) dx + \int 1\, dx$$

Consider $\int \frac{x}{x^2-x+1} dx$

Let x = 1/2 (2x – 1) + 1/2 and split,

$$\Rightarrow \int \left(\frac{2x-1}{2(x^2-x+1)} + \frac{1}{2(x^2-x+1)}\right) dx$$

$$\Rightarrow \frac{1}{2}\int \frac{2x-1}{(x^2-x+1)} dx + \frac{1}{2}\int \frac{1}{(x^2-x+1)} dx$$

Consider $\int \frac{2x-1}{(x^2-x+1)} dx$

Let u = x^2 – x + 1 → dx = du/2x – 1

$$\Rightarrow \int \frac{2x-1}{(x^2-x+1)} dx = \int \frac{2x-1}{u} \frac{du}{2x-1}$$

$$= \int \frac{1}{u} du$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{u} du = \log|u| = \log|x^2 - x + 1|$$

Now consider $\int \frac{1}{(x^2-x+1)} dx$

$$\Rightarrow \int \frac{1}{(x^2-x+1)} dx = \int \frac{1}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}} dx$$

Let $u = \frac{2x-1}{\sqrt{3}} \to dx = \frac{\sqrt{3}}{2} du$

$$\Rightarrow \int \frac{1}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}} dx = \int \frac{2\sqrt{3}}{3u^2+3} du$$

$$= \frac{2}{\sqrt{3}} \int \frac{1}{u^2+1} du$$

We know that $\int \frac{1}{x^2+1} dx = \tan^{-1} x + c$

$$\Rightarrow \frac{2}{\sqrt{3}} \int \frac{1}{u^2+1} du = \frac{2\tan^{-1} u}{\sqrt{3}} = \frac{2\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)}{\sqrt{3}}$$

Then,

$$\Rightarrow \int \frac{x}{x^2-x+1}dx = \frac{1}{2}\int \frac{2x-1}{(x^2-x+1)}dx + \frac{1}{2}\int \frac{1}{(x^2-x+1)}dx$$

$$= \frac{1}{2}(\log|x^2-x+1|) + \frac{1}{2}\left(\frac{2\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)}{\sqrt{3}}\right)$$

$$= \frac{\log|x^2-x+1|}{2} + \frac{\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)}{\sqrt{3}}$$

Now $2\int\left(\frac{x}{x^2-x+1}\right)dx + \int 1\,dx$

We know that $\int 1\ dx = x + c$

$$\Rightarrow 2\int\left(\frac{x}{x^2-x+1}\right)dx + \int 1\,dx = 2\left(\frac{\log|x^2-x+1|}{2} + \frac{\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)}{\sqrt{3}}\right) + x + c$$

$$= (\log|x^2-x+1|) + \left(\frac{2\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)}{\sqrt{3}}\right) + x + c$$

$$\therefore I = \int \frac{x^2+x+1}{x^2-x+1}dx = (\log|x^2-x+1|) + \left(\frac{2\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)}{\sqrt{3}}\right) + x + c$$

7. Question

Evaluate the following integrals:

$$\int \frac{(x-1)^2}{x^2+2x+2}dx$$

Answer

Given $I = \int \frac{(x-1)^2}{x^2+2x+2}dx$

Expressing the integral $\int \frac{P(x)}{ax^2+bx+c}dx = \int Q(x)\,dx + \int \frac{R(x)}{ax^2+bx+c}dx$

$$\Rightarrow \int \frac{(x-1)^2}{x^2+2x+2}dx = \int\left(\frac{-4x-1}{x^2+2x+2} + 1\right)dx$$

$$= -\int \frac{4x+1}{x^2+2x+2}dx + \int 1\,dx$$

Consider $\int \frac{4x+1}{x^2+2x+2}dx$

Let 4x + 1 = 2 (2x + 2) – 3 and split,

$$\Rightarrow \int \frac{4x+1}{x^2+2x+2}dx = \int\left(\frac{2(2x+2)}{x^2+2x+2} - \frac{3}{x^2+2x+2}\right)dx$$

$$= 4\int \frac{x+1}{x^2+2x+2}dx - 3\int \frac{1}{x^2+2x+2}dx$$

Consider $\int \frac{x+1}{x^2+2x+2}dx$

Let $u = x^2 + 2x + 2 \rightarrow dx = \frac{1}{2x+2}du$

$$\Rightarrow \int \frac{x+1}{(x^2+2x+2)}dx = \int \frac{x+1}{u}\frac{1}{2x+2}du$$

$$= \int \frac{1}{2u}du$$

We know that $\int \frac{1}{x}dx = \log|x| + c$

$$\Rightarrow \frac{1}{2}\int \frac{1}{u}du = \frac{\log|u|}{2} = \frac{\log|x^2+2x+2|}{2}$$

Now consider $\int \frac{1}{x^2+2x+2}dx$

$$\Rightarrow \int \frac{1}{x^2+2x+2}dx = \int \frac{1}{(x+1)^2+1}dx$$

Let u = x + 1 → dx = du

$$\Rightarrow \int \frac{1}{(x+1)^2+1}dx = \int \frac{1}{u^2+1}du$$

We know that $\int \frac{1}{x^2+1}dx = \tan^{-1}x + c$

$$\Rightarrow \int \frac{1}{u^2+1}du = \tan^{-1}u = \tan^{-1}(x+1)$$

Then,

$$\Rightarrow \int \frac{4x+1}{x^2+2x+2}dx = 4\int \frac{x+1}{x^2+2x+2}dx - 3\int \frac{1}{x^2+2x+2}dx$$

$$= 4\left(\frac{\log|x^2+2x+2|}{2}\right) - 3(\tan^{-1}(x+1))$$

$$= 2\log|x^2+2x+2| - 3\tan^{-1}(x+1)$$

Then,

$$\Rightarrow \int \frac{(x-1)^2}{x^2+2x+2}dx = -\int \frac{4x+1}{x^2+2x+2}dx + \int 1\,dx$$

We know that $\int 1\ dx = x + c$

$$\Rightarrow -\int \frac{4x+1}{x^2+2x+2}dx + \int 1\,dx = -2\log|x^2+2x+2| + 3\tan^{-1}(x+1) + x + c$$

$$\therefore I = \int \frac{(x-1)^2}{x^2+2x+2}dx = -2\log|x^2+2x+2| + 3\tan^{-1}(x+1) + x + c$$

8. Question

Evaluate the following integrals:

$$\int \frac{x^3+x^2+2x+1}{x^2-x+1}dx$$

Answer

Given $I = \int \frac{x^3+x^2+2x+1}{x^2-x+1}dx$

Expressing the integral $\int \frac{P(x)}{ax^2+bx+c}dx = \int Q(x)\,dx + \int \frac{R(x)}{ax^2+bx+c}dx$

$$\Rightarrow \int \frac{x^3+x^2+2x+1}{x^2-x+1}dx = \int \frac{3x-1}{x^2-x+1} + x + 2\,dx$$

$$= \int \frac{3x-1}{x^2-x+1}dx + \int x\,dx + 2\int 1\,dx$$

Consider $\int \frac{3x-1}{x^2-x+1}dx$

Let $3x-1 = \frac{3}{2}(2x-1) + \frac{1}{2}$ and split,

$$\Rightarrow \int \frac{3x-1}{x^2-x+1}dx = \int \left(\frac{3(2x-1)}{2(x^2-x+1)} + \frac{1}{2(x^2-x+1)}\right)dx$$

$$= \frac{3}{2}\int \frac{(2x-1)}{(x^2-x+1)}dx + \frac{1}{2}\int \frac{1}{(x^2-x+1)}dx$$

Consider $\int \frac{(2x-1)}{(x^2-x+1)}dx$

Let $u = x^2-x+1 \rightarrow dx = \frac{1}{2x-1}du$

$$\Rightarrow \int \frac{(2x-1)}{(x^2-x+1)}dx = \int \frac{(2x-1)}{u}\frac{1}{2x-1}du$$

$$= \int \frac{1}{u}du$$

We know that $\int \frac{1}{x}dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{u}du = \log|u| = \log|x^2-x+1|$$

Consider $\int \frac{1}{(x^2-x+1)}dx$

$$\Rightarrow \int \frac{1}{(x^2-x+1)}dx = \int \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}dx$$

Let $u = \frac{2x-1}{\sqrt{3}} \rightarrow dx = \frac{\sqrt{3}}{2}du$

$$\Rightarrow \int \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}dx = \int \frac{2\sqrt{3}}{3u^2+3}du$$

$$= \frac{2}{\sqrt{3}}\int \frac{1}{u^2+1}du$$

We know that $\int \frac{1}{x^2+1}dx = \tan^{-1}x + c$

$$\Rightarrow \frac{2}{\sqrt{3}}\int \frac{1}{u^2+1}du = \frac{2\tan^{-1}u}{\sqrt{3}} = \frac{2\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)}{\sqrt{3}}$$

Then,

$$\Rightarrow \int \frac{3x-1}{x^2-x+1}dx = \frac{3}{2}\int \frac{2x-1}{(x^2-x+1)}dx + \frac{1}{2}\int \frac{1}{(x^2-x+1)}dx$$

$$= \frac{3}{2}(\log|x^2-x+1|) + \frac{1}{2}\left(\frac{2\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)}{\sqrt{3}}\right)$$

$$= \frac{3\log|x^2 - x + 1|}{2} + \frac{\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)}{\sqrt{3}}$$

Then,

$$\Rightarrow \int \frac{x^3 + x^2 + 2x + 1}{x^2 - x + 1} dx = \int \frac{3x - 1}{x^2 - x + 1} dx + \int x\, dx + 2\int 1\, dx$$

We know that $\int x^n\, dx = \frac{x^{n+1}}{n+1} + c$ and $\int 1\ dx = x + c$

$$\Rightarrow \int \frac{3x - 1}{x^2 - x + 1} dx + \int x\, dx + 2\int 1\, dx$$

$$= \frac{3\log|x^2 - x + 1|}{2} + \frac{\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)}{\sqrt{3}} + \frac{x^2}{2} + 2x + c$$

$$= \frac{3\log|x^2 - x + 1| + x^2 + 4x}{2} + \frac{\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)}{\sqrt{3}} + c$$

$$\therefore I = \int \frac{x^3 + x^2 + 2x + 1}{x^2 - x + 1} dx = \frac{3\log|x^2 - x + 1| + x^2 + 4x}{2} + \frac{\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)}{\sqrt{3}} + c$$

9. Question

Evaluate the following integrals:

$$\int \frac{x^2\left(x^4 + 4\right)}{x^2 + 4} dx$$

Answer

Given $I = \int \frac{x^2(x^4+4)}{x^2+4} dx$

Expressing the integral $\int \frac{P(x)}{ax^2+bx+c} dx = \int Q(x)\, dx + \int \frac{R(x)}{ax^2+bx+c} dx$

$$\Rightarrow \int \frac{x^2(x^4 + 4)}{x^2 + 4} dx = \int \left(-\frac{80}{x^2 + 4} + x^4 - 4x^2 + 20\right) dx$$

$$= -80\int \frac{1}{x^2 + 4} dx + \int x^4\, dx - 4\int x^2\, dx + 20\int 1\, dx$$

Consider $\int \frac{1}{x^2+4} dx$

Let u = 1/2 x → dx = 2du

$$\Rightarrow \int \frac{1}{x^2 + 4} dx = \int \frac{2}{4u^2 + 4} du$$

$$= \frac{1}{2}\int \frac{1}{u^2 + 1} du$$

We know that $\int \frac{1}{x^2+1} dx = \tan^{-1} x + c$

$$\Rightarrow \frac{1}{2}\int \frac{1}{u^2 + 1} du = \frac{\tan^{-1} u}{2} = \frac{\tan^{-1}\left(\frac{x}{2}\right)}{2}$$

Then,

$$\Rightarrow \int \frac{x^2(x^4 + 4)}{x^2 + 4} dx = -80\int \frac{1}{x^2 + 4} dx + \int x^4\, dx - 4\int x^2\, dx + 20\int 1\, dx$$

We know that $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$ and $\int 1\ dx = x + c$

$$\Rightarrow -80\left(\frac{\tan^{-1}\left(\frac{x}{2}\right)}{2}\right) + \frac{x^5}{5} - \frac{4x^3}{3} + 20x + c$$

$$\Rightarrow -40\tan^{-1}\left(\frac{x}{2}\right) + \frac{x^5}{5} - \frac{4x^3}{3} + 20x + c$$

$$\therefore I = \int \frac{x^2(x^4+4)}{x^2+4}dx = -40\tan^{-1}\left(\frac{x}{2}\right) + \frac{x^5}{5} - \frac{4x^3}{3} + 20x + c$$

10. Question

Evaluate the following integrals:

$$\int \frac{x^2}{x^2+6x+12}dx$$

Answer

Given $I = \int \frac{x^2}{x^2+6x+12}dx$

Expressing the integral $\int \frac{P(x)}{ax^2+bx+c}dx = \int Q(x)\,dx + \int \frac{R(x)}{ax^2+bx+c}dx$

$$\Rightarrow \int \frac{x^2}{x^2+6x+12}dx = \int \left(\frac{-6x-12}{x^2+6x+12} + 1\right)dx$$

$$= -6\int \frac{x+2}{x^2+6x+12}dx + \int 1\,dx$$

Consider $\int \frac{x+2}{x^2+6x+12}dx$

Let x + 2 = 1/2(2x + 6) – 1 and split,

$$\Rightarrow \int \frac{x+2}{x^2+6x+12}dx = \int \left(\frac{(2x+6)}{2(x^2+6x+12)} - \frac{1}{(x^2+6x+12)}\right)dx$$

$$= \int \frac{x+3}{x^2+6x+12}dx - \int \frac{1}{x^2+6x+12}dx$$

Consider $\int \frac{x+3}{x^2+6x+12}dx$

Let $u = x^2 + 6x + 12 \rightarrow dx = \frac{1}{2x+6}du$

$$\Rightarrow \int \frac{x+3}{(x^2+6x+12)}dx = \int \frac{x+3}{u}\frac{1}{2x+6}du$$

$$= \int \frac{1}{2u}du$$

We know that $\int \frac{1}{x}dx = \log|x| + c$

$$\Rightarrow \frac{1}{2}\int \frac{1}{u}du = \frac{\log|u|}{2} = \frac{\log|x^2+6x+12|}{2}$$

Now consider $\int \frac{1}{x^2+6x+12}dx$

$$\Rightarrow \int \frac{1}{x^2+6x+12}dx = \int \frac{1}{(x+3)^2+3}dx$$

Let $u = \frac{x+3}{\sqrt{3}} \rightarrow dx = \sqrt{3}du$

$$\Rightarrow \int \frac{1}{(x+3)^2+3}dx = \frac{\sqrt{3}}{3u^2+3}$$

$$= \frac{1}{\sqrt{3}}\int \frac{1}{u^2+1}du$$

We know that $\int \frac{1}{x^2+1}dx = \tan^{-1}x + c$

$$\Rightarrow \frac{1}{\sqrt{3}}\int \frac{1}{u^2+1}du = \frac{\tan^{-1}u}{\sqrt{3}} = \frac{\tan^{-1}(\frac{x+3}{\sqrt{3}})}{\sqrt{3}}$$

Then,

$$\Rightarrow \int \frac{x+2}{x^2+6x+12}dx = \int \frac{x+3}{x^2+6x+12}dx - \int \frac{1}{x^2+6x+12}dx$$

$$= \frac{\log|x^2+6x+12|}{2} - \frac{\tan^{-1}(\frac{x+3}{\sqrt{3}})}{\sqrt{3}}$$

Then,

$$\Rightarrow \int \frac{x^2}{x^2+6x+12}dx = -6\int \frac{x+2}{x^2+6x+12}dx + \int 1\,dx$$

We know that $\int 1\ dx = x + c$

$$\Rightarrow -6\int \frac{x+2}{x^2+6x+12}dx + \int 1\,dx$$

$$= -3\log|x^2+6x+12| + \frac{6\tan^{-1}(\frac{x+3}{\sqrt{3}})}{\sqrt{3}} + x + c$$

$$= -3\log|x^2+6x+12| + 2.\sqrt{3}\tan^{-1}(\frac{x+3}{\sqrt{3}}) + x + c$$

$$\therefore I = \int \frac{x^2}{x^2+6x+12}dx = -3\log|x^2+6x+12| + 2.\sqrt{3}\tan^{-1}(\frac{x+3}{\sqrt{3}}) + x + c$$

Exercise 19.21

1. Question

Evaluate the following integrals:

$$\int \frac{x}{\sqrt{x^2+6x+10}}dx$$

Answer

Given $I = \int \frac{x}{\sqrt{x^2+6x+10}}dx$

Integral is of form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Writing numerator as $px + q = \lambda\left\{\frac{d}{dx}(ax^2+bx+c)\right\} + \mu$

$\Rightarrow px + q = \lambda(2ax + b) + \mu$

$\Rightarrow x = \lambda(2x + 6) + \mu$

∴ λ = 1/2 and μ = -3

Let x = 1/2(2x + 6) – 3 and split,

$$\Rightarrow \int \frac{x}{\sqrt{x^2+6x+10}}dx = \int \left(\frac{2x+6}{2\sqrt{x^2+6x+10}} - \frac{3}{\sqrt{x^2+6x+10}}\right)dx$$

$$= \int \frac{x+3}{\sqrt{x^2+6x+10}}dx - 3\int \frac{1}{\sqrt{x^2+6x+10}}dx$$

Consider $\int \frac{x+3}{\sqrt{x^2+6x+10}}dx$

Let $u = x^2 + 6x + 10 \rightarrow dx = \frac{1}{2x+6}du$

$$\Rightarrow \int \frac{x+3}{\sqrt{x^2+6x+10}}dx = \int \frac{1}{2\sqrt{u}}du$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{u}}du$$

We know that $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow \frac{1}{2}\int \frac{1}{\sqrt{u}}du = \frac{1}{2}(2\sqrt{u})$$

$$= \sqrt{u} = \sqrt{x^2+6x+10}$$

Consider $\int \frac{1}{\sqrt{x^2+6x+10}}dx$

$$\Rightarrow \int \frac{1}{\sqrt{x^2+6x+10}}dx = \int \frac{1}{\sqrt{(x+3)^2+1}}dx$$

Let u = x + 3 → dx = du

$$\Rightarrow \int \frac{1}{\sqrt{(x+3)^2+1}}dx = \int \frac{1}{\sqrt{(u)^2+1}}du$$

We know that $\int \frac{1}{\sqrt{x^2+1}}dx = \sinh^{-1}x + c$

$$\Rightarrow \int \frac{1}{\sqrt{u^2+1}}du = \sinh^{-1}(u)$$

$$= \sinh^{-1}(x+3)$$

Then,

$$\Rightarrow \int \frac{x}{\sqrt{x^2+6x+10}}dx = \int \frac{x+3}{\sqrt{x^2+6x+10}}dx - 3\int \frac{1}{\sqrt{x^2+6x+10}}dx$$

$$= \sqrt{x^2+6x+10} - 3\sinh^{-1}(x+3) + c$$

$$\therefore I = \int \frac{x}{\sqrt{x^2+6x+10}}dx = \sqrt{x^2+6x+10} - 3\sinh^{-1}(x+3) + c$$

2. Question

Evaluate the following integrals:

$$\int \frac{2x+1}{\sqrt{x^2+2x-1}}dx$$

Answer

Given $I = \int \frac{2x+1}{\sqrt{x^2+2x-1}}dx$

Integral is of form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Writing numerator as $px + q = \lambda\left\{\frac{d}{dx}(ax^2 + bx + c)\right\} + \mu$

$\Rightarrow px + q = \lambda(2ax + b) + \mu$

$\Rightarrow 2x + 1 = \lambda (2x + 2) + \mu$

$\therefore \lambda = 1$ and $\mu = -1$

Let 2x + 1 = 2x + 2 – 1 and split,

$$\Rightarrow \int \frac{2x+1}{\sqrt{x^2+2x-1}}dx = \int \left(\frac{2x+2}{\sqrt{x^2+2x-1}} - \frac{1}{\sqrt{x^2+2x-1}}\right)dx$$

$$= 2\int \frac{x+1}{\sqrt{x^2+2x-1}}dx - \int \frac{1}{\sqrt{x^2+2x-1}}dx$$

Consider $\int \frac{x+1}{\sqrt{x^2+2x-1}}dx$

Let $u = x^2 + 2x - 1 \rightarrow dx = \frac{1}{2x+2}du$

$$\Rightarrow \int \frac{x+1}{\sqrt{x^2+2x-1}}dx = \int \frac{1}{2\sqrt{u}}du$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{u}}du$$

We know that $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow \frac{1}{2}\int \frac{1}{\sqrt{u}}du = \frac{1}{2}\left(2\sqrt{u}\right)$$

$$= \sqrt{u} = \sqrt{x^2 + 2x - 1}$$

Consider $\int \frac{1}{\sqrt{x^2+2x-1}}dx$

$$\Rightarrow \int \frac{1}{\sqrt{x^2+2x-1}}dx = \int \frac{1}{\sqrt{(x+1)^2-2}}dx$$

Let $u = \frac{x+1}{\sqrt{2}} \rightarrow dx = \sqrt{2}du$

$$\Rightarrow \int \frac{1}{\sqrt{(x+1)^2-2}}dx = \int \frac{\sqrt{2}}{\sqrt{2u^2-2}}du$$

$$= \int \frac{1}{\sqrt{u^2-1}}du$$

We know that $\int \frac{1}{\sqrt{x^2-1}}dx = \cosh^{-1}x + c$

$$\Rightarrow \int \frac{1}{\sqrt{u^2-1}}du = \cosh^{-1}(u)$$

$$= \cosh^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$$

Then,

$$\Rightarrow \int \frac{2x+1}{\sqrt{x^2+2x-1}}dx = 2\int \frac{x+1}{\sqrt{x^2+2x-1}}dx - \int \frac{1}{\sqrt{x^2+2x-1}}dx$$

$$= 2\sqrt{x^2+2x-1} - \cosh^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$$

$$\therefore I = \int \frac{2x+1}{\sqrt{x^2+2x-1}}dx = 2\sqrt{x^2+2x-1} - \cosh^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$$

3. Question

Evaluate the following integrals:

$$\int \frac{x+1}{\sqrt{x+5x-x^2}}dx$$

Answer

Given $I = \int \frac{x+1}{\sqrt{4+5x-x^2}}dx$

Integral is of form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Writing numerator as $px+q = \lambda\left\{\frac{d}{dx}(ax^2+bx+c)\right\}+\mu$

⇒ px + q = λ(2ax + b) + μ

⇒ x + 1 = λ (-2x + 5) + μ

∴ λ = -1/2 and μ = 7/2

Let x + 1 = – 1/2(–2x + 5) + 7/2

$$\Rightarrow \int \frac{x+1}{\sqrt{-x^2+5x+4}}dx = \int \left(\frac{-2x+5}{2\sqrt{-x^2+5x+4}} + \frac{7}{2\sqrt{-x^2+5x+4}}\right)dx$$

$$= \frac{1}{2}\int \frac{-2x+5}{\sqrt{-x^2+5x+4}}dx + \frac{7}{2}\int \frac{1}{\sqrt{-x^2+5x+4}}dx$$

Consider $\int \frac{-2x+5}{\sqrt{-x^2+5x+4}}dx$

Let $u = -x^2+5x+4 \rightarrow dx = \frac{1}{-2x+5}du$

$$\Rightarrow \int \frac{-2x+5}{\sqrt{-x^2+5x+4}}dx = -\int \frac{1}{\sqrt{u}}du$$

We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow -\int \frac{1}{\sqrt{u}}du = -(2\sqrt{u})$$

$$= -2\sqrt{x^2+6x+10}$$

Consider $\int \frac{1}{\sqrt{-x^2+5x+4}}dx$

$$\Rightarrow \int \frac{1}{\sqrt{-x^2+5x+4}}dx = \int \frac{1}{\sqrt{-\left(x-\frac{5}{2}\right)^2+\frac{41}{4}}}dx$$

Let $u = \frac{2x-5}{\sqrt{41}} \rightarrow dx = \frac{\sqrt{41}}{2}du$

$$\Rightarrow \int \frac{1}{\sqrt{-\left(x-\frac{5}{2}\right)^2+\frac{41}{4}}}dx = \int \frac{\sqrt{41}}{\sqrt{41-41u^2}}du$$

$$= \int \frac{1}{\sqrt{1-u^2}}du$$

We know that $\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}(x) + c$

$$\Rightarrow \int \frac{1}{\sqrt{1-u^2}}du = \sin^{-1}\left(\frac{2x-5}{\sqrt{41}}\right)$$

Then,

$$\Rightarrow \int \frac{x+1}{\sqrt{-x^2+5x+4}}dx = \frac{1}{2}\int \frac{-2x+5}{\sqrt{-x^2+5x+4}}dx + \frac{7}{2}\int \frac{1}{\sqrt{-x^2+5x+4}}dx$$

$$= -\sqrt{-x^2+5x+4} + \frac{7}{2}\left(\sin^{-1}\left(\frac{2x-5}{\sqrt{41}}\right)\right) + c$$

$$\therefore I = \int \frac{x+1}{\sqrt{-x^2+5x+4}}dx = -\sqrt{-x^2+5x+4} + \frac{7}{2}\left(\sin^{-1}\left(\frac{2x-5}{\sqrt{41}}\right)\right) + c$$

4. Question

Evaluate the following integrals:

$$\int \frac{6x-5}{\sqrt{3x^2-5x+1}}dx$$

Answer

Given $I = \int \frac{6x-5}{\sqrt{3x^2-5x+1}}dx$

Integral is of form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Writing numerator as $px + q = \lambda\left\{\frac{d}{dx}(ax^2+bx+c)\right\} + \mu$

$\Rightarrow px + q = \lambda(2ax + b) + \mu$

$\Rightarrow 6x - 5 = \lambda (6x - 5) + \mu$

$\therefore \lambda = 1$ and $\mu = 0$

Let $u = 3x^2 - 5x + 1 \rightarrow dx = \frac{1}{6x-5}du$

$$\Rightarrow \int \frac{6x-5}{\sqrt{3x^2-5x+1}}dx = \int \frac{1}{\sqrt{u}}du$$

We know that $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow \int \frac{1}{\sqrt{u}}du = (2\sqrt{u}) + c$$

$$= 2\sqrt{3x^2-5x+1} + c$$

$$\therefore I = \int \frac{6x-5}{\sqrt{3x^2-5x+1}}dx = 2\sqrt{3x^2-5x+1} + c$$

5. Question

Evaluate the following integrals:

$$\int \frac{3x+1}{\sqrt{5-2x-x^2}}dx$$

Answer

Given $I = \int \frac{3x+1}{\sqrt{-x^2-2x+5}}dx$

Integral is of form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Writing numerator as $px + q = \lambda\left\{\frac{d}{dx}(ax^2 + bx + c)\right\} + \mu$

⇒ px + q = λ(2ax + b) + μ

⇒ 3x + 1 = λ (-2x – 2) + μ

∴ λ = -3/2 and μ = -2

Let 3x + 1 = – (3/2)(–2x – 2) – 2

$$\Rightarrow \int \frac{3x+1}{\sqrt{-x^2-2x+5}}dx = \int \left(\frac{-3(-2x-2)}{2\sqrt{-x^2-2x+5}} - \frac{2}{\sqrt{-x^2-2x+5}}\right)dx$$

$$= 3\int \frac{x+1}{\sqrt{-x^2-2x+5}}dx - 2\int \frac{1}{\sqrt{-x^2-2x+5}}dx$$

Consider $\int \frac{x+1}{\sqrt{-x^2-2x+5}}dx$

Let $u = -x^2 - 2x + 5 \rightarrow dx = \frac{1}{-2x-2}du$

$$\Rightarrow \int \frac{x+1}{\sqrt{-x^2-2x+5}}dx = \int -\frac{1}{2\sqrt{u}}du$$

$$= -\frac{1}{2}\int \frac{1}{\sqrt{u}}du$$

We know that $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow -\frac{1}{2}\int \frac{1}{\sqrt{u}}du = -(\sqrt{u})$$

$$= -\sqrt{-x^2-2x+5}$$

Consider $\int \frac{1}{\sqrt{-x^2-2x+5}}dx$

$$\Rightarrow \int \frac{1}{\sqrt{-x^2-2x+5}}dx = \int \frac{1}{\sqrt{6-(x+1)^2}}dx$$

Let $u = \frac{x+1}{\sqrt{6}} \rightarrow dx = \sqrt{6}du$

$$\Rightarrow \int \frac{1}{\sqrt{6-(x+1)^2}}dx = \int \frac{\sqrt{6}}{\sqrt{6-6u^2}}du$$

$$= \int \frac{1}{\sqrt{1-u^2}}du$$

We know that $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}(x) + c$

$$\Rightarrow \int \frac{1}{\sqrt{1-u^2}} du = \sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right)$$

Then,

$$\Rightarrow \int \frac{3x+1}{\sqrt{-x^2-2x+5}} dx = 3\int \frac{x+1}{\sqrt{-x^2-2x+5}} dx - 2\int \frac{1}{\sqrt{-x^2-2x+5}} dx$$

$$= -3\sqrt{-x^2-2x+5} - 2\left(\sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right)\right) + c$$

$$\therefore I = \int \frac{3x+1}{\sqrt{-x^2-2x+5}} dx = -3\sqrt{-x^2-2x+5} - 2\sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right) + c$$

6. Question

Evaluate the following integrals:

$$\int \frac{x}{\sqrt{8+x-x^2}} dx$$

Answer

Given $I = \int \frac{x}{\sqrt{-x^2+x+8}} dx$

Integral is of form $\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$

Writing numerator as $px + q = \lambda\left\{\frac{d}{dx}(ax^2+bx+c)\right\} + \mu$

$\Rightarrow px + q = \lambda(2ax + b) + \mu$

$\Rightarrow x = \lambda(-2x + 1) + \mu$

$\therefore \lambda = -1/2$ and $\mu = -1/2$

Let $x = -1/2(-2x + 1) - 1/2$ and split,

$$\Rightarrow \int \frac{x}{\sqrt{-x^2+x+8}} dx = \int \left(\frac{-(-2x+1)}{2\sqrt{-x^2+x+8}} - \frac{1}{2\sqrt{-x^2+x+8}}\right) dx$$

$$= \frac{1}{2}\int \frac{2x-1}{\sqrt{-x^2+x+8}} dx - \frac{1}{2}\int \frac{1}{\sqrt{-x^2+x+8}} dx$$

Consider $\int \frac{2x-1}{\sqrt{-x^2+x+8}} dx$

Let $u = -x^2 + x + 8 \rightarrow dx = \frac{1}{-2x+1} du$

$$\Rightarrow \int \frac{2x-1}{\sqrt{-x^2+x+8}} dx = \int -\frac{1}{\sqrt{u}} du$$

$$= -\int \frac{1}{\sqrt{u}} du$$

We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow -\int \frac{1}{\sqrt{u}} du = -(2\sqrt{u})$$

$$= -2\sqrt{-x^2+x+8}$$

Consider $\int \frac{1}{\sqrt{-x^2+x+8}}dx$

$$\Rightarrow \int \frac{1}{\sqrt{-x^2+x+8}}dx = \int \frac{1}{\sqrt{\frac{33}{4}-\left(x-\frac{1}{2}\right)^2}}dx$$

Let $u = \frac{2x-1}{\sqrt{33}} \rightarrow dx = \frac{\sqrt{33}}{2}du$

$$\Rightarrow \int \frac{1}{\sqrt{\frac{33}{4}-\left(x-\frac{1}{2}\right)^2}}dx = \int \frac{\sqrt{33}}{\sqrt{33-33u^2}}du$$

$$= \int \frac{1}{\sqrt{1-u^2}}du$$

We know that $\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}(x) + c$

$$\Rightarrow \int \frac{1}{\sqrt{1-u^2}}du = \sin^{-1}(u)$$

$$= \sin^{-1}\left(\frac{2x-1}{\sqrt{33}}\right)$$

Then,

$$\Rightarrow \int \frac{x}{\sqrt{-x^2+x+8}}dx = \frac{1}{2}\int \frac{2x-1}{\sqrt{-x^2+x+8}}dx - \frac{1}{2}\int \frac{1}{\sqrt{-x^2+x+8}}dx$$

$$= -\sqrt{-x^2+x+8} - \frac{1}{2}\left(\sin^{-1}\left(\frac{2x-1}{\sqrt{33}}\right)\right) + c$$

$$\therefore I = \int \frac{x}{\sqrt{-x^2+x+8}}dx = -\sqrt{-x^2+x+8} - \frac{\sin^{-1}\left(\frac{2x-1}{\sqrt{33}}\right)}{2} + c$$

7. Question

Evaluate the following integrals:

$$\int \frac{x+2}{\sqrt{x^2+2x-1}}dx$$

Answer

Given $I = \int \frac{x+2}{\sqrt{x^2+2x-1}}dx$

Integral is of form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Writing numerator as $px + q = \lambda\left\{\frac{d}{dx}(ax^2+bx+c)\right\} + \mu$

$\Rightarrow px + q = \lambda(2ax + b) + \mu$

$\Rightarrow x + 2 = \lambda (2x + 2) + \mu$

$\therefore \lambda = 1/2$ and $\mu = 1$

Let $x + 2 = 1/2(2x + 2) + 1$ and split,

$$\Rightarrow \int \frac{x+2}{\sqrt{x^2+2x-1}}dx = \int \left(\frac{2x+2}{2\sqrt{x^2+2x-1}} + \frac{1}{\sqrt{x^2+2x-1}}\right)dx$$

$$= \int \frac{x+1}{\sqrt{x^2+2x-1}}dx + \int \frac{1}{\sqrt{x^2+2x-1}}dx$$

Consider $\int \frac{x+1}{\sqrt{x^2+2x-1}}dx$

Let $u = x^2 + 2x - 1 \rightarrow dx = \frac{1}{2x+2}du$

$$\Rightarrow \int \frac{x+1}{\sqrt{x^2+2x-1}}dx = \int \frac{1}{2\sqrt{u}}du$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{u}}du$$

We know that $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow \frac{1}{2}\int \frac{1}{\sqrt{u}}du = \frac{1}{2}(2\sqrt{u})$$

$$= \sqrt{u} = \sqrt{x^2+2x-1}$$

Consider $\int \frac{1}{\sqrt{x^2+2x-1}}dx$

$$\Rightarrow \int \frac{1}{\sqrt{x^2+2x-1}}dx = \int \frac{1}{\sqrt{(x+1)^2-2}}dx$$

Let $u = \frac{x+1}{\sqrt{2}} \rightarrow dx = \sqrt{2}du$

$$\Rightarrow \int \frac{1}{\sqrt{(x+1)^2-2}}dx = \int \frac{\sqrt{2}}{\sqrt{2u^2-2}}du$$

$$= \int \frac{1}{\sqrt{u^2-1}}du$$

We know that $\int \frac{1}{\sqrt{x^2-1}}dx = \log\left(\sqrt{x^2-1} + x\right) + c$

$$\Rightarrow \int \frac{1}{\sqrt{u^2-1}}du = \log(\sqrt{u^2-1} + u)$$

$$= \log\left(\sqrt{\frac{(x+1)^2}{2} - 1} + \frac{x+1}{\sqrt{2}}\right)$$

Then,

$$\Rightarrow \int \frac{x+2}{\sqrt{x^2+2x-1}}dx = \int \frac{x+1}{\sqrt{x^2+2x-1}}dx + \int \frac{1}{\sqrt{x^2+2x-1}}dx$$

$$= \sqrt{x^2+2x-1} + \log\left(\sqrt{\frac{(x+1)^2}{2} - 1} + \frac{x+1}{\sqrt{2}}\right) + c$$

$$= \sqrt{x^2+2x-1} + \log\left(\sqrt{(x+1)^2-2} + x + 1\right) + c$$

$$\therefore I = \int \frac{2x+1}{\sqrt{x^2+2x-1}}dx = \sqrt{x^2+2x-1} + \log\left(\sqrt{(x+1)^2-2} + x + 1\right) + c$$

8. Question

Evaluate the following integrals:

$\int \frac{x+2}{\sqrt{x^2-1}} dx$

Answer

Given $I = \int \frac{x+2}{\sqrt{x^2-1}} dx$

Integral is of form $\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$

Writing numerator as $px + q = \lambda\left\{\frac{d}{dx}(ax^2 + bx + c)\right\} + \mu$

⇒ px + q = λ(2ax + b) + μ

⇒ x + 2 = λ (2x) + μ

∴ λ = 1/2 and μ = 2

Let x + 2 = 1/2(2x) + 2 and split,

$$\Rightarrow \int \frac{x+2}{\sqrt{x^2-1}} dx = \int \left(\frac{2x}{2\sqrt{x^2-1}} + \frac{2}{\sqrt{x^2-1}}\right) dx$$

$$= \int \frac{x}{\sqrt{x^2-1}} dx + 2\int \frac{1}{\sqrt{x^2-1}} dx$$

Consider $\int \frac{x}{\sqrt{x^2-1}} dx$

Let $u = x^2 - 1 \rightarrow dx = \frac{1}{2x} du$

$$\Rightarrow \int \frac{x}{\sqrt{x^2-1}} dx = \int \frac{1}{2\sqrt{u}} du$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{u}} du$$

We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow \frac{1}{2}\int \frac{1}{\sqrt{u}} du = \frac{1}{2}(2\sqrt{u})$$

$$= \sqrt{u} = \sqrt{x^2-1}$$

Consider $\int \frac{1}{\sqrt{x^2-1}} dx$

We know that $\int \frac{1}{\sqrt{x^2-1}} dx + c = \cosh^{-1} x + c$

$$\Rightarrow \int \frac{1}{\sqrt{x^2-1}} dx = \cosh^{-1}(x)$$

Then,

$$\Rightarrow \int \frac{x+2}{\sqrt{x^2-1}} dx = \int \frac{x}{\sqrt{x^2-1}} dx + 2\int \frac{1}{\sqrt{x^2-1}} dx$$

$$= \sqrt{x^2-1} + \cosh^{-1}(x) + c$$

$$\therefore I = \int \frac{x+2}{\sqrt{x^2-1}} dx = \sqrt{x^2-1} + \cosh^{-1}(x) + c$$

9. Question

Evaluate the following integrals:

Answer

Given $I = \int \frac{x-1}{\sqrt{x^2+1}} dx$

Integral is of form $\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$

Writing numerator as $px + q = \lambda\left\{\frac{d}{dx}(ax^2 + bx + c)\right\} + \mu$

⇒ px + q = λ(2ax + b) + μ

⇒ x – 1 = λ (2x) + μ

∴ λ = 1/2 and μ = -1

Let x – 1 = 1/2(2x) – 1 and split,

$$\Rightarrow \int \frac{x-1}{\sqrt{x^2+1}} dx = \int \left(\frac{2x}{2\sqrt{x^2+1}} - \frac{1}{\sqrt{x^2+1}}\right) dx$$

$$= \int \frac{x}{\sqrt{x^2+1}} dx - \int \frac{1}{\sqrt{x^2+1}} dx$$

Consider $\int \frac{x}{\sqrt{x^2+1}} dx$

Let $u = x^2 + 1 \rightarrow dx = \frac{1}{2x} du$

$$\Rightarrow \int \frac{x}{\sqrt{x^2+1}} dx = \int \frac{1}{2\sqrt{u}} du$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{u}} du$$

We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow \frac{1}{2}\int \frac{1}{\sqrt{u}} du = \frac{1}{2}(2\sqrt{u})$$

$$= \sqrt{u} = \sqrt{x^2+1}$$

Consider $\int \frac{1}{\sqrt{x^2+1}} dx$

We know that $\int \frac{1}{\sqrt{x^2+1}} dx + c = \sinh^{-1} x + c$

$$\Rightarrow \int \frac{1}{\sqrt{x^2+1}} dx = \sinh^{-1}(x)$$

Then,

$$\Rightarrow \int \frac{x-1}{\sqrt{x^2+1}} dx = \int \frac{x}{\sqrt{x^2+1}} dx - \int \frac{1}{\sqrt{x^2+1}} dx$$

$$= \sqrt{x^2+1} - \sinh^{-1}(x) + c$$

$$\therefore I = \int \frac{x-1}{\sqrt{x^2+1}} dx = \sqrt{x^2+1} - \sinh^{-1}(x) + c$$

10. Question

Evaluate the following integrals:

$$\int \frac{x}{\sqrt{x^2+x+1}} dx$$

Answer

Given $I = \int \frac{x}{\sqrt{x^2+x+1}} dx$

Integral is of form $\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$

Writing numerator as $px + q = \lambda\left\{\frac{d}{dx}(ax^2 + bx + c)\right\} + \mu$

$\Rightarrow$ px + q = λ(2ax + b) + μ

$\Rightarrow$ x = λ (2x + 1) + μ

∴ λ = 1/2 and μ = -1/2

Let x = 1/2(2x + 1) – 1/2 and split,

$$\Rightarrow \int \frac{x}{\sqrt{x^2+x+1}} dx = \int \left(\frac{2x+1}{2\sqrt{x^2+x+1}} - \frac{1}{2\sqrt{x^2+x+1}}\right) dx$$

$$= \frac{1}{2}\int \frac{2x+1}{\sqrt{x^2+x+1}} dx - \frac{1}{2}\int \frac{1}{\sqrt{x^2+x+1}} dx$$

Consider $\int \frac{2x+1}{\sqrt{x^2+x+1}} dx$

Let $u = x^2 + x + 1 \rightarrow dx = \frac{1}{2x+1} du$

$$\Rightarrow \int \frac{2x+1}{\sqrt{x^2+x+1}} dx = \int \frac{1}{\sqrt{u}} du$$

$$= \int \frac{1}{\sqrt{u}} du$$

We know that $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow \int \frac{1}{\sqrt{u}} du = (2\sqrt{u})$$

$$= 2\sqrt{u} = 2\sqrt{x^2 + x + 1}$$

Consider $\int \frac{1}{\sqrt{x^2+x+1}} dx$

$$\Rightarrow \int \frac{1}{\sqrt{x^2+x+1}} dx = \int \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}} dx$$

Let $u = \frac{2x+1}{\sqrt{3}} \rightarrow dx = \frac{\sqrt{3}}{2} du$

$$\Rightarrow \int \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}} dx = \int \frac{\sqrt{3}}{\sqrt{3u^2+3}} du$$

$$= \int \frac{1}{\sqrt{u^2+1}} du$$

We know that $\int \frac{1}{\sqrt{x^2+1}} dx = \sinh^{-1} x + c$

$$\Rightarrow \int \frac{1}{\sqrt{u^2+1}} du = \sinh^{-1}(u)$$

$$= \sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

Then,

$$\Rightarrow \int \frac{x}{\sqrt{x^2+x+1}} dx = \frac{1}{2}\int \frac{2x+1}{\sqrt{x^2+x+1}} dx - \frac{1}{2}\int \frac{1}{\sqrt{x^2+x+1}} dx$$

$$= \sqrt{x^2+x+1} - \frac{\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)}{2} + c$$

$$\therefore I = \int \frac{x}{\sqrt{x^2+x+1}} dx = \sqrt{x^2+x+1} - \frac{\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)}{2} + c$$

11. Question

Evaluate the following integrals:

$$\int \frac{x+1}{\sqrt{x^2+1}} dx$$

Answer

Given $I = \int \frac{x+1}{\sqrt{x^2+1}} dx$

Integral is of form $\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$

Writing numerator as $px + q = \lambda\left\{\frac{d}{dx}(ax^2+bx+c)\right\} + \mu$

$\Rightarrow px + q = \lambda(2ax + b) + \mu$

$\Rightarrow x + 1 = \lambda(2x) + \mu$

$\therefore \lambda = 1/2$ and $\mu = 1$

Let $x + 1 = 1/2(2x) + 1$ and split,

$$\Rightarrow \int \frac{x+1}{\sqrt{x^2+1}} dx = \int \left(\frac{2x}{2\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}}\right) dx$$

$$= \int \frac{x}{\sqrt{x^2+1}} dx + \int \frac{1}{\sqrt{x^2+1}} dx$$

Consider $\int \frac{x}{\sqrt{x^2+1}} dx$

Let $u = x^2 + 1 \rightarrow dx = \frac{1}{2x} du$

$$\Rightarrow \int \frac{x}{\sqrt{x^2+1}} dx = \int \frac{1}{2\sqrt{u}} du$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{u}} du$$

We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow \frac{1}{2}\int \frac{1}{\sqrt{u}} du = \frac{1}{2}(2\sqrt{u})$$

$= \sqrt{u} = \sqrt{x^2 + 1}$

Consider $\int \frac{1}{\sqrt{x^2+1}} dx$

We know that $\int \frac{1}{\sqrt{x^2+1}} dx + c = \sinh^{-1} x + c$

$\Rightarrow \int \frac{1}{\sqrt{x^2 + 1}} dx = \sinh^{-1}(x)$

Then,

$\Rightarrow \int \frac{x + 1}{\sqrt{x^2 + 1}} dx = \int \frac{x}{\sqrt{x^2 + 1}} dx + \int \frac{1}{\sqrt{x^2 + 1}} dx$

$= \sqrt{x^2 + 1} + \sinh^{-1}(x) + c$

$\therefore I = \int \frac{x + 1}{\sqrt{x^2 + 1}} dx = \sqrt{x^2 + 1} + \sinh^{-1}(x) + c$

12. Question

Evaluate the following integrals:

$$\int \frac{2x + 5}{\sqrt{x^2 + 2x + 5}} dx$$

Answer

Given $I = \int \frac{2x+5}{\sqrt{x^2+2x+5}} dx$

Integral is of form $\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$

Writing numerator as $px + q = \lambda \left\{ \frac{d}{dx} (ax^2 + bx + c) \right\} + \mu$

$\Rightarrow px + q = \lambda(2ax + b) + \mu$

$\Rightarrow 2x + 5 = \lambda (2x + 2) + \mu$

$\therefore \lambda = 1$ and $\mu = 3$

Let $2x + 5 = 2x + 2 + 3$ and split,

$\Rightarrow \int \frac{2x + 5}{\sqrt{x^2 + 2x + 5}} dx = \int \left(\frac{2x + 2}{\sqrt{x^2 + 2x + 5}} + \frac{3}{\sqrt{x^2 + 2x + 5}} \right) dx$

$= 2 \int \frac{x + 1}{\sqrt{x^2 + 2x + 5}} dx + 3 \int \frac{1}{\sqrt{x^2 + 2x + 5}} dx$

Consider $\int \frac{x+1}{\sqrt{x^2+2x+5}} dx$

Let $u = x^2 + 2x + 5 \rightarrow dx = \frac{1}{2x+2} du$

$\Rightarrow \int \frac{x + 1}{\sqrt{x^2 + 2x + 5}} dx = \int \frac{1}{2\sqrt{u}} du$

$= \frac{1}{2} \int \frac{1}{\sqrt{u}} du$

We know that $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow \frac{1}{2}\int \frac{1}{\sqrt{u}}du = \frac{1}{2}(2\sqrt{u})$$

$$= \sqrt{u} = \sqrt{x^2 + 2x + 5}$$

Consider $\int \frac{1}{\sqrt{x^2+2x+5}}dx$

$$\Rightarrow \int \frac{1}{\sqrt{x^2 + 2x + 5}}dx = \int \frac{1}{\sqrt{(x+1)^2 + 4}}dx$$

Let $u = \frac{x+1}{2} \rightarrow dx = 2du$

$$\Rightarrow \int \frac{1}{\sqrt{(x+1)^2 + 4}}dx = \int \frac{2}{\sqrt{4u^2 + 4}}du$$

$$= \int \frac{1}{\sqrt{u^2 + 1}}du$$

We know that $\int \frac{1}{\sqrt{x^2+1}}dx = \sinh^{-1}x + c$

$$\Rightarrow \int \frac{1}{\sqrt{u^2 + 1}}du = \sinh^{-1}(u)$$

$$= \sinh^{-1}\left(\frac{x+1}{2}\right)$$

Then,

$$\Rightarrow \int \frac{2x+5}{\sqrt{x^2 + 2x + 5}}dx = 2\int \frac{x+1}{\sqrt{x^2 + 2x + 5}}dx + 3\int \frac{1}{\sqrt{x^2 + 2x + 5}}dx$$

$$= 2\sqrt{x^2 + 2x + 5} + 3\sinh^{-1}\left(\frac{x+1}{2}\right) + c$$

$$\therefore I = \int \frac{2x+5}{\sqrt{x^2 + 2x + 5}}dx = 2\sqrt{x^2 + 2x + 5} + 3\sinh^{-1}\left(\frac{x+1}{2}\right) + c$$

13. Question

Evaluate the following integrals:

$$\int \frac{3x+1}{\sqrt{5 - 2x - x^2}}dx$$

Answer

Given $I = \int \frac{3x+1}{\sqrt{-x^2-2x+5}}dx$

Integral is of form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Writing numerator as $px + q = \lambda\left\{\frac{d}{dx}(ax^2 + bx + c)\right\} + \mu$

$\Rightarrow px + q = \lambda(2ax + b) + \mu$

$\Rightarrow 3x + 1 = \lambda(-2x - 2) + \mu$

$\therefore \lambda = -3/2$ and $\mu = -2$

Let $3x + 1 = -(3/2)(-2x - 2) - 2$

$$\Rightarrow \int \frac{3x+1}{\sqrt{-x^2-2x+5}}dx = \int \left(\frac{-3(-2x-2)}{2\sqrt{-x^2-2x+5}} - \frac{2}{\sqrt{-x^2-2x+5}}\right)dx$$

$$= 3\int \frac{x+1}{\sqrt{-x^2-2x+5}}dx - 2\int \frac{1}{\sqrt{-x^2-2x+5}}dx$$

Consider $\int \frac{x+1}{\sqrt{-x^2-2x+5}}dx$

Let $u = -x^2 - 2x + 5 \rightarrow dx = \frac{1}{-2x-2}du$

$$\Rightarrow \int \frac{x+1}{\sqrt{-x^2-2x+5}}dx = \int -\frac{1}{2\sqrt{u}}du$$

$$= -\frac{1}{2}\int \frac{1}{\sqrt{u}}du$$

We know that $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow -\frac{1}{2}\int \frac{1}{\sqrt{u}}du = -(\sqrt{u})$$

$$= -\sqrt{-x^2-2x+5}$$

Consider $\int \frac{1}{\sqrt{-x^2-2x+5}}dx$

$$\Rightarrow \int \frac{1}{\sqrt{-x^2-2x+5}}dx = \int \frac{1}{\sqrt{6-(x+1)^2}}dx$$

Let $u = \frac{x+1}{\sqrt{6}} \rightarrow dx = \sqrt{6}du$

$$\Rightarrow \int \frac{1}{\sqrt{6-(x+1)^2}}dx = \int \frac{\sqrt{6}}{\sqrt{6-6u^2}}du$$

$$= \int \frac{1}{\sqrt{1-u^2}}du$$

We know that $\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}(x) + c$

$$\Rightarrow \int \frac{1}{\sqrt{1-u^2}}du = \sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right)$$

Then,

$$\Rightarrow \int \frac{3x+1}{\sqrt{-x^2-2x+5}}dx = 3\int \frac{x+1}{\sqrt{-x^2-2x+5}}dx - 2\int \frac{1}{\sqrt{-x^2-2x+5}}dx$$

$$= -3\sqrt{-x^2-2x+5} - 2\left(\sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right)\right) + c$$

$$\therefore I = \int \frac{3x+1}{\sqrt{-x^2-2x+5}}dx = -3\sqrt{-x^2-2x+5} - 2\sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right) + c$$

14. Question

Evaluate the following integrals:

$$\int\sqrt{\frac{1-x}{1+x}}dx$$

Answer

Given $I=\int\frac{\sqrt{1-x}}{\sqrt{1+x}}dx$

Rationalizing the denominator,

$$\Rightarrow\int\sqrt{\frac{1-x}{1+x}}dx=\int\sqrt{\frac{1-x}{1+x}\times\frac{1-x}{1-x}}dx$$

$$=\int\frac{1-x}{\sqrt{1-x^2}}dx$$

Integral is of form $\int\frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Writing numerator as $px+q=\lambda\left\{\frac{d}{dx}(ax^2+bx+c)\right\}+\mu$

$\Rightarrow px + q = \lambda(2ax + b) + \mu$

$\Rightarrow -x + 1 = \lambda(-2x) + \mu$

$\therefore \lambda = 1/2$ and $\mu = 1$

Let $-x + 1 = 1/2(-2x) + 1$ and split,

$$\Rightarrow\int\frac{1-x}{\sqrt{1-x^2}}dx=\int\left(\frac{-2x}{2\sqrt{1-x^2}}+\frac{1}{\sqrt{1-x^2}}\right)dx$$

$$=-\int\frac{x}{\sqrt{1-x^2}}dx+\int\frac{1}{\sqrt{1-x^2}}dx$$

Consider $\int\frac{x}{\sqrt{1-x^2}}dx$

Let $u=1-x^2\rightarrow dx=\frac{-1}{2x}du$

$$\Rightarrow\int\frac{x}{\sqrt{1-x^2}}dx=\int\frac{-1}{2\sqrt{u}}du$$

$$=\frac{-1}{2}\int\frac{1}{\sqrt{u}}du$$

We know that $\int x^n\,dx=\frac{x^{n+1}}{n+1}+c$

$$\Rightarrow\frac{1}{2}\int\frac{1}{\sqrt{u}}du=\frac{1}{2}\left(2\sqrt{u}\right)$$

$$=\sqrt{u}=-\sqrt{1-x^2}$$

Consider $\int\frac{1}{\sqrt{1-x^2}}dx$

We know that $\int\frac{1}{\sqrt{1-x^2}}dx+c=\sin^{-1}x+c$

$$\Rightarrow\int\frac{1}{\sqrt{1-x^2}}dx=\sin^{-1}(x)$$

Then,

$$\Rightarrow\int\frac{1-x}{\sqrt{1-x^2}}dx=-\int\frac{x}{\sqrt{1-x^2}}dx+\int\frac{1}{\sqrt{1-x^2}}dx$$

$$=\sqrt{1-x^2}+\sin^{-1}(x)+c$$

$\therefore I = \int \sqrt{\frac{1-x}{1+x}} dx = \sqrt{1-x^2} + \sin^{-1}(x) + c$

15. Question

Evaluate the following integrals:

$$\int \frac{2x+1}{\sqrt{x^2+4x+3}} dx$$

Answer

Given $I = \int \frac{2x+1}{\sqrt{x^2+4x+3}} dx$

Integral is of form $\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$

Writing numerator as $px + q = \lambda\left\{\frac{d}{dx}(ax^2+bx+c)\right\} + \mu$

⇒ px + q = λ(2ax + b) + μ

⇒ 2x + 1 = λ (2x + 4) + μ

∴ λ = 1 and μ = -3

Let 2x + 1 = 2x + 4 – 3 and split,

$$\Rightarrow \int \frac{2x+1}{\sqrt{x^2+4x+3}} dx = \int \left(\frac{2x+4}{\sqrt{x^2+4x+3}} - \frac{3}{\sqrt{x^2+4x+3}}\right) dx$$

$$= 2\int \frac{x+2}{\sqrt{x^2+4x+3}} dx - 3\int \frac{1}{\sqrt{x^2+4x+3}} dx$$

Consider $\int \frac{x+2}{\sqrt{x^2+4x+3}} dx$

Let $u = x^2 + 4x + 3 \rightarrow dx = \frac{1}{2x+4} du$

$$\Rightarrow \int \frac{x+2}{\sqrt{x^2+4x+3}} dx = \int \frac{1}{2\sqrt{u}} du$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{u}} du$$

We know that $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow \frac{1}{2}\int \frac{1}{\sqrt{u}} du = \frac{1}{2}(2\sqrt{u})$$

$$= \sqrt{u} = \sqrt{x^2+4x+3}$$

Consider $\int \frac{1}{\sqrt{x^2+4x+3}} dx$

$$\Rightarrow \int \frac{1}{\sqrt{x^2+4x+3}} dx = \int \frac{1}{\sqrt{(x+2)^2-1}} dx$$

Let u = x + 2 → dx = du

$$\Rightarrow \int \frac{1}{\sqrt{(x+2)^2-1}} dx = \int \frac{1}{\sqrt{u^2-1}} du$$

We know that $\int \frac{1}{\sqrt{x^2-1}}dx = \log\left(\sqrt{x^2-1}+x\right)+c$

$$\Rightarrow \int \frac{1}{\sqrt{u^2-1}}du = \log(\sqrt{u^2-1}+u)$$

$$= \log\left(\sqrt{(x+2)^2-1}+x+2\right)$$

Then,

$$\Rightarrow \int \frac{2x+1}{\sqrt{x^2+4x+3}}dx = 2\int \frac{x+2}{\sqrt{x^2+4x+3}}dx - 3\int \frac{1}{\sqrt{x^2+4x+3}}dx$$

$$= 2\sqrt{x^2+4x+3} - 3\log\left(\sqrt{(x+2)^2-1}+x+2\right)+c$$

$$= 2\sqrt{x^2+4x+3} - 3\log\left(\sqrt{x^2+4x+3}+x+2\right)+c$$

$$= 2\sqrt{(x+1)(x+3)} - 3\log\left(\left|\sqrt{(x+1)(x+3)}+x+2\right|\right)+c$$

$$\therefore I = \int \frac{2x+1}{\sqrt{x^2+4x+3}}dx$$
$$= 2\sqrt{(x+1)(x+3)} - 3\log\left(\left|\sqrt{(x+1)(x+3)}+x+2\right|\right)+c$$

16. Question

Evaluate the following integrals:

$$\int \frac{2x+3}{\sqrt{x^2+4x+5}}dx$$

Answer

Given $I = \int \frac{2x+3}{\sqrt{x^2+4x+5}}dx$

Integral is of form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Writing numerator as $px+q = \lambda\left\{\frac{d}{dx}(ax^2+bx+c)\right\}+\mu$

$\Rightarrow px + q = \lambda(2ax + b) + \mu$

$\Rightarrow 2x + 3 = \lambda (2x + 4) + \mu$

$\therefore \lambda = 1/2$ and $\mu = -1$

Let 2x + 3 = 2x + 4 – 1 and split,

$$\Rightarrow \int \frac{2x+3}{\sqrt{x^2+4x+5}}dx = \int \left(\frac{2x+4}{\sqrt{x^2+4x+5}} - \frac{1}{\sqrt{x^2+4x+5}}\right)dx$$

$$= 2\int \frac{x+2}{\sqrt{x^2+4x+5}}dx - \int \frac{1}{\sqrt{x^2+4x+5}}dx$$

Consider $\int \frac{x+2}{\sqrt{x^2+4x+5}}dx$

Let $u = x^2+4x+5 \rightarrow dx = \frac{1}{2x+4}du$

$$\Rightarrow \int \frac{x+2}{\sqrt{x^2+4x+5}}dx = \int \frac{1}{2\sqrt{u}}du$$

$= \frac{1}{2}\int \frac{1}{\sqrt{u}} du$

We know that $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$

$\Rightarrow \frac{1}{2}\int \frac{1}{\sqrt{u}} du = \frac{1}{2}(2\sqrt{u})$

$= \sqrt{u} = \sqrt{x^2 + 4x + 5}$

Consider $\int \frac{1}{\sqrt{x^2+4x+5}} dx$

$\Rightarrow \int \frac{1}{\sqrt{x^2 + 4x + 5}} dx = \int \frac{1}{\sqrt{(x+2)^2 + 1}} dx$

Let u = x + 2 → dx = du

$\Rightarrow \int \frac{1}{\sqrt{(x+2)^2 + 1}} dx = \int \frac{1}{\sqrt{u^2 + 1}} du$

We know that $\int \frac{1}{\sqrt{x^2+1}} dx = \sinh^{-1} x + c$

$\Rightarrow \int \frac{1}{\sqrt{u^2 + 1}} du = \sinh^{-1}(u)$

$= \sinh^{-1}(x + 2)$

Then,

$\Rightarrow \int \frac{2x + 3}{\sqrt{x^2 + 4x + 5}} dx = 2\int \frac{x + 2}{\sqrt{x^2 + 4x + 5}} dx - \int \frac{1}{\sqrt{x^2 + 4x + 5}} dx$

$= 2\sqrt{x^2 + 4x + 5} - \sinh^{-1}(x + 2) + c$

$\therefore I = \int \frac{2x + 3}{\sqrt{x^2 + 4x + 5}} dx = 2\sqrt{x^2 + 4x + 5} - \sinh^{-1}(x + 2) + c$

17. Question

Evaluate the following integrals:

$$\int \frac{5x + 3}{\sqrt{x^2 + 4x + 10}} dx$$

Answer

Given $I = \int \frac{5x+3}{\sqrt{x^2+4x+10}} dx$

Integral is of form $\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$

Writing numerator as $px + q = \lambda\left\{\frac{d}{dx}(ax^2 + bx + c)\right\} + \mu$

⇒ px + q = λ(2ax + b) + μ

⇒ 5x + 3 = λ (2x + 4) + μ

∴ λ = 5/2 and μ = -7

Let $5x + 3 = \frac{5}{2}(2x + 4) - 7$ and split,

$$\Rightarrow \int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \int \left(\frac{5(2x+4)}{2\sqrt{x^2+4x+10}} - \frac{7}{\sqrt{x^2+4x+10}}\right)dx$$

$$= 5\int \frac{x+2}{\sqrt{x^2+4x+10}}dx - 7\int \frac{1}{\sqrt{x^2+4x+10}}dx$$

Consider $\int \frac{x+2}{\sqrt{x^2+4x+10}}dx$

Let $u = x^2 + 4x + 10 \rightarrow dx = \frac{1}{2x+4}du$

$$\Rightarrow \int \frac{x+2}{\sqrt{x^2+4x+10}}dx = \int \frac{1}{2\sqrt{u}}du$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{u}}du$$

We know that $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow \frac{1}{2}\int \frac{1}{\sqrt{u}}du = \frac{1}{2}(2\sqrt{u})$$

$$= \sqrt{u} = \sqrt{x^2+4x+10}$$

Consider $\int \frac{1}{\sqrt{x^2+4x+10}}dx$

$$\Rightarrow \int \frac{1}{\sqrt{x^2+4x+10}}dx = \int \frac{1}{\sqrt{(x+2)^2+6}}dx$$

Let $u = \frac{x+2}{\sqrt{6}} \rightarrow dx = \sqrt{6}\,du$

$$\Rightarrow \int \frac{1}{\sqrt{(x+2)^2+6}}dx = \int \frac{\sqrt{6}}{\sqrt{6u^2+6}}du$$

$$= \int \frac{1}{\sqrt{u^2+1}}du$$

We know that $\int \frac{1}{\sqrt{x^2+1}}dx = \sinh^{-1}x + c$

$$\Rightarrow \int \frac{1}{\sqrt{u^2+1}}du = \sinh^{-1}(u)$$

$$= \sinh^{-1}\left(\frac{x+2}{\sqrt{6}}\right)$$

Then,

$$\Rightarrow \int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = 5\int \frac{x+2}{\sqrt{x^2+4x+10}}dx - 7\int \frac{1}{\sqrt{x^2+4x+10}}dx$$

$$= 5\sqrt{x^2+4x+10} - 7\sinh^{-1}\left(\frac{x+2}{\sqrt{6}}\right) + c$$

$$\therefore I = \int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = 5\sqrt{x^2+4x+10} - 7\sinh^{-1}\left(\frac{x+2}{\sqrt{6}}\right) + c$$

18. Question

Evaluate the following integrals:

$$\int \frac{x+2}{\sqrt{x^2+2x+3}}$$

Answer

Given $I = \int \frac{x+2}{\sqrt{x^2+2x+3}} dx$

Integral is of form $\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$

Writing numerator as $px + q = \lambda\left\{\frac{d}{dx}(ax^2 + bx + c)\right\} + \mu$

⇒ px + q = λ(2ax + b) + μ

⇒ x + 2 = λ (2x + 2) + μ

∴ λ = 1/2 and μ = 1

Let x + 2 = 1/2(2x + 2) + 1 and split,

$$\Rightarrow \int \frac{x+2}{\sqrt{x^2+2x+3}} dx = \int \left(\frac{2x+2}{2\sqrt{x^2+2x+3}} + \frac{1}{\sqrt{x^2+2x+3}}\right) dx$$

$$= \int \frac{x+1}{\sqrt{x^2+2x+3}} dx + \int \frac{1}{\sqrt{x^2+2x+3}} dx$$

Consider $\int \frac{x+1}{\sqrt{x^2+2x+3}} dx$

Let $u = x^2 + 2x + 3 \rightarrow dx = \frac{1}{2x+2} du$

$$\Rightarrow \int \frac{x+1}{\sqrt{x^2+2x+3}} dx = \int \frac{1}{2\sqrt{u}} du$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{u}} du$$

We know that $\int x^n\, dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow \frac{1}{2}\int \frac{1}{\sqrt{u}} du = \frac{1}{2}(2\sqrt{u})$$

$$= \sqrt{u} = \sqrt{x^2+2x+3}$$

Consider $\int \frac{1}{\sqrt{x^2+2x+3}} dx$

$$\Rightarrow \int \frac{1}{\sqrt{x^2+2x\mp 3}} dx = \int \frac{1}{\sqrt{(x+1)^2+2}} dx$$

Let $u = \frac{x+1}{\sqrt{2}} \rightarrow dx = \sqrt{2} du$

$$\Rightarrow \int \frac{1}{\sqrt{(x+1)^2+2}} dx = \int \frac{\sqrt{2}}{\sqrt{2u^2+2}} du$$

$$= \int \frac{1}{\sqrt{u^2+1}} du$$

We know that $\int \frac{1}{\sqrt{x^2+1}} dx = \sinh^{-1} x + c$

$$\Rightarrow \int \frac{1}{\sqrt{u^2+1}} du = \sinh^{-1}(u)$$

$$= \sinh^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$$

Then,

$$\Rightarrow \int \frac{x+2}{\sqrt{x^2+2x+3}} dx = \int \frac{x+1}{\sqrt{x^2+2x+3}} dx + \int \frac{1}{\sqrt{x^2+2x+3}} dx$$

$$= \sqrt{x^2+2x+3} + \sinh^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$$

$$\therefore I = \int \frac{x+2}{\sqrt{x^2+2x+3}} dx = \sqrt{x^2+2x+3} + \sinh^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$$

Exercise 19.22

1. Question

Evaluate the following integrals:

$$\int \frac{1}{4\cos^2 x + 9\sin^2 x} dx$$

Answer

Given $I = \int \frac{1}{4\cos^2 x + 9\sin^2 x} dx$

Dividing the numerator and denominator of the given integrand by $\cos^2 x$, we get

$$\Rightarrow I = \int \frac{1}{4\cos^2 x + 9\sin^2 x} dx = \int \frac{\sec^2 x}{4 + 9\tan^2 x} dx$$

Putting tanx = t and $\sec^2 x$ dx = dt, we get

$$\Rightarrow I = \int \frac{dt}{4+9t^2} = \frac{1}{9}\int \frac{dt}{\frac{4}{9}+t^2}$$

We know that $\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

$$\Rightarrow \frac{1}{9}\int \frac{dt}{\frac{4}{9}+t^2} = \frac{1}{9} \times \frac{1}{\frac{2}{3}}\tan^{-1}\left(\frac{t}{\frac{2}{3}}\right) + c$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3t}{2}\right) + c$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3\tan x}{2}\right) + c$$

$$\therefore I = \int \frac{1}{4\cos^2 x + 9\sin^2 x} dx = \frac{1}{6}\tan^{-1}\left(\frac{3\tan x}{2}\right) + c$$

2. Question

Evaluate the following integrals:

$$\int \frac{1}{4\sin^2 x + 5\cos^2 x} dx$$

Answer

Given $I = \int \frac{1}{4\sin^2 x + 5\cos^2 x} dx$

Dividing the numerator and denominator of the given integrand by $\cos^2 x$, we get

$$\Rightarrow I = \int \frac{1}{4\sin^2 x + 5\cos^2 x} dx = \int \frac{\sec^2 x}{4\tan^2 x + 5} dx$$

Putting $\tan x = t$ and $\sec^2 x\, dx = dt$, we get

$$\Rightarrow I = \int \frac{dt}{4t^2 + 5} = \frac{1}{4} \int \frac{dt}{t^2 + \left(\frac{5}{4}\right)}$$

We know that $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$

$$\Rightarrow \frac{1}{4} \int \frac{dt}{t^2 + \left(\frac{5}{4}\right)} = \frac{1}{4} \times \frac{1}{\frac{\sqrt{5}}{2}} \tan^{-1}\left(\frac{t}{\frac{\sqrt{5}}{2}}\right) + c$$

$$= \frac{1}{2\sqrt{5}} \tan^{-1}\left(\frac{2t}{\sqrt{5}}\right) + c$$

$$= \frac{1}{2\sqrt{5}} \tan^{-1}\left(\frac{2\tan x}{\sqrt{5}}\right) + c$$

$$\therefore I = \int \frac{1}{4\sin^2 x + 5\cos^2 x} dx = \frac{1}{2\sqrt{5}} \tan^{-1}\left(\frac{2\tan x}{\sqrt{5}}\right) + c$$

3. Question

Evaluate the following integrals:

$$\cdot \int \frac{2}{2 + \sin 2x} dx \cdot$$

Answer

Given $I = \int \frac{2}{2 + \sin 2x} dx$

We know that $\sin 2x = 2 \sin x \cos x$

$$\Rightarrow \int \frac{2}{2 + \sin 2x} dx = \int \frac{2}{2 + 2\sin x \cos x} dx$$

$$= \int \frac{1}{1 + \sin x \cos x} dx$$

Dividing the numerator and denominator by $\cos^2 x$,

$$\Rightarrow \int \frac{1}{1 + \sin x \cos x} dx = \int \frac{\sec^2 x}{\sec^2 x + \tan x} dx$$

Replacing $\sec^2 x$ in denominator by $1 + \tan^2 x$,

$$\Rightarrow \int \frac{\sec^2 x}{\sec^2 x + \tan x} dx = \int \frac{\sec^2 x}{1 + \tan^2 x + \tan x} dx$$

Putting $\tan x = t$ so that $\sec^2 x\, dx = dt$,

$$\Rightarrow \int \frac{\sec^2 x}{\tan^2 x + \tan x + 1} dx = \int \frac{dt}{t^2 + t + 1}$$

$$= \int \frac{dt}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

We know that $\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

$$\Rightarrow \int \frac{dt}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right) + c$$

$$\therefore I = \int \frac{2}{2+\sin 2x} dx = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right) + c$$

4. Question

Evaluate the following integrals:

$$\int \frac{\cos x}{\cos 3x} dx$$

Answer

Given $I = \int \frac{\cos x}{\cos 3x} dx$

$$\Rightarrow \int \frac{\cos x}{\cos 3x} dx = \int \frac{\cos x}{4\cos^3 x - 3\cos x} dx$$

$$= \int \frac{1}{4\cos^2 x - 3} dx$$

Dividing numerator and denominator by $\cos^2 x$,

$$\Rightarrow \int \frac{1}{4\cos^2 x - 3} dx = \int \frac{\sec^2 x}{4 - 3\sec^2 x} dx$$

Replacing $\sec^2 x$ by $1 + \tan^2 x$ in denominator,

$$\Rightarrow \int \frac{\sec^2 x}{4 - 3\sec^2 x} dx = \int \frac{\sec^2 x}{4 - 3 - 3\tan^2 x} dx$$

$$= \int \frac{\sec^2 x}{1 - 3\tan^2 x} dx$$

Putting $\tan x = t$ and $\sec^2 x \, dx = dt$, we get

$$I = \int \frac{dt}{1-3t^2} = \frac{1}{3}\int \frac{1}{\frac{1}{3} - t^2} dt$$

We know that $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

$$\Rightarrow \frac{1}{3}\int \frac{1}{\frac{1}{3} - t^2} dt = \frac{1}{3} \times \frac{1}{2\sqrt{3}}\log\left|\frac{\frac{1}{\sqrt{3}} + t}{\frac{1}{\sqrt{3}} - t}\right| + c$$

$$= \frac{1}{6\sqrt{3}}\log\left|\frac{1+\sqrt{3}t}{1-\sqrt{3}t}\right| + c$$

$$= \frac{1}{6\sqrt{3}} \log\left|\frac{1 + \sqrt{3}\tan x}{1 - \sqrt{3}\tan x}\right| + c$$

$$\therefore I = \int \frac{\cos x}{\cos 3x} dx = \frac{1}{6\sqrt{3}} \log\left|\frac{1 + \sqrt{3}\tan x}{1 - \sqrt{3}\tan x}\right| + c$$

5. Question

Evaluate the following integrals:

$$\int \frac{1}{1+3\sin^2 x} dx$$

Answer

Given $I = \int \frac{1}{1+3\sin^2 x} dx$

Divide numerator and denominator by $\cos^2 x$,

$$\Rightarrow I = \int \frac{1}{1 + 3\sin^2 x} dx = \int \frac{\sec^2 x}{\sec^2 x + 3\tan^2 x} dx$$

Replacing $\sec^2 x$ in denominator by $1 + \tan^2 x$,

$$\Rightarrow \int \frac{\sec^2 x}{\sec^2 x + 3\tan^2 x} dx = \int \frac{\sec^2 x}{1 + \tan^2 x + 3\tan^2 x} dx$$

$$= \int \frac{\sec^2 x}{1 + 4\tan^2 x} dx$$

Putting $\tan x = t$ so that $\sec^2 x \, dx = dt$,

$$\Rightarrow \int \frac{\sec^2 x}{1 + 4\tan^2 x} dx = \int \frac{dt}{1 + 4t^2}$$

$$= \frac{1}{4} \int \frac{1}{\frac{1}{4} + t^2} dt$$

We know that $\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

$$\Rightarrow \frac{1}{4} \int \frac{1}{\frac{1}{4} + t^2} dt = \frac{1}{4} \times \frac{1}{2} \tan^{-1}\left(\frac{t}{2}\right) + c$$

$$= \frac{1}{8}\tan^{-1}\left(\frac{\tan x}{2}\right) + c$$

$$\therefore I = \int \frac{1}{1 + 3\sin^2 x} dx = \frac{1}{8}\tan^{-1}\left(\frac{\tan x}{2}\right) + c$$

6. Question

Evaluate the following integrals:

$$\int \frac{1}{3+2\cos^2 x} dx$$

Answer

Given $I = \int \frac{1}{3+2\cos^2 x} dx$

Divide numerator and denominator by $\cos^2 x$,

$$\Rightarrow I = \int \frac{1}{3 + 2\cos^2 x} dx = \int \frac{\sec^2 x}{3\sec^2 x + 2} dx$$

Replacing $\sec^2 x$ in denominator by $1 + \tan^2 x$,

$$\Rightarrow \int \frac{\sec^2 x}{3\sec^2 x + 2} dx = \int \frac{\sec^2 x}{3 + 3\tan^2 x + 2} dx$$

$$= \int \frac{\sec^2 x}{5 + 3\tan^2 x} dx$$

Putting $\tan x = t$ so that $\sec^2 x\ dx = dt$,

$$\Rightarrow \int \frac{\sec^2 x}{5 + 3\tan^2 x} dx = \int \frac{dt}{5 + 3t^2}$$

$$= \frac{1}{3} \int \frac{1}{\frac{5}{3} + t^2} dt$$

We know that $\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

$$\Rightarrow \frac{1}{3} \int \frac{1}{\frac{5}{3} + t^2} dt = \frac{1}{3} \times \sqrt{\frac{5}{3}} \tan^{-1}\left(\frac{t}{\sqrt{\frac{5}{3}}}\right) + c$$

$$= \frac{\sqrt{5}}{3\sqrt{3}} \tan^{-1}\left(\frac{\sqrt{3}\tan x}{\sqrt{5}}\right) + c$$

$$\therefore I = \int \frac{1}{3 + 2\cos^2 x} dx = \frac{\sqrt{5}}{3\sqrt{3}} \tan^{-1}\left(\frac{\sqrt{3}\tan x}{\sqrt{5}}\right) + c$$

7. Question

Evaluate the following integrals:

$$\int \frac{1}{(\sin x - 2\cos x)(2\sin x + \cos x)} dx$$

Answer

Given $I = \int \frac{1}{(\sin x - 2\cos x)(2\sin x + \cos x)} dx$

$$\Rightarrow \int \frac{1}{(\sin x - 2\cos x)(2\sin x + \cos x)} dx$$

$$= \int \frac{1}{2\sin^2 x + \sin x\cos x - 4\sin x\cos x - 2\cos^2 x} dx$$

Dividing the numerator and denominator by $\cos^2 x$,

$$\Rightarrow \int \frac{1}{2\sin^2 x + \sin x\cos x - 4\sin x\cos x - 2\cos^2 x} dx$$

$$= \int \frac{\sec^2 x}{2\tan^2 x - 3\tan x - 2} dx$$

Putting $\tan x = t$ so that $\sec^2 x\ dx = dt$.

$$\Rightarrow \int \frac{\sec^2 x}{2\tan^2 x - 3\tan x - 2} dx = \int \frac{dt}{2t^2 - 3t - 2}$$

$$=\frac{1}{2}\int \frac{1}{t^2-\frac{3}{2}-1}dt$$

$$=\frac{1}{2}\int \frac{1}{\left(t-\frac{3}{4}\right)^2-\left(\frac{5}{4}\right)^2}dt$$

We know that $\int \frac{1}{x^2-a^2}dx=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c$

$$\Rightarrow \frac{1}{2}\int \frac{1}{\left(t-\frac{3}{4}\right)^2-\left(\frac{5}{4}\right)^2}dt=\frac{1}{2}\times\frac{1}{2\left(\frac{5}{4}\right)}\log\left|\frac{t-\frac{3}{4}-\frac{5}{4}}{t-\frac{3}{4}+\frac{5}{4}}\right|+c$$

$$=\frac{1}{5}\log\left|\frac{t-2}{t+\frac{1}{2}}\right|+c$$

$$=\frac{1}{5}\log\left|\frac{2\tan x-4}{2\tan x+1}\right|+c$$

$$\therefore I=\int \frac{1}{(\sin x-2\cos x)(2\sin x+\cos x)}dx=\frac{1}{5}\log\left|\frac{2\tan x-4}{2\tan x+1}\right|+c$$

8. Question

Evaluate the following integrals:

$$\int \frac{\sin 2x}{\sin^4 x+\cos^4 x}dx$$

Answer

Given $I=\int \frac{\sin 2x}{\sin^4 x+\cos^4 x}dx$

Dividing the numerator and denominator by $\cos^4 x$,

$$\Rightarrow \int \frac{\sin 2x}{\sin^4 x+\cos^4 x}dx=\int \frac{2\tan x\sec^2 x}{\tan^4 x+1}dx$$

Putting $\tan^2 x = t$ so that $2\tan x \sec^2 x\, dx = dt$

$$\Rightarrow \int \frac{2\tan x\sec^2 x}{\tan^4 x+1}dx=\int \frac{dt}{t^2+1}$$

We know that $\int \frac{1}{1+x^2}dx=\tan^{-1}(x)+c$

$$\Rightarrow \int \frac{dt}{t^2+1}=\tan^{-1}(t)+c$$

$$=\tan^{-1}(\tan^2 x)+c$$

$$\therefore I=\int \frac{\sin 2x}{\sin^4 x+\cos^4 x}dx=\tan^{-1}(\tan^2 x)+c$$

9. Question

Evaluate the following integrals:

.w $\int \frac{1}{\cos x(\sin x+2\cos x)}dx$.

Answer

Given $I = \int \frac{1}{\cos x(\sin x + 2\cos x)} dx$

$$\Rightarrow I = \int \frac{1}{\cos x(\sin x + 2\cos x)} dx = \int \frac{1}{\cos x \sin x + 2\cos^2 x} dx$$

Dividing the numerator and denominator by $\cos^2 x$,

$$\Rightarrow \int \frac{1}{\cos x \sin x + 2\cos^2 x} dx = \int \frac{\sec^2 x}{\tan x + 2} dx$$

Putting $\tan x + 2 = t$ so that $\sec^2 x\, dx = dt$,

$$\Rightarrow \int \frac{\sec^2 x}{\tan x + 2} dx = \int \frac{dt}{t}$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{t} dt = \log|t| + c$$

$$= \log|\tan x + 2| + x$$

$$\therefore I = \int \frac{1}{\cos x(\sin x + 2\cos x)} dx = \log|\tan x + 2| + c$$

10. Question

Evaluate the following integrals:

$$\int \frac{1}{\sin^2 x + \sin 2x} dx$$

Answer

Given $I = \int \frac{1}{\sin^2 x + \sin 2x} dx$

We know that $\sin 2x = 2 \sin x \cos x$

$$\Rightarrow I = \int \frac{1}{\sin^2 x + 2\sin x \cos x} dx$$

Dividing numerator and denominator by $\cos^2 x$,

$$\Rightarrow \int \frac{1}{\sin^2 x + 2\sin x \cos x} dx = \int \frac{\sec^2 x}{\tan^2 x + 2\tan x} dx$$

Putting $\tan x = t$ so that $\sec^2 x\, dx = dt$,

$$\Rightarrow \int \frac{\sec^2 x}{\tan^2 x + 2\tan x} dx = \int \frac{dt}{t^2 + 2t}$$

$$= \int \frac{1}{t^2 + 2t + 1^2 - 1^2} dt$$

$$= \int \frac{1}{(t+1)^2 - 1^2} dt$$

We know that $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + c$

$$\Rightarrow \int \frac{1}{(t+1)^2 - 1^2} dt = \frac{1}{2} \log\left|\frac{t+1-1}{t+1+1}\right| + c$$

$$=\frac{1}{2}\log\left|\frac{t}{t+2}\right|+c$$

$$=\frac{1}{2}\log\left|\frac{\tan x}{\tan x+2}\right|+c$$

$$\therefore I=\int\frac{1}{\sin^2 x+\sin 2x}dx=\frac{1}{2}\log\left|\frac{\tan x}{\tan x+2}\right|+c$$

11. Question

Evaluate the following integrals:

$$\int\frac{1}{\cos 2x+3\sin^2 x}dx$$

Answer

Given $I=\int\frac{1}{\cos 2x+3\sin^2 x}dx$

We know that cos 2x = 1 – 2sin^2 x.

$$\Rightarrow\int\frac{1}{\cos 2x+3\sin^2 x}dx=\int\frac{1}{1-2\sin^2 x+3\sin^2 x}dx$$

$$=\int\frac{1}{1+\sin^2 x}dx$$

Dividing numerator and denominator by cos^{2}x,

$$\Rightarrow\int\frac{1}{1+\sin^2 x}dx=\int\frac{\sec^2 x}{\sec^2 x+\tan^2 x}dx$$

Replacing sec^{2}x in denominator by 1 + tan^{2}x,

$$\Rightarrow\int\frac{\sec^2 x}{\sec^2 x+\tan^2 x}dx=\int\frac{\sec^2 x}{1+2\tan^2 x}dx$$

Putting tan x = t so that sec^{2}x dx = dt,

$$\Rightarrow\int\frac{\sec^2 x}{1+2\tan^2 x}dx=\int\frac{dt}{1+2t^2}$$

$$=\frac{1}{2}\int\frac{1}{\frac{1}{2}+t^2}dt$$

We know that $\int\frac{1}{a^2+x^2}dx=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$

$$\Rightarrow\frac{1}{2}\int\frac{1}{\frac{1}{2}+t^2}dt=\frac{1}{2}\times\frac{1}{\frac{1}{\sqrt{2}}}\tan^{-1}\left(\frac{t}{\frac{1}{\sqrt{2}}}\right)+c$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x)+c$$

$$\therefore I=\int\frac{1}{\cos 2x+3\sin^2 x}dx=\frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x)+c$$

Exercise 19.23

1. Question

Evaluate the following integrals:

$$\int \frac{1}{5+4\cos x} dx$$

Answer

Given $I = \int \frac{1}{5+4\cos x} dx$

We know that $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$\Rightarrow \int \frac{1}{5+4\cos x} dx = \int \frac{1}{5+4\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)} dx$$

$$= \int \frac{1+\tan^2\frac{x}{2}}{5\left(1+\tan^2\frac{x}{2}\right)+4\left(1-\tan^2\frac{x}{2}\right)} dx$$

Replacing 1 + $\tan^2 x/2$ in numerator by $\sec^2 x/2$,

$$\Rightarrow \int \frac{1+\tan^2\frac{x}{2}}{5\left(1+\tan^2\frac{x}{2}\right)+4\left(1-\tan^2\frac{x}{2}\right)} dx = \int \frac{\sec^2\frac{x}{2}}{\tan^2\frac{x}{2}+9} dx$$

Putting tanx/2 = t and $\sec^2(x/2)dx = 2dt$,

$$\Rightarrow \int \frac{\sec^2\frac{x}{2}}{\tan^2\frac{x}{2}+9} dx = \int \frac{2dt}{t^2+9}$$

$$= 2\int \frac{1}{t^2+9} dt$$

We know that $\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

$$\Rightarrow 2\int \frac{1}{t^2+9} dt = 2\left(\frac{1}{3}\right)\tan^{-1}\left(\frac{t}{3}\right) + c$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{\tan x}{3}\right) + c$$

$$\therefore I = \int \frac{1}{5+4\cos x} dx = \frac{2}{3}\tan^{-1}\left(\frac{\tan x}{3}\right) + c$$

2. Question

Evaluate the following integrals:

$$\int \frac{1}{5-4\sin x} dx$$

Answer

Given $I = \int \frac{1}{5-4\sin x} dx$

We know that $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$\Rightarrow \int \frac{1}{5-4\sin x} dx = \int \frac{1}{5-4\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)} dx$$

$$= \int \frac{1 + \tan^2\frac{x}{2}}{5\left(1 + \tan^2\frac{x}{2}\right) - 4\left(2\tan\frac{x}{2}\right)} dx$$

Replacing 1 + tan^{2}x/2 in numerator by sec^{2}x/2,

$$\Rightarrow \int \frac{1 + \tan^2\frac{x}{2}}{5\left(1 + \tan^2\frac{x}{2}\right) - 4\left(2\tan\frac{x}{2}\right)} dx = \int \frac{\sec^2\frac{x}{2}}{5 + 5\tan^2\frac{x}{2} - 8\tan\frac{x}{2}} dx$$

Putting tanx/2 = t and sec^2(x/2)dx = 2dt,

$$\Rightarrow \int \frac{\sec^2\frac{x}{2}}{5 + 5\tan^2\frac{x}{2} - 8\tan\frac{x}{2}} dx = \int \frac{2dt}{5 + 5t^2 - 8t}$$

$$= \frac{2}{5}\int \frac{1}{t^2 - \frac{8}{5}t + 1} dt$$

$$= \frac{2}{5}\int \frac{1}{\left(t - \frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2} dt$$

We know that $\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

$$\Rightarrow \frac{2}{5}\int \frac{1}{\left(t - \frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2} dt = \frac{2}{5}\left(\frac{1}{\frac{3}{5}}\right)\tan^{-1}\left(\frac{t - \frac{4}{5}}{\frac{3}{5}}\right) + c$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{5\tan x - 4}{3}\right) + c$$

$$\therefore I = \int \frac{1}{5 - 4\sin x} dx = \frac{2}{3}\tan^{-1}\left(\frac{5\tan x - 4}{3}\right) + c$$

3. Question

Evaluate the following integrals:

$$\int \frac{1}{1 - 2\sin x} dx$$

Answer

Given $I = \int \frac{1}{1-2\sin x} dx$

We know that $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$\Rightarrow \int \frac{1}{1 - 2\sin x} dx = \int \frac{1}{1 - 2\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)} dx$$

$$= \int \frac{1 + \tan^2\frac{x}{2}}{1\left(1 + \tan^2\frac{x}{2}\right) - 2\left(2\tan\frac{x}{2}\right)} dx$$

Replacing 1 + tan^{2}x/2 in numerator by sec^{2}x/2,

$$\Rightarrow \int \frac{1 + \tan^2\frac{x}{2}}{1\left(1 + \tan^2\frac{x}{2}\right) - 2\left(2\tan\frac{x}{2}\right)} dx = \int \frac{\sec^2\frac{x}{2}}{1 + \tan^2\frac{x}{2} - 4\tan\frac{x}{2}} dx$$

Putting tanx/2 = t and $\sec^2(x/2)dx = 2dt$,

$$\Rightarrow \int \frac{\sec^2\frac{x}{2}}{1+\tan^2\frac{x}{2}-4\tan\frac{x}{2}}dx = \int \frac{2dt}{1+t^2-4t}$$

$$= 2\int \frac{1}{t^2-4t+1}dt$$

$$= 2\int \frac{1}{(t-2)^2-\left(\sqrt{3}\right)^2}dt$$

We know that $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$

$$\Rightarrow 2\int \frac{1}{(t-2)^2-\left(\sqrt{3}\right)^2}dt = 2\left(\frac{1}{2\sqrt{3}}\right)\tan^{-1}\left(\frac{t-2-\sqrt{3}}{t+2+\sqrt{3}}\right) + c$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\tan x-(2+\sqrt{3})}{\tan x+\left(2+\sqrt{3}\right)}\right) + c$$

$$\therefore I = \int \frac{1}{1-2\sin x}dx = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\tan x-(2+\sqrt{3})}{\tan x+\left(2+\sqrt{3}\right)}\right) + c$$

4. Question

Evaluate the following integrals:

$$\int \frac{1}{4\cos x-1}dx$$

Answer

Given $I = \int \frac{1}{4\cos x-1}dx$

We know that $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$\Rightarrow \int \frac{1}{-1+4\cos x}dx = \int \frac{1}{-1+4\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}dx$$

$$= \int \frac{1+\tan^2\frac{x}{2}}{-1\left(1+\tan^2\frac{x}{2}\right)+4(1-\tan^2\frac{x}{2})}dx$$

Replacing 1 + $\tan^2 x/2$ in numerator by $\sec^2 x/2$,

$$\Rightarrow \int \frac{1+\tan^2\frac{x}{2}}{-1\left(1+\tan^2\frac{x}{2}\right)+4(1-\tan^2\frac{x}{2})}dx = \int \frac{\sec^2\frac{x}{2}}{-5\tan^2\frac{x}{2}+3}dx$$

Putting $\tan\frac{x}{2} = t$ and $\frac{1}{2}\sec^2(\frac{x}{2})dx = dt$,

$$\Rightarrow \int \frac{\sec^2\frac{x}{2}}{-5\tan^2\frac{x}{2}+3}dx = \int \frac{dt}{3-5t^2}$$

$$= \frac{1}{5}\int \frac{1}{\frac{3}{5}-t^2}dt$$

We know that $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

$$\Rightarrow \frac{1}{5}\int \frac{1}{\frac{3}{5}-t^2}dt = \frac{1}{5}\left(\frac{1}{\sqrt{\frac{3}{5}}}\right)\log\left|\frac{\sqrt{\frac{3}{5}}+t}{\sqrt{\frac{3}{5}}-t}\right| + c$$

$$= \frac{1}{\sqrt{15}}\log\left|\frac{\sqrt{3}+\sqrt{5}\tan\frac{x}{2}}{\sqrt{3}-\sqrt{5}\tan\frac{x}{2}}\right| + c$$

$$\therefore I = \int \frac{1}{4\cos x - 1}dx = \frac{1}{\sqrt{15}}\log\left|\frac{\sqrt{3}+\sqrt{5}\tan\frac{x}{2}}{\sqrt{3}-\sqrt{5}\tan\frac{x}{2}}\right| + c$$

5. Question

Evaluate the following integrals:

$$\int \frac{1}{1-\sin x+\cos x}dx$$

Answer

Given $I = \int \frac{1}{1-\sin x+\cos x}dx$

We know that $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$\Rightarrow \int \frac{1}{1-\sin x+\cos x}dx = \int \frac{1}{1-\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}+\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}dx$$

$$= \int \frac{1+\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}-2\tan\frac{x}{2}+1-\tan^2\frac{x}{2}}dx$$

Replacing $1 + \tan^2 x/2$ in numerator by $\sec^2 x/2$ and putting $\tan x/2 = t$ and $\sec^2 x/2\, dx = 2dt$,

$$\Rightarrow \int \frac{1+\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}-2\tan\frac{x}{2}+1-\tan^2\frac{x}{2}}dx = \int \frac{\sec^2\frac{x}{2}}{2-2\tan\frac{x}{2}}dx$$

$$= \int \frac{2dt}{2-2t}$$

$$= \int \frac{1}{1-t}dt$$

We know that $\int \frac{1}{x}dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{1-t}dt = -\log|1-t| + c$$

$$= -\log\left|1-\tan\frac{x}{2}\right| + c$$

$$\therefore I = \int \frac{1}{1-\sin x+\cos x}dx = -\log\left|1-\tan\frac{x}{2}\right| + c$$

6. Question

Evaluate the following integrals:

$$\int \frac{1}{3+2\sin x+\cos x} dx$$

Answer

Given $I = \int \frac{1}{3+2\sin x+\cos x} dx$

We know that $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$\Rightarrow \int \frac{1}{3+2\sin x+\cos x} dx = \int \frac{1}{3+2\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)+\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}} dx$$

$$= \int \frac{1+\tan^2\frac{x}{2}}{3+3\tan^2\frac{x}{2}+4\tan\frac{x}{2}+1-\tan^2\frac{x}{2}} dx$$

Replacing $1 + \tan^2 x/2$ in numerator by $\sec^2 x/2$ and putting $\tan x/2 = t$ and $\sec^2 x/2\, dx = 2dt$,

$$\Rightarrow \int \frac{1+\tan^2\frac{x}{2}}{3+3\tan^2\frac{x}{2}+4\tan\frac{x}{2}+1-\tan^2\frac{x}{2}} dx = \int \frac{\sec^2\frac{x}{2}}{2\tan^2\frac{x}{2}+4\tan\frac{x}{2}+4} dx$$

$$= \int \frac{2dt}{2t^2+4t+4}$$

$$= \int \frac{1}{t^2+2t+2} dt$$

$$= \int \frac{1}{(t+1)^2+1^2} dt$$

We know that $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$

$$\Rightarrow \int \frac{1}{(t+1)^2+1^2} dt = \tan^{-1}(t+1) + c$$

$$= \tan^{-1}(\tan\frac{x}{2}+1) + c$$

$$\therefore I = \int \frac{1}{3+2\sin x+\cos x} dx = \tan^{-1}(\tan\frac{x}{2}+1) + c$$

7. Question

Evaluate the following integrals:

$$\int \frac{1}{13+3\cos x+4\sin x} dx$$

Answer

Given $I = \int \frac{1}{13+3\cos x+4\sin x} dx$

We know that $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$\Rightarrow \int \frac{1}{13+4\sin x+3\cos x} dx = \int \frac{1}{13+4\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)+3\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)} dx$$

$$= \int \frac{1+\tan^2\frac{x}{2}}{13+13\tan^2\frac{x}{2}+8\tan\frac{x}{2}+3-3\tan^2\frac{x}{2}}dx$$

Replacing 1 + tan^{2}x/2 in numerator by sec^{2}x/2 and putting tan x/2 = t and sec^2 x/2 dx = 2dt,

$$\Rightarrow \int \frac{1+\tan^2\frac{x}{2}}{13+13\tan^2\frac{x}{2}+8\tan\frac{x}{2}+3-3\tan^2\frac{x}{2}}dx = \int \frac{\sec^2\frac{x}{2}}{10\tan^2\frac{x}{2}+8\tan\frac{x}{2}+16}dx$$

$$= \int \frac{2dt}{10t^2+8t+16}$$

$$= \frac{2}{10}\int \frac{1}{t^2+\frac{4}{5}t+\frac{8}{5}}dt$$

$$= \frac{1}{5}\int \frac{1}{\left(t+\frac{2}{5}\right)^2+\frac{6}{5}^2}dt$$

We know that $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$

$$\Rightarrow \frac{1}{5}\int \frac{1}{\left(t+\frac{2}{5}\right)^2+\frac{6}{5}^2}dt = \frac{1}{5}\left(\frac{1}{\frac{6}{5}}\right)\tan^{-1}\frac{t+\frac{2}{5}}{\frac{6}{5}}+c$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{5\tan\frac{x}{2}+2}{6}\right)+c$$

$$\therefore I = \int \frac{1}{13+3\cos x+4\sin x}dx = \frac{1}{6}\tan^{-1}\left(\frac{5\tan\frac{x}{2}+2}{6}\right)+c$$

8. Question

Evaluate the following integrals:

$$\int \frac{1}{\cos x-\sin x}dx$$

Answer

Given $I = \int \frac{1}{\cos x-\sin x}dx$

We know that $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$\Rightarrow \int \frac{1}{-\sin x+\cos x}dx = \int \frac{1}{-\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}+\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}dx$$

$$= \int \frac{1+\tan^2\frac{x}{2}}{-2\tan\frac{x}{2}+1-\tan^2\frac{x}{2}}dx$$

Replacing 1 + tan^{2}x/2 in numerator by sec^{2}x/2 and putting tan x/2 = t and sec^2 x/2 dx = 2dt,

$$\Rightarrow \int \frac{1+\tan^2\frac{x}{2}}{-2\tan\frac{x}{2}+1-\tan^2\frac{x}{2}}dx = \int \frac{\sec^2\frac{x}{2}}{-\tan^2\frac{x}{2}-2\tan\frac{x}{2}+1}dx$$

$$= -\int \frac{2dt}{t^2+2t-1}$$

$$= -2\int \frac{1}{(t+1)^2 - (\sqrt{2})^2} dt$$

$$= 2\int \frac{1}{(\sqrt{2})^2 - (t+1)^2} dt$$

We know that $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

$$\Rightarrow 2\int \frac{1}{(\sqrt{2})^2 - (t+1)^2} dt = \frac{2}{2\sqrt{2}}\log\left|\frac{\sqrt{2}+t+1}{\sqrt{2}-t-1}\right| + c$$

$$= \frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}+\tan\frac{x}{2}+1}{\sqrt{2}-\tan\frac{x}{2}-1}\right| + c$$

$$\therefore I = \int \frac{1}{\cos x - \sin x} dx = \frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}+\tan\frac{x}{2}+1}{\sqrt{2}-\tan\frac{x}{2}-1}\right| + c$$

9. Question

Evaluate the following integrals:

$$\int \frac{1}{\sin x + \cos x} dx$$

Answer

Given $I = \int \frac{1}{\sin x + \cos x} dx$

We know that $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$\Rightarrow \int \frac{1}{\sin x + \cos x} dx = \int \frac{1}{\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} + \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}} dx$$

$$= \int \frac{1+\tan^2\frac{x}{2}}{2\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2}} dx$$

Replacing 1 + $\tan^2 x/2$ in numerator by $\sec^2 x/2$ and putting tan x/2 = t and $\sec^2$ x/2 dx = 2dt,

$$\Rightarrow \int \frac{1+\tan^2\frac{x}{2}}{2\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2}} dx = \int \frac{\sec^2\frac{x}{2}}{-\tan^2\frac{x}{2} + 2\tan\frac{x}{2} + 1} dx$$

$$= -\int \frac{2dt}{t^2 - 2t - 1}$$

$$= -2\int \frac{1}{(t-1)^2 - (\sqrt{2})^2} dt$$

$$= 2\int \frac{1}{(\sqrt{2})^2 - (t-1)^2} dt$$

We know that $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

$$\Rightarrow 2\int \frac{1}{(\sqrt{2})^2 - (t-1)^2} dt = \frac{2}{2\sqrt{2}}\log\left|\frac{\sqrt{2}+t-1}{\sqrt{2}-t+1}\right| + c$$

$$= \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} + \tan\frac{x}{2} - 1}{\sqrt{2} - \tan\frac{x}{2} + 1} \right| + c$$

$$\therefore I = \int \frac{1}{\sin x + \cos x} dx = \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} + \tan\frac{x}{2} - 1}{\sqrt{2} - \tan\frac{x}{2} + 1} \right| + c$$

10. Question

Evaluate the following integrals:

$$\int \frac{1}{5 - 4\cos x} dx$$

Answer

Given $I = \int \frac{1}{5-4\cos x} dx$

We know that $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$\Rightarrow \int \frac{1}{5 - 4\cos x} dx = \int \frac{1}{5 - 4\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)} dx$$

$$= \int \frac{1 + \tan^2\frac{x}{2}}{5\left(1 + \tan^2\frac{x}{2}\right) - 4(1 - \tan^2\frac{x}{2})} dx$$

Replacing 1 + $\tan^2 x/2$ in numerator by $\sec^2 x/2$,

$$\Rightarrow \int \frac{1 + \tan^2\frac{x}{2}}{5\left(1 + \tan^2\frac{x}{2}\right) - 4(1 - \tan^2\frac{x}{2})} dx = \int \frac{\sec^2\frac{x}{2}}{9\tan^2\frac{x}{2} + 1} dx$$

Putting tanx/2 = t and $\sec^2(x/2)dx = 2dt$,

$$\Rightarrow \int \frac{\sec^2\frac{x}{2}}{9\tan^2\frac{x}{2} + 1} dx = \int \frac{2dt}{9t^2 + 1}$$

$$= \frac{2}{9} \int \frac{1}{t^2 + \frac{1}{9}} dt$$

We know that $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$

$$\Rightarrow \frac{2}{9} \int \frac{1}{t^2 + \frac{1}{9}} dt = \frac{2}{9}\left(\frac{1}{\frac{1}{3}}\right) \tan^{-1}\left(\frac{t}{\frac{1}{3}}\right) + c$$

$$= \frac{2}{3} \tan^{-1}\ (3\tan x) + c$$

$$\therefore I = \int \frac{1}{5 - 4\cos x} dx = \frac{2}{3} \tan^{-1}\ (3\tan x) + c$$

11. Question

Evaluate the following integrals:

$$\int \frac{1}{2 + \sin x + \cos x} dx$$

Answer

Given $I = \int \frac{1}{2+\sin x+\cos x} dx$

We know that $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$\Rightarrow \int \frac{1}{2+\sin x+\cos x} dx = \int \frac{1}{2+\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}+\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}} dx$$

$$= \int \frac{1+\tan^2\frac{x}{2}}{2+2\tan^2\frac{x}{2}-2\tan\frac{x}{2}+1-\tan^2\frac{x}{2}} dx$$

Replacing 1 + tan^{2}x/2 in numerator by sec^{2}x/2 and putting tan x/2 = t and sec^2 x/2 dx = 2dt,

$$\Rightarrow \int \frac{1+\tan^2\frac{x}{2}}{2+2\tan^2\frac{x}{2}-2\tan\frac{x}{2}+1-\tan^2\frac{x}{2}} dx = \int \frac{\sec^2\frac{x}{2}}{\tan^2\frac{x}{2}-2\tan\frac{x}{2}+3} dx$$

$$= \int \frac{2dt}{t^2-2t+3}$$

$$= 2\int \frac{1}{(t+1)^2+\left(\sqrt{2}\right)^2} dt$$

We know that $\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$

$$\Rightarrow 2\int \frac{1}{(t+1)^2+\left(\sqrt{2}\right)^2} dt = 2\left(\frac{1}{\sqrt{2}}\right)\tan^{-1}\left(\frac{t+1}{\sqrt{2}}\right)$$

$$= \sqrt{2}\tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{\sqrt{2}}\right)$$

$$\therefore I = \int \frac{1}{2+\sin x+\cos x} dx = \sqrt{2}\tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{\sqrt{2}}\right)$$

12. Question

Evaluate the following integrals:

$$\int \frac{1}{\sin x+\sqrt{3}\cos x} dx$$

Answer

Given $I = \int \frac{1}{\sin x+\sqrt{3}\cos x} dx$

We know that $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$\Rightarrow \int \frac{1}{\sin x+\sqrt{3}\cos x} dx = \int \frac{1}{\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}+\sqrt{3}\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)} dx$$

$$= \int \frac{1+\tan^2\frac{x}{2}}{2\tan\frac{x}{2}+\sqrt{3}-\sqrt{3}\tan^2\frac{x}{2}} dx$$

Replacing 1 + tan^{2}x/2 in numerator by sec^{2}x/2 and putting tan x/2 = t and sec^2 x/2 dx = 2dt,

$$\Rightarrow \int \frac{1+\tan^2\frac{x}{2}}{2\tan\frac{x}{2}+\sqrt{3}-\sqrt{3}\tan^2\frac{x}{2}}dx = \int \frac{\sec^2\frac{x}{2}}{-\sqrt{3}\tan^2\frac{x}{2}+2\tan\frac{x}{2}+\sqrt{3}}dx$$

$$= -\int \frac{2dt}{\sqrt{3}t^2-2t-\sqrt{3}}$$

$$= -\frac{2}{\sqrt{3}}\int \frac{1}{\left(t-\frac{1}{\sqrt{3}}\right)^2-\left(\frac{2}{\sqrt{3}}\right)^2}dt$$

$$= \frac{2}{\sqrt{3}}\int \frac{1}{\left(\frac{2}{\sqrt{3}}\right)^2-\left(t-\frac{1}{\sqrt{3}}\right)^2}dt$$

We know that $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

$$\Rightarrow \frac{2}{\sqrt{3}}\int \frac{1}{\left(\frac{2}{\sqrt{3}}\right)^2-\left(t-\frac{1}{\sqrt{3}}\right)^2}dt = \frac{2}{\sqrt{3}}\left(\frac{1}{2\left(\frac{2}{\sqrt{3}}\right)}\right)\log\left|\frac{\frac{2}{\sqrt{3}}+t-\frac{1}{\sqrt{3}}}{\frac{2}{\sqrt{3}}-t+\frac{1}{\sqrt{3}}}\right| + c$$

$$= \frac{1}{2}\log\left|\frac{\frac{2}{\sqrt{3}}+\tan\frac{x}{2}-\frac{1}{\sqrt{3}}}{\frac{2}{\sqrt{3}}-\tan\frac{x}{2}+\frac{1}{\sqrt{3}}}\right| + c$$

$$\therefore I = \int \frac{1}{\sin x+\sqrt{3}\cos x}dx = \frac{1}{2}\log\left|\frac{\frac{2}{\sqrt{3}}+\tan\frac{x}{2}-\frac{1}{\sqrt{3}}}{\frac{2}{\sqrt{3}}-\tan\frac{x}{2}+\frac{1}{\sqrt{3}}}\right| + c$$

13. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{3}\sin x+\cos x}dx$$

Answer

Given $I = \int \frac{1}{\sqrt{3}\sin x+\cos x}dx$

Let √3 = r cosθ and 1 = r sinθ

$r = \sqrt{3+1} = 2$

And tan θ = 1/√3 → θ = π/6

$$\Rightarrow \int \frac{1}{\sqrt{3}\sin x+\cos x}dx = \int \frac{1}{r\cos\theta\sin x+r\sin\theta\cos x}dx$$

$$= \frac{1}{r}\int \frac{1}{\sin(x+\theta)}dx$$

$$= \frac{1}{r}\int \mathrm{cosec}(x+\theta)dx$$

We know that $\int \mathrm{cosec}\,x\,dx = \log\left|\tan\frac{x}{2}\right| + c$

$$\Rightarrow \frac{1}{r}\int \mathrm{cosec}(x+\theta)dx = \frac{1}{2}\log\left|\tan\left(\frac{x}{2}+\frac{\theta}{2}\right)\right| + c$$

$$= \frac{1}{2}\log\left|\tan\left(\frac{x}{2}+\frac{\pi}{12}\right)\right| + c$$

$$\therefore I = \int \frac{1}{\sqrt{3}\sin x + \cos x} dx = \frac{1}{2}\log\left|\tan\left(\frac{x}{2} + \frac{\pi}{12}\right)\right| + c$$

14. Question

Evaluate the following integrals:

$$\int \frac{1}{\sin x - \sqrt{3}\cos x} dx$$

Answer

Given $I = \int \frac{1}{\sin x - \sqrt{3}\cos x} dx$

Let 1 = r cosθ and √3 = r sinθ

$r = \sqrt{3+1} = 2$

And tan θ = √3 → θ = π/3

$$\Rightarrow \int \frac{1}{\sin x - \sqrt{3}\cos x} dx = \int \frac{1}{r\cos\theta \sin x - r\sin\theta\cos x} dx$$

$$= \frac{1}{r}\int \frac{1}{\sin(x-\theta)} dx$$

$$= \frac{1}{r}\int \operatorname{cosec}(x-\theta) dx$$

We know that $\int \operatorname{cosec} x\, dx = \log\left|\tan\frac{x}{2}\right| + c$

$$\Rightarrow \frac{1}{r}\int \operatorname{cosec}(x-\theta) dx = \frac{1}{2}\log\left|\tan\left(\frac{x}{2} - \frac{\theta}{2}\right)\right| + c$$

$$= \frac{1}{2}\log\left|\tan\left(\frac{x}{2} - \frac{\pi}{6}\right)\right| + c$$

$$\therefore I = \int \frac{1}{\sin x - \sqrt{3}\cos x} dx = \frac{1}{2}\log\left|\tan\left(\frac{x}{2} - \frac{\pi}{6}\right)\right| + c$$

15. Question

Evaluate the following integrals:

$$\int \frac{1}{5 + 7\cos x + \sin x} dx$$

Answer

Given $I = \int \frac{1}{5+7\cos x+\sin x} dx$

We know that $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$\Rightarrow \int \frac{1}{5 + \sin x + 7\cos x} dx = \int \frac{1}{5 + \left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right) + 7\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)} dx$$

$$= \int \frac{1 + \tan^2\frac{x}{2}}{5 + 5\tan^2\frac{x}{2} + 2\tan\frac{x}{2} + 7 - 7\tan^2\frac{x}{2}} dx$$

Replacing 1 + $\tan^2 x/2$ in numerator by $\sec^2 x/2$ and putting tan x/2 = t and $\sec^2$ x/2 dx = 2dt,

$$\Rightarrow \int \frac{1+\tan^2\frac{x}{2}}{5+5\tan^2\frac{x}{2}+2\tan\frac{x}{2}+7-7\tan^2\frac{x}{2}}dx = \int \frac{\sec^2\frac{x}{2}}{-2\tan^2\frac{x}{2}+2\tan\frac{x}{2}+12}dx$$

$$= \int \frac{2dt}{-2t^2+2t+12}$$

$$= -\int \frac{1}{t^2-t-6}dt$$

$$= -\int \frac{1}{\left(t-\frac{1}{2}\right)^2-\frac{5}{2}^2}dt$$

We know that $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$

$$\Rightarrow -\int \frac{1}{\left(t-\frac{1}{2}\right)^2-\frac{5}{2}^2}dt = -\left(\frac{1}{2\left(\frac{5}{2}\right)}\right)\log\left|\frac{t-\frac{1}{2}-\frac{5}{2}}{t-\frac{1}{2}+\frac{5}{2}}\right| + c$$

$$= \frac{-1}{5}\log\left|\frac{\tan\frac{x}{2}-3}{\tan\frac{x}{2}+2}\right| + c$$

$$\therefore I = \int \frac{1}{5+7\cos x+\sin x}dx = \frac{-1}{5}\log\left|\frac{\tan\frac{x}{2}-3}{\tan\frac{x}{2}+2}\right| + c$$

Exercise 19.24

1. Question

Evaluate the integral

$$\int \frac{1}{1-\cot x}dx$$

Answer

Ideas required to solve the problems:

* Integration by substitution: A change in the variable of integration often reduces an integral to one of the fundamental integration. If derivative of a function is present in an integration or if chances of its presence after few modification is possible then we apply integration by substitution method.

* Knowledge of integration of fundamental functions like sin, cos ,polynomial, log etc and formula for some special functions.

Let, I = $\int \frac{1}{1-\cot x}dx$

To solve such integrals involving trigonometric terms in numerator and denominators. We use the basic substitution method and to apply this simply we follow the undermentioned procedure-

If I has the form $\int \frac{a\sin x+b\cos x+c}{d\sin x+e\cos x+f}dx$

Then substitute numerator as -

$$a\sin x+b\cos x+c = A\frac{d}{dx}(d\sin x+e\cos x+f)+B(d\sin x+e\cos x+c)+C$$

Where A, B and C are constants

We have, I = $\int \frac{1}{1-\cot x}dx = \int \frac{1}{1-\frac{\cos x}{\sin x}}dx = \int \frac{\sin x}{\sin x-\cos x}dx$

As I matches with the form described above, So we will take the steps as described.

$\therefore \sin x = A\frac{d}{dx}(\sin x - \cos x) + B(\sin x - \cos x) + C$

$\Rightarrow \sin x = A(\cos x + \sin x) + B(\sin x - \cos x) + C \ \{\because \frac{d}{dx}\cos x = -\sin x\}$

$\Rightarrow \sin x = \sin x\,(B + A) + \cos x\,(A - B) + C$

Comparing both sides we have:

C = 0

A – B = 0 ⇒ A = B

B + A = 1 ⇒ 2A = 1 ⇒ A = 1/2

∴ A = B = 1/2

Thus I can be expressed as:

$I = \int \frac{\frac{1}{2}(\cos x+\sin x)+\frac{1}{2}(\sin x-\cos x)}{\sin x-\cos x}dx$

$I = \int \frac{\frac{1}{2}(\cos x+\sin x)}{\sin x-\cos x}dx + \int \frac{\frac{1}{2}(\sin x-\cos x)}{\sin x-\cos x}dx$

∴ Let $I_1 = \frac{1}{2}\int \frac{(\cos x+\sin x)}{\sin x-\cos x}dx$ and $I_2 = \frac{1}{2}\int \frac{(\sin x-\cos x)}{\sin x-\cos x}dx$

⇒ $I = I_1 + I_2$ ....equation 1

$I_1 = \frac{1}{2}\int \frac{(\cos x+\sin x)}{\sin x-\cos x}dx$

Let, u = sin x – cos x ⇒ du = (cos x + sin x)dx

So, I_1 reduces to:

$I_1 = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\log|u| + C_1$

∴ $I_1 = \frac{1}{2}\log|\sin x - \cos x| + C_1$ .....equation 2

As, $I_2 = \frac{1}{2}\int \frac{(\sin x-\cos x)}{\sin x-\cos x}dx = \frac{1}{2}\int dx$

∴ $I_2 = \frac{x}{2} + C_2$ .....equation 3

From equation 1 ,2 and 3 we have:

$I = \frac{1}{2}\log|\sin x - \cos x| + C_1 + \frac{x}{2} + C_2$

∴ $I = \frac{1}{2}\log|\sin x - \cos x| + \frac{x}{2} + C$

2. Question

Evaluate the integral

$$\int \frac{1}{1-\tan x}dx$$

Answer

Ideas required to solve the problems:

* Integration by substitution: A change in the variable of integration often reduces an integral to one of the fundamental integration. If derivative of a function is present in an integration or if chances of its presence after few modification is possible then we apply integration by substitution method.

* Knowledge of integration of fundamental functions like sin, cos ,polynomial, log etc and formula for some

special functions.

Let, I = $\int \frac{1}{1\text{-}\tan x}dx$

To solve such integrals involving trigonometric terms in numerator and denominators. We use the basic substitution method and to apply this simply we follow the undermentioned procedure-

If I has the form $\int \frac{a\sin x+b\cos x+c}{d\sin x+e\cos x+f}\,dx$

Then substitute numerator as -

$$a\sin x + b\cos x + c = A\frac{d}{dx}(d\sin x + e\cos x + f) + B(d\sin x + e\cos x + c) + C$$

Where A, B and C are constants

We have, I = $\int \frac{1}{1\text{-}\tan x}dx = \int \frac{1}{1-\frac{\sin x}{\cos x}}\,dx = \int \frac{\cos x}{\cos x-\sin x}\,dx$

As I matches with the form described above, So we will take the steps as described.

$\therefore \cos x = A\frac{d}{dx}(\cos x - \sin x) + B(\cos x - \sin x) + C$

$\Rightarrow \cos x = A(-\sin x - \cos x) + B(\cos x - \sin x) + C\ \{\because \frac{d}{dx}\cos x = -\sin x\}$

$\Rightarrow \cos x = -\sin x\,(B + A) + \cos x\,(B - A) + C$

Comparing both sides we have:

C = 0

B – A = 1 ⇒ A = B - 1

B + A = 0 ⇒ 2B - 1 = 0 ⇒ B = 1/2

∴ A = B - 1 = -1/2

Thus I can be expressed as:

I = $\int \frac{\frac{1}{2}(\cos x+\sin x)+\frac{1}{2}(\cos x-\sin x)}{(\cos x-\sin x)}dx$

I = $\int \frac{\frac{1}{2}(\cos x+\sin x)}{(\cos x-\sin x)}dx + \int \frac{\frac{1}{2}(\cos x-\sin x)}{(\cos x-\sin x)}dx$

∴ Let $I_1 = \frac{1}{2}\int \frac{(\cos x+\sin x)}{(\cos x-\sin x)}dx$ and $I_2 = \frac{1}{2}\int \frac{(\cos x-\sin x)}{(\cos x-\sin x)}dx$

⇒ I = $I_1 + I_2$ ....equation 1

$I_1 = \frac{1}{2}\int \frac{(\cos x+\sin x)}{(\cos x-\sin x)}dx$

Let, u = cos x – sin x ⇒ du = -(cos x + sin x)dx

So, I_1 reduces to:

$I_1 = -\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\log|u| + C_1$

∴ $I_1 = -\frac{1}{2}\log|\cos x - \sin x| + C_1$ .....equation 2

As, $I_2 = \frac{1}{2}\int \frac{(\cos x-\sin x)}{(\cos x-\sin x)}dx = \frac{1}{2}\int dx$

∴ $I_2 = \frac{x}{2} + C_2$ .....equation 3

From equation 1 ,2 and 3 we have:

$I = -\frac{1}{2}\log|\cos x - \sin x| + C_1 + \frac{x}{2} + C_2$

$\therefore I = -\frac{1}{2}\log|\cos x - \sin x| + \frac{x}{2} + C$

3. Question

Evaluate the integral

$$\int \frac{3 + 2\cos x + 4\sin x}{2\sin x + \cos x + 3}dx$$

Answer

Ideas required to solve the problems:

* Integration by substitution: A change in the variable of integration often reduces an integral to one of the fundamental integration. If derivative of a function is present in an integration or if chances of its presence after few modification is possible then we apply integration by substitution method.

* Knowledge of integration of fundamental functions like sin, cos ,polynomial, log etc and formula for some special functions.

Let, $I = \int \frac{3+2\cos x+4\sin x}{2\sin x+\cos x+3}dx$

To solve such integrals involving trigonometric terms in numerator and denominators. We use the basic substitution method and to apply this simply we follow the undermentioned procedure-

If I has the form $\int \frac{a\sin x+b\cos x+c}{d\sin x+e\cos x+f}dx$

Then substitute numerator as -

$$a\sin x + b\cos x + c = A\frac{d}{dx}(d\sin x + e\cos x + f) + B(d\sin x + e\cos x + c) + C$$

Where A, B and C are constants

We have, $I = \int \frac{3+2\cos x+4\sin x}{2\sin x+\cos x+3}dx$

As I matches with the form described above, So we will take the steps as described.

$\therefore 3 + 2\cos x + 4\sin x = A\frac{d}{dx}(2\sin x + \cos x + 3) + B(2\sin x + \cos x + 3) + C$

$\Rightarrow 3 + 2\cos x + 4\sin x = A(2\cos x - \sin x) + B(2\sin x + \cos x + 3) + C \quad \{\because \frac{d}{dx}\cos x = -\sin x\}$

$\Rightarrow 3 + 2\cos x + 4\sin x = \sin x\,(2B - A) + \cos x\,(B + 2A) + 3B + C$

Comparing both sides we have:

3B+ C = 3

B + 2A = 2

2B - A = 4

On solving for A ,B and C we have:

A = 0, B = 2 and C = -3

Thus I can be expressed as:

$I = \int \frac{2(2\sin x+\cos x+3)-3}{2\sin x+\cos x+3}dx$

$I = \int \frac{2(2\sin x+\cos x+3)}{2\sin x+\cos x+3}dx + \int \frac{-3}{2\sin x+\cos x+3}dx$

$\therefore$ Let $I_1 = 2\int \frac{(2\sin x+\cos x+3)}{2\sin x+\cos x+3}dx$ and $I_2 = -3\int \frac{1}{2\sin x+\cos x+3}dx$

$\Rightarrow I = I_1 + I_2$ ....equation 1

$I_1 = 2\int \frac{(2\sin x+\cos x+3)}{2\sin x+\cos x+3}dx$

So, I_1 reduces to:

$I_1 = 2\int dx = 2x + C_1$ .....equation 2

As, $I_2 = -3\int \frac{1}{2\sin x+\cos x+3}dx$

To solve the integrals of the form $\int \frac{1}{a\sin x+b\cos x+c}dx$

To apply substitution method we take following procedure.

We substitute:

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} \text{ and } \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

$\therefore I_2 = -3\int \frac{1}{2\sin x+\cos x+3}dx$

$\Rightarrow I_2 = -3\int \frac{1}{2\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)+3\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)+3}dx$

$\Rightarrow I_2 = -3\int \frac{1+\tan^2\frac{x}{2}}{4\tan\frac{x}{2}+1-\tan^2\frac{x}{2}+3(1+\tan^2\frac{x}{2})}dx$

$\Rightarrow I_2 = -3\int \frac{\sec^2\frac{x}{2}}{2(2\tan\frac{x}{2}+2+1\tan^2\frac{x}{2})}dx$

Let, $t = \tan\frac{x}{2} \Rightarrow dt = \frac{1}{2}\sec^2\frac{x}{2}dx$

$\therefore I_2 = -3\int \frac{1}{(2t+2+t^2)}dt$

As, the denominator is polynomial without any square root term. So one of the special integral will be used to solve I_2.

$I_2 = -3\int \frac{1}{(2t+2+t^2)}dt$

$\Rightarrow I_2 = -3\int \frac{1}{(t^2+2(1)t+1)+1}dt$

$\therefore I_2 = -3\int \frac{1}{(t+1)^2+1}dt$ $\{\because a^2 + 2ab + b^2 = (a+b)^2\}$

As, I_2 matches with the special integral form

$$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$I_2 = -3\tan^{-1}(t+1)$

Putting value of t we have:

$\therefore I_2 = -3\tan^{-1}\left(\tan\frac{x}{2}+1\right) + C_2$ ......equation 3

From equation 1,2 and 3:

$I = 2x + C_1 - 3\tan^{-1}\left(\tan\frac{x}{2}+1\right) + C_2$

$\therefore$ I = $2x - 3\tan^{-1}\left(\tan\frac{x}{2} + 1\right)$ + C ....ans

4. Question

Evaluate the integral

$$\int \frac{1}{p + q\tan x}dx$$

Answer

Ideas required to solve the problems:

* Integration by substitution: A change in the variable of integration often reduces an integral to one of the fundamental integration. If derivative of a function is present in an integration or if chances of its presence after few modification is possible then we apply integration by substitution method.

* Knowledge of integration of fundamental functions like sin, cos ,polynomial, log etc and formula for some special functions.

Let, I = $\int \frac{1}{p+q\tan x}dx$

To solve such integrals involving trigonometric terms in numerator and denominators. We use the basic substitution method and to apply this simply we follow the undermentioned procedure-

If I has the form $\int \frac{a\sin x+b\cos x+c}{d\sin x+e\cos x+f}dx$

Then substitute numerator as -

$$a\sin x + b\cos x + c = A\frac{d}{dx}(d\sin x + e\cos x + f) + B(d\sin x + e\cos x + c) + C$$

Where A, B and C are constants

We have, I = $\int \frac{1}{p+q\tan x}dx = \int \frac{1}{p+q\frac{\sin x}{\cos x}}dx = \int \frac{\cos x}{p\cos x+q\sin x}dx$

As I matches with the form described above, So we will take the steps as described.

$\therefore \cos x = A\frac{d}{dx}(p\cos x + q\sin x) + B(p\cos x + q\sin x) + C$

$\Rightarrow \cos x = A(-p\sin x + q\cos x) + B(p\cos x - q\sin x) + C \ \{\because \frac{d}{dx}\cos x = -\sin x\}$

$\Rightarrow \cos x = -\sin x\,(Bq + Ap) + \cos x\,(Bp + Aq) + C$

Comparing both sides we have:

C = 0

Bp + Aq = 1

Bq + Ap = 0

On solving above equations, we have:

A = $\frac{q}{p^2+q^2}$ B = $\frac{p}{p^2+q^2}$ and C = 0

Thus I can be expressed as:

I = $\int \frac{\frac{q}{p^2+q^2}(-p\sin x+q\sin x)+\frac{p}{p^2+q^2}(p\cos x+q\sin x)}{(p\cos x+q\sin x)}dx$

I = $\int \frac{\frac{q}{p^2+q^2}(-p\sin x+q\sin x)}{(p\cos x+q\sin x)}dx + \int \frac{\frac{p}{p^2+q^2}(p\cos x+q\sin x)}{(p\cos x+q\sin x)}dx$

$\therefore$ Let $I_1 = \frac{q}{p^2+q^2}\int \frac{(-p\sin x+q\sin x)}{(p\cos x+q\sin x)}dx$ and $I_2 = \frac{p}{p^2+q^2}\int \frac{(p\cos x+q\sin x)}{(p\cos x+q\sin x)}dx$

$\Rightarrow I = I_1 + I_2$ ....equation 1

$I_1 = \frac{q}{p^2+q^2}\int \frac{(-p\sin x+q\sin x)}{(p\cos x+q\sin x)}dx$

Let, u = pcos x + qsin x $\Rightarrow$ du = (-psin x + qcos x)dx

So, I_1 reduces to:

$I_1 = \frac{q}{p^2+q^2}\int \frac{du}{u} = \frac{q}{p^2+q^2}\log|u| + C_1$

$\therefore I_1 = \frac{q}{p^2+q^2}\log|(p\cos x + q\sin x)| + C_1$ .....equation 2

As, $I_2 = \frac{p}{p^2+q^2}\int \frac{(p\cos x+q\sin x)}{(p\cos x+q\sin x)}dx = \frac{p}{p^2+q^2}\int dx$

$\therefore I_2 = \frac{px}{p^2+q^2} + C_2$ .....equation 3

From equation 1 ,2 and 3 we have:

$I = \frac{q}{p^2+q^2}\log|(p\cos x + q\sin x)| + C_1 + \frac{px}{p^2+q^2} + C_2$

$\therefore I = \frac{q}{p^2+q^2}\log|(p\cos x + q\sin x)| + \frac{px}{p^2+q^2} + C$

5. Question

Evaluate the integral

$$\int \frac{5\cos x + 6}{2\cos x + \sin x + 3}dx$$

Answer

Ideas required to solve the problems:

* Integration by substitution: A change in the variable of integration often reduces an integral to one of the fundamental integration. If derivative of a function is present in an integration or if chances of its presence after few modification is possible then we apply integration by substitution method.

* Knowledge of integration of fundamental functions like sin, cos ,polynomial, log etc and formula for some special functions.

Let, $I = \int \frac{5\cos x+6}{2\cos x+\sin x+3}dx$

To solve such integrals involving trigonometric terms in numerator and denominators. We use the basic substitution method and to apply this simply we follow the undermentioned procedure-

If I has the form $\int \frac{a\sin x+b\cos x+c}{d\sin x+e\cos x+f}dx$

Then substitute numerator as -

$$a\sin x + b\cos x + c = A\frac{d}{dx}(d\sin x + e\cos x + f) + B(d\sin x + e\cos x + c) + C$$

Where A, B and C are constants

We have, $I = \int \frac{5\cos x+6}{2\cos x+\sin x+3}dx$

As I matches with the form described above, So we will take the steps as described.

$\therefore 5\cos x + 6 = A\frac{d}{dx}(2\cos x + \sin x + 3) + B(2\cos x + \sin x + 3) + C$

$\Rightarrow 5\cos x + 6 = A(-2\sin x + \cos x) + B(2\cos x + \sin x + 3) + C \quad \{\because \frac{d}{dx}\cos x = -\sin x\}$

$\Rightarrow 5\cos x + 6 = \sin x\,(B - 2A) + \cos x\,(2B + A) + 3B + C$

Comparing both sides we have:

3B+ C = 6

2B + A = 5

B - 2A = 0

On solving for A ,B and C we have:

A = 1, B = 2 and C = 0

Thus I can be expressed as:

$I = \int \frac{(-2\sin x+\cos x)+2(2\cos x+\sin x+3)}{2\cos x+\sin x+3} dx$

$I = \int \frac{(-2\sin x+\cos x)}{2\cos x+\sin x+3} dx + \int \frac{2(2\cos x+\sin x+3)}{2\cos x+\sin x+3} dx$

$\therefore$ Let $I_1 = \int \frac{(-2\sin x+\cos x)}{2\cos x+\sin x+3} dx$ and $I_2 = \int \frac{2(2\cos x+\sin x+3)}{2\cos x+\sin x+3} dx$

$\Rightarrow I = I_1 + I_2$ ....equation 1

$I_1 = \int \frac{(-2\sin x+\cos x)}{2\cos x+\sin x+3} dx$

Let, 2 cos x + sin x + 3 = u

⇒ (-2sin x + cos x)dx = du

So, I_1 reduces to:

$I_1 = \int \frac{du}{u} = \log|u| + C_1$

$\therefore I_1 = \log|2\cos x + \sin x + 3| + C_1$ .....equation 2

As, $I_2 = \int \frac{2(2\cos x+\sin x+3)}{2\cos x+\sin x+3} dx$

$\Rightarrow I_2 = 2\int dx = 2x + C_2$ .....equation 3

From equation 1, 2 and 3 we have:

$I = \log|2\cos x + \sin x + 3| + C_1 + 2x + C_2$

$\therefore I = \log|2\cos x + \sin x + 3| + 2x + C$

6. Question

Evaluate the integral

$$\int \frac{2\sin x + 3\cos x}{3\sin x + 4\cos x} dx$$

Answer

Ideas required to solve the problems:

* Integration by substitution: A change in the variable of integration often reduces an integral to one of the fundamental integration. If derivative of a function is present in an integration or if chances of its presence after few modification is possible then we apply integration by substitution method.

* Knowledge of integration of fundamental functions like sin, cos ,polynomial, log etc and formula for some special functions.

Let, I = $\int \frac{2\sin x+3\cos x}{4\cos x+3\sin x}dx$

To solve such integrals involving trigonometric terms in numerator and denominators. We use the basic substitution method and to apply this simply we follow the undermentioned procedure-

If I has the form $\int \frac{a\sin x+b\cos x+c}{d\sin x+e\cos x+f}\,dx$

Then substitute numerator as -

$$a\sin x+b\cos x+c = A\frac{d}{dx}(d\sin x+e\cos x+f)+B(d\sin x+e\cos x+c)+C$$

Where A, B and C are constants

We have, I = $\int \frac{2\sin x+3\cos x}{4\cos x+3\sin x}dx$

As I matches with the form described above, So we will take the steps as described.

$\therefore 2\sin x+\ 3\cos x = A\frac{d}{dx}(3\sin x+4\cos x)+B(4\cos x+3\sin x)+C$

$\Rightarrow 2\sin x+3\cos x = A(3\cos x-4\sin x)+B(4\cos x+3\sin x)+C \quad \{\because \frac{d}{dx}\cos x = -\sin x\}$

$\Rightarrow 2\sin x+3\cos x = \sin x\,(3B-4A)+\cos x\,(3A+4B)+C$

Comparing both sides we have:

C = 0

3B - 4A = 2

4B + 3A = 3

On solving for A ,B and C we have:

A = 1/25 , B = 18/25 and C = 0

Thus I can be expressed as:

I = $\int \frac{\frac{1}{25}(3\cos x-4\sin x)+\frac{18}{25}(4\cos x+3\sin x)}{4\cos x+3\sin x}dx$

I = $\int \frac{\frac{1}{25}(3\cos x-4\sin x)}{4\cos x+3\sin x}dx+\int \frac{\frac{18}{25}(4\cos x+3\sin x)}{4\cos x+3\sin x}dx$

$\therefore$ Let $I_1 = \frac{1}{25}\int \frac{(3\cos x-4\sin x)}{4\cos x+3\sin x}dx$ and $I_2 = \frac{18}{25}\int \frac{(4\cos x+3\sin x)}{4\cos x+3\sin x}dx$

$\Rightarrow I = I_1 + I_2$ ....equation 1

$I_1 = \frac{1}{25}\int \frac{(3\cos x-4\sin x)}{4\cos x+3\sin x}dx$

Let, 4 cos x + 3sin x = u

⇒ (-4sin x + 3cos x)dx = du

So, I_1 reduces to:

$I_1 = \frac{1}{25}\int \frac{du}{u} = \frac{1}{25}\log|u| + C_1$

$\therefore I_1 = \frac{1}{25}\log|4\cos x+3\sin x| + C_1$ .....equation 2

As, $I_2 = \frac{18}{25}\int \frac{(4\cos x+3\sin x)}{4\cos x+3\sin x}dx$

$\Rightarrow I_2 = \frac{18}{25}\int dx = \frac{18x}{25} + C_2$ .....equation 3

From equation 1, 2 and 3 we have:

$I = \frac{1}{25}\log|4\cos x + 3\sin x| + C_1 + \frac{18x}{25} + C_2$

$\therefore I = \frac{1}{25}\log|4\cos x + 3\sin x| + \frac{18x}{25} + C$

7. Question

Evaluate the integral

$$\int \frac{1}{3+4\cot x}dx$$

Answer

Ideas required to solve the problems:

* Integration by substitution: A change in the variable of integration often reduces an integral to one of the fundamental integration. If derivative of a function is present in an integration or if chances of its presence after few modification is possible then we apply integration by substitution method.

* Knowledge of integration of fundamental functions like sin, cos ,polynomial, log etc and formula for some special functions.

Let, $I = \int \frac{1}{3+4\cot x}dx$

To solve such integrals involving trigonometric terms in numerator and denominators. We use the basic substitution method and to apply this simply we follow the undermentioned procedure-

If I has the form $\int \frac{a\sin x + b\cos x + c}{d\sin x + e\cos x + f}dx$

Then substitute numerator as -

$$a\sin x + b\cos x + c = A\frac{d}{dx}(d\sin x + e\cos x + f) + B(d\sin x + e\cos x + c) + C$$

Where A, B and C are constants

We have, $I = \int \frac{1}{3+4\cot x}dx = \int \frac{1}{3+4\frac{\cos x}{\sin x}}dx = \int \frac{\sin x}{3\sin x + 4\cos x}dx$

As I matches with the form described above, So we will take the steps as described.

$\therefore \sin x = A\frac{d}{dx}(3\sin x + 4\cos x) + B(4\cos x + 3\sin x) + C$

$\Rightarrow \sin x = A(3\cos x - 4\sin x) + B(4\cos x + 3\sin x) + C \quad \{\because \frac{d}{dx}\cos x = -\sin x\}$

$\Rightarrow \sin x = \sin x\,(3B - 4A) + \cos x\,(3A + 4B) + C$

Comparing both sides we have:

C = 0

3B - 4A = 1

4B + 3A = 0

On solving for A ,B and C we have:

A = -4/25 , B = 3/25 and C = 0

Thus I can be expressed as:

$I = \int \frac{\frac{-4}{25}(3\cos x - 4\sin x) + \frac{3}{25}(4\cos x + 3\sin x)}{4\cos x + 3\sin x}dx$

$I = \int \frac{\frac{-4}{25}(3\cos x - 4\sin x)}{4\cos x + 3\sin x} dx + \int \frac{\frac{3}{25}(4\cos x + 3\sin x)}{4\cos x + 3\sin x} dx$

$\therefore$ Let $I_1 = -\frac{4}{25}\int \frac{(3\cos x - 4\sin x)}{4\cos x + 3\sin x} dx$ and $I_2 = \frac{3}{25}\int \frac{(4\cos x + 3\sin x)}{4\cos x + 3\sin x} dx$

$\Rightarrow I = I_1 + I_2$ ....equation 1

$I_1 = -\frac{4}{25}\int \frac{(3\cos x - 4\sin x)}{4\cos x + 3\sin x} dx$

Let, 4 cos x + 3sin x = u

⇒ (-4sin x + 3cos x)dx = du

So, I_1 reduces to:

$I_1 = -\frac{4}{25}\int \frac{du}{u} = \frac{-4}{25}\log|u| + C_1$

$\therefore I_1 = -\frac{4}{25}\log|4\cos x + 3\sin x| + C_1$ .....equation 2

As, $I_2 = \frac{3}{25}\int \frac{(4\cos x + 3\sin x)}{4\cos x + 3\sin x} dx$

$\Rightarrow I_2 = \frac{3}{25}\int dx = \frac{3x}{25} + C_2$ .....equation 3

From equation 1, 2 and 3 we have:

$I = -\frac{4}{25}\log|4\cos x + 3\sin x| + C_1 + \frac{3x}{25} + C_2$

$\therefore I = -\frac{4}{25}\log|4\cos x + 3\sin x| + \frac{3x}{25} + C$

8. Question

Evaluate the integral

$$\int \frac{2\tan x + 3}{3\tan x + 4} dx$$

Answer

Ideas required to solve the problems:

* Integration by substitution: A change in the variable of integration often reduces an integral to one of the fundamental integration. If derivative of a function is present in an integration or if chances of its presence after few modification is possible then we apply integration by substitution method.

* Knowledge of integration of fundamental functions like sin, cos ,polynomial, log etc and formula for some special functions.

Let, $I = \int \frac{2\tan x + 3}{3\tan x + 4} dx$

To solve such integrals involving trigonometric terms in numerator and denominators. We use the basic substitution method and to apply this simply we follow the undermentioned procedure-

If I has the form $\int \frac{a\sin x + b\cos x + c}{d\sin x + e\cos x + f} dx$

Then substitute numerator as -

$$a\sin x + b\cos x + c = A\frac{d}{dx}(d\sin x + e\cos x + f) + B(d\sin x + e\cos x + c) + C$$

Where A, B and C are constants

We have, $I = \int \frac{2\tan x + 3}{3\tan x + 4} dx = \int \frac{2\frac{\sin x}{\cos x} + 3}{3\frac{\sin x}{\cos x} + 4} = \int \frac{2\sin x + 3\cos x}{4\cos x + 3\sin x} dx$

As I matches with the form described above, So we will take the steps as described.

$\therefore 2\sin x + 3\cos x = A\frac{d}{dx}(3\sin x + 4\cos x) + B(4\cos x + 3\sin x) + C$

$\Rightarrow 2\sin x + 3\cos x = A(3\cos x - 4\sin x) + B(4\cos x + 3\sin x) + C \quad \{\because \frac{d}{dx}\cos x = -\sin x\}$

$\Rightarrow 2\sin x + 3\cos x = \sin x\,(3B - 4A) + \cos x\,(3A + 4B) + C$

Comparing both sides we have:

C = 0

3B - 4A = 2

4B + 3A = 3

On solving for A ,B and C we have:

A = 1/25 , B = 18/25 and C = 0

Thus I can be expressed as:

$I = \int \frac{\frac{1}{25}(3\cos x - 4\sin x) + \frac{18}{25}(4\cos x + 3\sin x)}{4\cos x + 3\sin x}dx$

$I = \int \frac{\frac{1}{25}(3\cos x - 4\sin x)}{4\cos x + 3\sin x}dx + \int \frac{\frac{18}{25}(4\cos x + 3\sin x)}{4\cos x + 3\sin x}dx$

$\therefore$ Let $I_1 = \frac{1}{25}\int \frac{(3\cos x - 4\sin x)}{4\cos x + 3\sin x}dx$ and $I_2 = \frac{18}{25}\int \frac{(4\cos x + 3\sin x)}{4\cos x + 3\sin x}dx$

$\Rightarrow I = I_1 + I_2$ ....equation 1

$I_1 = \frac{1}{25}\int \frac{(3\cos x - 4\sin x)}{4\cos x + 3\sin x}dx$

Let, 4 cos x + 3sin x = u

⇒ (-4sin x + 3cos x)dx = du

So, I_1 reduces to:

$I_1 = \frac{1}{25}\int \frac{du}{u} = \frac{1}{25}\log|u| + C_1$

$\therefore I_1 = \frac{1}{25}\log|4\cos x + 3\sin x| + C_1$ .....equation 2

As, $I_2 = \frac{18}{25}\int \frac{(4\cos x + 3\sin x)}{4\cos x + 3\sin x}dx$

$\Rightarrow I_2 = \frac{18}{25}\int dx = \frac{18x}{25} + C_2$ .....equation 3

From equation 1, 2 and 3 we have:

$I = \frac{1}{25}\log|4\cos x + 3\sin x| + C_1 + \frac{18x}{25} + C_2$

$\therefore I = \frac{1}{25}\log|4\cos x + 3\sin x| + \frac{18x}{25} + C$

9. Question

Evaluate the integral

$$\int \frac{1}{4 + 3\tan x}dx$$

Answer

Ideas required to solve the problems:

* Integration by substitution: A change in the variable of integration often reduces an integral to one of the fundamental integration. If derivative of a function is present in an integration or if chances of its presence after few modification is possible then we apply integration by substitution method.

* Knowledge of integration of fundamental functions like sin, cos ,polynomial, log etc and formula for some special functions.

Let, $I = \int \frac{1}{4+3\tan x} dx$

To solve such integrals involving trigonometric terms in numerator and denominators. We use the basic substitution method and to apply this simply we follow the undermentioned procedure-

If I has the form $\int \frac{a\sin x+b\cos x+c}{d\sin x+e\cos x+f} dx$

Then substitute numerator as -

$$a\sin x + b\cos x + c = A\frac{d}{dx}(d\sin x + e\cos x + f) + B(d\sin x + e\cos x + c) + C$$

Where A, B and C are constants

We have, $I = \int \frac{1}{4+3\tan x} dx = \int \frac{1}{4+3\frac{\sin x}{\cos x}} dx = \int \frac{\cos x}{3\sin x+4\cos x} dx$

As I matches with the form described above, So we will take the steps as described.

$\therefore \cos x = A\frac{d}{dx}(3\sin x + 4\cos x) + B(4\cos x + 3\sin x) + C$

$\Rightarrow \cos x = A(3\cos x - 4\sin x) + B(4\cos x + 3\sin x) + C \quad \{\because \frac{d}{dx}\cos x = -\sin x\}$

$\Rightarrow \cos x = \sin x\,(3B - 4A) + \cos x\,(3A + 4B) + C$

Comparing both sides we have:

C = 0

3B - 4A = 0

4B + 3A = 1

On solving for A ,B and C we have:

A = 3/25 , B = 4/25 and C = 0

Thus I can be expressed as:

$I = \int \frac{\frac{3}{25}(3\cos x-4\sin x)+\frac{4}{25}(4\cos x+3\sin x)}{4\cos x+3\sin x} dx$

$I = \int \frac{\frac{3}{25}(3\cos x-4\sin x)}{4\cos x+3\sin x} dx + \int \frac{\frac{4}{25}(4\cos x+3\sin x)}{4\cos x+3\sin x} dx$

$\therefore$ Let $I_1 = \frac{3}{25}\int \frac{(3\cos x-4\sin x)}{4\cos x+3\sin x} dx$ and $I_2 = \frac{4}{25}\int \frac{(4\cos x+3\sin x)}{4\cos x+3\sin x} dx$

$\Rightarrow I = I_1 + I_2$ ....equation 1

$I_1 = \frac{3}{25}\int \frac{(3\cos x-4\sin x)}{4\cos x+3\sin x} dx$

Let, 4 cos x + 3sin x = u

$\Rightarrow$ (-4sin x + 3cos x)dx = du

So, I_1 reduces to:

$I_1 = \frac{3}{25}\int \frac{du}{u} = \frac{3}{25}\log|u| + C_1$

$\therefore I_1 = \frac{3}{25}\log|4\cos x + 3\sin x| + C_1$ .....equation 2

As, $I_2 = \frac{4}{25}\int \frac{(4\cos x + 3\sin x)}{4\cos x + 3\sin x}dx$

$\Rightarrow I_2 = \frac{4}{25}\int dx\ = \frac{3x}{25} + C_2$ .....equation 3

From equation 1, 2 and 3 we have:

$I = \frac{3}{25}\log|4\cos x + 3\sin x| + C_1 + \frac{4x}{25} + C_2$

$\therefore I = \frac{3}{25}\ \log|4\cos x + 3\sin x| + \frac{4x}{25} + C$

10. Question

Evaluate the integral

$$\int \frac{8\cot x + 1}{3\cot x + 2}dx$$

Answer

Ideas required to solve the problems:

* Integration by substitution: A change in the variable of integration often reduces an integral to one of the fundamental integration. If derivative of a function is present in an integration or if chances of its presence after few modification is possible then we apply integration by substitution method.

* Knowledge of integration of fundamental functions like sin, cos ,polynomial, log etc and formula for some special functions.

Let, $I = \int \frac{8\cot x + 1}{3\cot x + 2}dx$

To solve such integrals involving trigonometric terms in numerator and denominators. We use the basic substitution method and to apply this simply we follow the undermentioned procedure-

If I has the form $\int \frac{a\sin x + b\cos x + c}{d\sin x + e\cos x + f}\ dx$

Then substitute numerator as -

$$a\sin x + b\cos x + c = A\frac{d}{dx}\ (d\sin x + e\ \cos x + f) + B(d\sin x + e\ \cos x + c) + C$$

Where A, B and C are constants

We have, $I = \int \frac{8\cot x + 1}{3\cot x + 2}dx = \int \frac{8\frac{\cos x}{\sin x} + 1}{3\frac{\cos x}{\sin x} + 2} = \ \int \frac{8\cos x + \sin x}{3\cos x + 2\sin x}dx$

As I matches with the form described above, So we will take the steps as described.

$\therefore \sin x +\ 8\cos x = A\frac{d}{dx}(3\cos x + 2\sin x) + B(3\cos x + 2\sin x) + C$

$\Rightarrow \sin x + 8\cos x = A(-3\sin x + 2\cos x) + B(3\cos x + 2\sin x) + C\ \ \{\because \frac{d}{dx}\cos x =\ -\sin x\}$

$\Rightarrow \sin x + 8\cos x = \sin x\ (\ 2B - 3A) + \cos x\ (2A + 3B) + C$

Comparing both sides we have:

C = 0

2B - 3A = 1

3B + 2A = 8

On solving for A ,B and C we have:

A = 1 , B = 2 and C = 0

Thus I can be expressed as:

$I = \int \frac{(-3\sin x+2\cos x)+2(3\cos x+2\sin x)}{3\cos x+2\sin x} dx$

$I = \int \frac{(-3\sin x+2\cos x)}{3\cos x+2\sin x} dx + \int \frac{2(3\cos x+2\sin x)}{3\cos x+2\sin x} dx$

∴ Let $I_1 = \int \frac{(-3\sin x+2\cos x)}{3\cos x+2\sin x} dx$ and $I_2 = \int \frac{2(3\cos x+2\sin x)}{3\cos x+2\sin x} dx$

⇒ $I = I_1 + I_2$ ....equation 1

$I_1 = \int \frac{(-3\sin x+2\cos x)}{3\cos x+2\sin x} dx$

Let, 3 cos x + 2 sin x = u

⇒ (-3sin x + 2cos x)dx = du

So, I_1 reduces to:

$I_1 = \int \frac{du}{u} = \log|u| + C_1$

∴ $I_1 = \log|3\cos x + 2\sin x| + C_1$ .....equation 2

As, $I_2 = \int \frac{2(3\cos x+2\sin x)}{3\cos x+2\sin x} dx$

⇒ $I_2 = 2\int dx = 2x + C_2$ .....equation 3

From equation 1, 2 and 3 we have:

$I = \frac{1}{25} \log|3\cos x + 2\sin x| + C_1 + 2x + C_2$

∴ $I = \frac{1}{25}\log|3\cos x + 2\sin x| + 2x + C$

11. Question

Evaluate the integral

$$\int \frac{4\sin x + 5\cos x}{5\sin x + 4\cos x} dx$$

Answer

Ideas required to solve the problems:

* Integration by substitution: A change in the variable of integration often reduces an integral to one of the fundamental integration. If derivative of a function is present in an integration or if chances of its presence after few modification is possible then we apply integration by substitution method.

* Knowledge of integration of fundamental functions like sin, cos ,polynomial, log etc and formula for some special functions.

Let, $I = \int \frac{4\sin x+5\cos x}{5\sin x+4\cos x} dx$

To solve such integrals involving trigonometric terms in numerator and denominators. We use the basic substitution method and to apply this simply we follow the undermentioned procedure-

If I has the form $\int \frac{a\sin x+b\cos x+c}{d\sin x+e\cos x+f} dx$

Then substitute numerator as -

$$a\sin x + b\cos x + c = A\frac{d}{dx}(d\sin x + e\cos x + f) + B(d\sin x + e\cos x + c) + C$$

Where A, B and C are constants

We have, I = $\int \frac{4\sin x+5\cos x}{5\sin x+4\cos x}dx$

As I matches with the form described above, So we will take the steps as described.

$\therefore 4\sin x + 5\cos x = A\frac{d}{dx}(5\sin x + 4\cos x) + B(4\cos x + 5\sin x) + C$

$\Rightarrow 4\sin x + 5\cos x = A(5\cos x - 4\sin x) + B(4\cos x + 5\sin x) + C \quad \{\because \frac{d}{dx}\cos x = -\sin x\}$

$\Rightarrow 4\sin x + 5\cos x = \sin x\,(5B - 4A) + \cos x\,(5A + 4B) + C$

Comparing both sides we have:

C = 0

5B - 4A = 4

4B + 5A = 5

On solving for A ,B and C we have:

A =9/41, B = 40/41 and C = 0

Thus I can be expressed as:

I = $\int \frac{\frac{9}{41}(5\cos x-4\sin x)+\frac{40}{41}(4\cos x+5\sin x)}{4\cos x+5\sin x}dx$

I = $\int \frac{\frac{9}{41}(5\cos x-4\sin x)}{4\cos x+5\sin x}dx + \int \frac{\frac{40}{41}(4\cos x+5\sin x)}{4\cos x+5\sin x}dx$

$\therefore$ Let $I_1 = \frac{9}{41}\int \frac{(5\cos x-4\sin x)}{4\cos x+5\sin x}$ and $I_2 = \frac{40}{41}\int \frac{(4\cos x+5\sin x)}{4\cos x+5\sin x}dx$

$\Rightarrow I = I_1 + I_2$ ....equation 1

$I_1 = \frac{9}{41}\int \frac{(5\cos x-4\sin x)}{4\cos x+5\sin x}$

Let, 4 cos x + 5sin x = u

⇒ (-4sin x + 5cos x)dx = du

So, I_1 reduces to:

$I_1 = \frac{9}{41}\int \frac{du}{u} = \frac{9}{41}\log|u| + C_1$

$\therefore I_1 = \frac{9}{41}\log|4\cos x + 5\sin x| + C_1$ .....equation 2

As, $I_2 = \frac{40}{41}\int \frac{(4\cos x+5\sin x)}{4\cos x+5\sin x}dx$

$\Rightarrow I_2 = \frac{40}{41}\int dx = \frac{40x}{41} + C_2$ .....equation 3

From equation 1, 2 and 3 we have:

$I = \frac{9}{41}\log|4\cos x + 5\sin x| + C_1 + \frac{40x}{41} + C_2$

$\therefore I = \frac{9}{41}\log|4\cos x + 5\sin x| + \frac{40x}{41} + C$

Exercise 19.25

1. Question

Evaluate the following integrals:

$\int$ x cos x dx

Answer

Let $I = \int x \cos x \, dx$

We know that, $\int UV = U \int Vdv - \int \frac{d}{dx} U \int Vdv$

Using integration by parts,

$$I = x \int \cos x \, dx - \int \frac{d}{dx} x \int \cos x \, dx I = \int x \cos x \, dx$$

We have, $\int \sin x = -\cos x, \int \cos x = \sin x$

$$= x \times \sin x — \int \sin x \, dx$$

$$= x\sin x + \cos x + c$$

2. Question

Evaluate the following integrals:

$\int$ log (x + 1) dx

Answer

Let $I = \int \log(x + 1) \, dx$

That is,

$$I = \int 1 \times \log(x + 1) \, dx$$

Using integration by parts,

$$I = \log(x + 1) \int 1 \, dx - \int \frac{d}{dx} \log(x + 1) \int 1 \, dx$$

We know that, $\int 1 \, dx = x$ and $\int \log x = \frac{1}{x}$

$$= \log(x + 1) \times x - \int \frac{1}{x + 1} \times x$$

$$\frac{x}{x + 1} = 1 - \frac{1}{x + 1}$$

$$= x \log(x + 1) - \int \left(1 - \frac{1}{x + 1}\right) dx$$

$$= x \log(x + 1) - x + \log(x + 1) + c$$

3. Question

Evaluate the following integrals:

$\int x^3$ log x dx

Answer

Let $I = \int x^3 \log x \, dx$

Using integration by parts,

$$I = \log x \int x^3 \, dx - \int \frac{d}{dx} \log x \int x^3 \, dx$$

We have, $\int x^n dx = \frac{x^{n+1}}{n+1}$ and $\int \log x = \frac{1}{x}$

$$= \log x \times \frac{x^4}{4} - \int \frac{1}{x} \times \frac{x^4}{4}$$

$$= \log x \times \frac{x^4}{4} - \frac{1}{4}\int x^3 dx$$

$$= \frac{x^4}{4}\log x - \frac{1}{4} \times \frac{x^4}{4}$$

$$= \frac{x^4}{4}\log x - \frac{x^4}{16} + c$$

4. Question

Evaluate the following integrals:

$\int xe^x\, dx$

Answer

Let $I = \int xe^x dx$

Using integration by parts,

$$I = x\int e^x dx - \int \frac{d}{dx}x \int e^x dx$$

We know that , $\int e^x dx = e^x$ and $\frac{d}{dx}x = 1$

$$= xe^x - \int e^x dx$$

$$= xe^x - e^x + c$$

5. Question

Evaluate the following integrals:

$\int xe^{2x}\, dx$

Answer

Let $I = \int xe^{2x} dx$

Using integration by parts,

$$I = x\int e^{2x} dx - \int \frac{d}{dx}x \int e^{2x} dx$$

We know that , $\int e^{nx} dx = \frac{e^x}{n}$ and $\frac{d}{dx}x = 1$

$$= \frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2} dx$$

$$= \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + c$$

$$I = \left(\frac{x}{2} - \frac{1}{4}\right)e^{2x} + c$$

6. Question

Evaluate the following integrals:

$\int x^2 e^{-x} dx$

Answer

Let $I = \int x^2 e^{-x} dx$

Using integration by parts,

$$= x^2 \int e^{-x} dx - \int \frac{d}{dx} x^2 \int e^{-x} dx$$

We know that, $\int e^{nx} dx = \frac{e^x}{n}$ and $\frac{d}{dx} x^n = nx^{n-1}$

$$= x^2 \times -e^{-x} - \int 2x \times - e^{-x} dx$$

Using integration by parts in second integral, $= -x^2 e^{-x} + 2\left(x \int e^{-x} dx - \int \frac{d}{dx} x \int e^{-x} dx\right)$

$$= -x^2 e^{-x} + 2(-xe^{-x} + (-e^{-x})) + c$$

$$= -x^2 e^{-x} + 2(-xe^{-x} - e^{-x}) + c$$

$$I = -e^{-x}(x^2 + 2x + 2) + c$$

7. Question

Evaluate the following integrals:

$\int x^2 \cos x \, dx$

Answer

Let $I = \int x^2 \cos x \, dx$

Using integration by parts,

$$= x^2 \int \cos x \, dx - \int \frac{d}{dx} x^2 \int \cos x \, dx$$

We know that, $\int \cos x \, dx = \sin x$ and $\frac{d}{dx} x^n = nx^{n-1}$

$$= x^2 \sin x - \int 2x \sin x \, dx$$

$$= x^2 \sin x - 2 \int x \sin x \, dx$$

We know that, $\int \sin x \, dx = -\cos x$

$$= x^2 \sin x - 2\left(x \int \sin x \, dx - \int \frac{d}{dx} x \int \sin x \, dx\right)$$

$$= x^2 \sin x - 2\left(-x \cos x + \int \cos x \, dx\right)$$

$$= x^2 \sin x - 2(-x \cos x + \sin x) + c$$

$$= x^2 \sin x + 2x \cos x - 2 \sin x + c$$

8. Question

Evaluate the following integrals:

$\int x^2 \cos 2x \, dx$

Answer

Let $I = \int x^2 \cos 2x \, dx$

Using integration by parts,

$$= x^2 \int \cos 2x \, dx - \int \frac{d}{dx} x^2 \int \cos 2x \, dx$$

We know that,

$$\int \cos 2x \, dx = \sin 2x \text{ and } \frac{d}{dx} x^2 = 2x$$

Then, $= \frac{x^2}{2} \sin 2x - \int 2x \frac{\sin 2x \, dx}{2}$

$$= \frac{x^2}{2} \sin 2x - \int x \sin 2x \, dx$$

Using integration by parts in $\int x \sin 2x \, dx$

$$= \frac{x^2}{2} \sin 2x - \left(x \int \sin 2x \, dx - \int \frac{d}{dx} x \int \sin 2x \, dx\right)$$

$$= \frac{x^2}{2} \sin 2x - \left(\frac{-x}{2} \cos 2x + \frac{1}{2} \int \cos 2x \, dx\right)$$

$$= \frac{x^2}{2} \sin 2x - \left(\frac{-x}{2} \cos 2x + \frac{1}{4} \sin 2x\right) + c$$

$$= \frac{x^2}{2} \sin 2x + \frac{x}{2} \cos 2x - \frac{1}{4} \sin 2x + c$$

9. Question

Evaluate the following integrals:

$\int x \sin 2x \, dx$

Answer

Let $I = \int x \sin 2x \, dx$

Using integration by parts,

$$= x \int \sin 2x \, dx - \int \frac{d}{dx} x \int \sin 2x \, dx$$

We know that, $\int \sin nx = \frac{-\cos nx}{n}$ and $\int \cos nx = \frac{\sin nx}{n}$

$$= \frac{x}{2} - \cos 2x + \int \frac{\cos 2x \, dx}{2}$$

$$= -\frac{x}{2} \cos 2x + \frac{1}{2} \frac{\sin 2x}{2} + c$$

$$= -\frac{x}{2} \cos 2x + \frac{1}{4} \sin 2x + c$$

10. Question

Evaluate the following integrals:

$$\int \frac{\log(\log x)}{x} dx$$

Answer

Let $I = \int \frac{\log(\log x)}{x} dx$

It can be written as, $= \int \left(\frac{1}{x}\right)(\log(\log x))dx$

Using integration by parts,

$$I = \log(\log x) \int \frac{1}{x} dx - \int \left(\frac{1}{x\log x} \int \frac{1}{x} dx\right) dx$$

We know that, $\int \log x = \frac{1}{x}$ and $\frac{d}{dx}\frac{1}{x} = \log x$

$$= \log x(\log x) \times \log x - \int \frac{1}{x\log x} \times \log x \, dx$$

$$= \log x(\log x) \times \log x - \int \frac{1}{x} dx$$

$$= \log x(\log x) \times \log x - \log x + c$$

$$= \log x(\log(\log x) - 1) + c$$

11. Question

Evaluate the following integrals:

$\int x^2 \cos x \, dx$

Answer

Let $I = \int x^2 \cos x \, dx$

Using integration by parts,

$$= x^2 \int \cos x \, dx - \int \frac{d}{dx} x^2 \int \cos x \, dx$$

We know that,

$$\int \cos nx = \frac{\sin nx}{n}$$

$$= x^2 \sin x - \int 2x \sin x \, dx$$

$$= x^2 \sin x - 2 \int x \sin x \, dx$$

Using integration by parts in second integral,

$$= x^2 \sin x - 2\left(x \int \sin x \, dx - \int \frac{d}{dx} x \int \sin x \, dx\right)$$

$$= x^2 \sin x - 2\left(-x\cos x + \int \cos x \, dx\right)$$

$$= x^2 \sin x - 2(-x\cos x + \sin x) + c$$

$$= x^2 \sin x + 2x\cos x - 2\sin x + c$$

12. Question

Evaluate the following integrals:

$\int x \operatorname{cosec}^2 x \, dx$

Answer

Let $I = \int x \operatorname{cosec}^2 x \, dx$

Using integration by parts,

$$I = x \int \operatorname{cosec}^2 x \, dx - \int \frac{d}{dx} x \int \operatorname{cosec}^2 x \, dx$$

We know that, $\int \operatorname{cosec}^2 x \, dx = -\cot x$ and $\int \cot x \, dx = \log|\sin x|$

$$= x \times -\cot x - \int -\cot x \, dx$$

$$= -x \cot x + \log|\sin x| + c$$

13. Question

Evaluate the following integrals:

$\int x \cos^2 x \, dx$

Answer

Let $I = \int x \cos^2 x \, dx$

Using integration by parts,

$$I = x \int \cos^2 x \, dx - \int \frac{d}{dx} x \int \cos^2 x \, dx$$

We know that, $\cos^2 x = \frac{\cos 2x + 1}{2}$

$$= x \int \left[\frac{\cos 2x + 1}{2}\right] dx - \int \left[1 \int \left[\frac{\cos 2x + 1}{2}\right] dx\right] dx$$

We know that,

$$\int \cos nx = \frac{\sin nx}{n}$$

$$= \frac{x}{2}\left[\frac{\sin 2x}{2} + x\right] - \frac{1}{2}\int \left(x + \frac{\sin 2x}{2}\right) dx$$

$$= \frac{x}{4}\sin 2x + \frac{x^2}{2} - \frac{1}{2} \times \frac{x^2}{2} - \frac{1}{4}\left(-\frac{\cos 2x}{2}\right) + c$$

$$I = \frac{x}{4}\sin 2x + \frac{x^2}{4} + \frac{1}{8}\cos 2x + c$$

14. Question

Evaluate the following integrals:

$\int x^n \log x \, dx$

Answer

Let $I = \int x^n \log x \, dx$

Using integration by parts,

$$I = \log x \int x^n \, dx - \int \frac{d}{dx} \log x \int x^n dx$$

We know that,

$\int x^n dx = \frac{x^{n+1}}{n+1}$ and $\frac{d}{dx} \log x = \frac{1}{x}$

$$= \log x \frac{x^{n+1}}{n+1} - \int \frac{1}{x} \times \frac{x^{n+1}}{n+1} \, dx$$

$$= \log x \frac{x^{n+1}}{n+1} - \int \frac{x^n}{n+1} \, dx$$

$$= \log x \frac{x^{n+1}}{n+1} - \frac{1}{n+1} \left[\int x^n dx \right]$$

We know that,

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$= \log x \frac{x^{n+1}}{n+1} - \frac{1}{(n+1)^2} x^{n+1} + c$$

15. Question

Evaluate the following integrals:

$$\int \frac{\log x}{x^n} dx$$

Answer

Let $I = \int \frac{\log x}{x^n} \, dx = \int \log x \frac{1}{x^n} dx$

Using integration by parts,

$$\int \log x \frac{1}{x^n} dx = \log x \int \frac{1}{x^n} dx - \int \frac{d}{dx} \log x \int \frac{1}{x^n} dx$$

We know that,

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$= \log x \left(\frac{x^{1-n}}{1-n} \right) - \int \frac{1}{x} \left(\frac{x^{1-n}}{1-n} \right) dx$$

$$= \log x \left(\frac{x^{1-n}}{1-n} \right) - \int \left(\frac{x^{-n}}{1-n} \right) dx$$

$$= \log x \left(\frac{x^{1-n}}{1-n} \right) - \left(\frac{1}{1-n} \right) \left(= \log x \left(\frac{x^{1-n}}{1-n} \right) - \right)$$

$$= \log x \left(\frac{x^{1-n}}{1-n} \right) - \left(\frac{x^{1-n}}{(1-n)^2} \right) + c$$

16. Question

Evaluate the following integrals:

$\int x^2 \sin^2 x \, dx$

Answer

Let $I = \int x^2 \sin^2 x \, dx$

We know that,

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$= \int x^2\left(\frac{1-\cos 2x}{2}\right)dx$$

Using integration by parts,

$$= \int \frac{x^2}{2}dx - \int \frac{x^2\cos 2x}{2}\,dx$$

$$= \frac{x^3}{6} - \frac{1}{2}\left[\int x^2\cos 2x\ dx\right]$$

Using integration by parts in second integral,

$$= \frac{x^3}{6} - \frac{1}{2}\left[x^2\int \cos 2x dx - \int \frac{d}{dx}x^2 \int \cos 2x\, dx\right]$$

$$= \frac{x^3}{6} - \frac{1}{2}\left(x^2\frac{\sin 2x}{2}\right) + \frac{1}{2}\times 2\int x\frac{\sin 2x}{2}dx$$

Using integration by parts again,

$$= \frac{x^3}{6} - \frac{1}{2}\left(x^2\frac{\sin 2x}{2}\right) + \frac{1}{2}\left[x\int \sin 2x\, dx - \int \frac{d}{dx}x\int \sin 2\,x\, dx\right]$$

$$= \frac{x^3}{6} - \frac{1}{2}\left(x^2\frac{\sin 2x}{2}\right) + \frac{1}{2}\left(\frac{x}{2} - \cos 2x + \int \frac{\cos 2x\, dx}{2}\right)$$

$$= \frac{x^3}{6} - \frac{1}{2}\left(x^2\frac{\sin 2x}{2}\right) + \frac{1}{2}\left(-\frac{x}{2}\cos 2x + \frac{1}{2}\frac{\sin 2x}{2}\right) + c$$

$$= \frac{x^3}{6} - \frac{1}{4}(x^2\sin 2x) - \frac{1}{4}x\cos 2x + \frac{1}{8}\sin 2x + c$$

17. Question

Evaluate the following integrals:

$$\int 2x^3 e^{x^2} dx$$

Answer

Let $I = \int 2x^3 e^{x^2} dx$

Put x^2=t

2xdx=dt

$$I = \int t\, e^t dt$$

Using integration by parts,

$$= t\int e^t\, dt - \int \frac{d}{dt}t\int e^t\, dt$$

We have, $\int e^x\, dx = e^x$

$$= te^t - e^t + c$$

$$= e^t(t-1) + c$$

Substitute value for t,

$$I = e^{x^2}(x^2-1) + c$$

18. Question

Evaluate the following integrals:

$\int x^3 \cos x^2\, dx$

Answer

Let $I = \int x^3 \cos x^2\, dx$

Put x^2=t

2xdx=dt

$I = \frac{1}{2}\int t\cos t\, dt$

Using integration by parts,

$I = \frac{1}{2}\left(t\int \cos t\, dt - \int \frac{d}{dt}t \int \cos t\, dt\right)$

$= \frac{1}{2}\left(t \times \sin t - \int \sin t\, dt\right)$

$= \frac{1}{2}(t\sin t + \cos t) + c$

Substitute value for t,

$= \frac{1}{2}(x^2\sin x^2 + \cos x^2) + c$

19. Question

Evaluate the following integrals:

$\int x \sin x \cos x\, dx$

Answer

Let $I = \int x \sin x \cos x\, dx = \frac{1}{2}\int x \times 2\sin x \cos x\, dx$

We know that, $\sin 2x = 2\sin x\cos x$

$= \frac{1}{2}\int x\sin 2x$

Using integration by parts,

$= \frac{1}{2}\left(x\int \sin 2x\, dx - \int \frac{d}{dx}x \int \sin 2x\, dx\right)$

We have,

$\int \sin nx = \frac{-\cos nx}{n}$ and $\int \cos nx = \frac{\sin nx}{n}$

$= \frac{1}{2}\left(\frac{x}{2} - \cos 2x + \int \frac{\cos 2x\, dx}{2}\right)$

$= \frac{1}{2}\left(-\frac{x}{2}\cos 2x + \frac{1}{2}\frac{\sin 2x}{2}\right) + c$

$= -\frac{x}{4}\cos 2x + \frac{1}{8}\sin 2x + c$

20. Question

Evaluate the following integrals:

$\int \sin x \log(\cos x)\, dx$

Answer

Let $I = \int \sin x \log(\cos x) dx$

Put cos x =t

-sinx dx=dt

$$I = \int -\log t\, dt$$

Using integration by parts,

$$= \int 1 \times -\log t\, dt$$

$$= -\left(\log t \int dt - \int \frac{d}{dt}\log t \int 1\, dt\right)$$

$$= -\left(t\log t - \int \frac{1}{t} \times t\, dt\right)$$

$$= -\left(t\log t - \int dt\right)$$

$$= -(t\log t - t) + c$$

$$= t(1 - \log t) + c$$

Replace t by cos x

$$I = \cos x(1 - \log(\cos x)) + c$$

21. Question

Evaluate the following integrals:

$\int (\log x)^2\, x\, dx$

Answer

Let $I = \int (\log x)^2 x\, dx$

Using integration by parts,

$$= (\log x)^2 \int x\, dx - \int \frac{d}{dx}(\log x)^2 \int x\, dx$$

$$= (\log x)^2 \frac{x^2}{2} - \int \left(2(\log x)\left(\frac{1}{x}\right) \int x dx\right) dx$$

$$= \frac{x^2}{2}(\log x)^2 - 2\int (\log x)\left(\frac{1}{x}\right)\left(\frac{x^2}{2}\right) dx$$

$$= \frac{x^2}{2}(\log x)^2 - \int x \log x\, dx$$

Using integration by integration by parts in second integral,

$$= \frac{x^2}{2}(\log x)^2 - \left[\log x \int x\, dx - \int \frac{d}{dx}\log x \int x\, dx\right]$$

We know that, $\int x\, dx = \frac{x^2}{2}$ and $\frac{d}{dx}\log x = \frac{1}{x}$

$= \frac{x^2}{2}(\log x)^2 - \log x\frac{x^2}{2} - \int \frac{1}{x} \times \frac{x^2}{2}$

$= \frac{x^2}{2}(\log x)^2 - \log x\frac{x^2}{2} - \frac{1}{2}\int x\,dx$

$= \frac{x^2}{2}(\log x)^2 - \log x\frac{x^2}{2} - \frac{1}{2}\frac{x^2}{2} + c$

$= \frac{x^2}{2}(\log x)^2 - \log x\frac{x^2}{2} - \frac{x^2}{4} + c$

$I = \frac{x^2}{2}\left[(\log x)^2 - \log x - \frac{1}{2}\right] + c$

22. Question

Evaluate the following integrals:

$\int e^{\sqrt{x}}\,dx$

Answer

Let $I = \int e^{\sqrt{x}}dx$

$\sqrt{x} = t; x = t^2$

dx=2tdt

$I = 2\int e^t t\,dt$

Using integration by parts,

$I = 2\left(t\int e^t dt - \int \frac{d}{dt}t\int e^t dt\right)$

$= 2\left(te^t - \int e^t dt\right)$

$= 2(te^t - e^t) + c$

$= 2e^t(t - 1) + c$

Replace the value of t

$= 2e^{\sqrt{x}}\left(\sqrt{x} - 1\right) + c$

23. Question

Evaluate the following integrals:

$\int \frac{\log(x+2)}{(x+2)^2}dx$

Answer

Let $I = \int \frac{\log(x+2)}{(x+2)^2}dx$

$\frac{1}{x+2} = t$

$\frac{-1}{(x+2)^2}dx = dt$

$$I = -\int \log\left(\frac{1}{t}\right) dt$$

Using integration by parts,

$$= -\int \log t^{-1}\, dt$$

$$= -\int 1 \times \log t^{-1}\, dt$$

We know that, $\frac{d}{dt}\log t = \frac{1}{t}$ and $\int dt = t$

$$I = \log t \int dt - \int \left(\frac{d}{dt}\log t \int dt\right) dt$$

$$= \log t \int dt - \int \left(\frac{1}{t}\int dt\right) dt$$

$$= t\log t - \int \frac{1}{t} \times t\, dt$$

$$= t\log t - t + c$$

Replace the value of t,

$$= \frac{1}{x+2}(\log(x+2)^{-1} - 1) + c$$

$$= -\frac{1}{x+2} - \frac{\log(x+2)}{x+2} + c$$

24. Question

Evaluate the following integrals:

$$\int \frac{x + \sin x}{1 + \cos x} dx$$

Answer

Let $I = \int \frac{x+\sin x}{1+\cos x} dx$

$1 + \cos x$ can be written as, $2\cos^2\frac{x}{2}$ and $\sin x$ can be written as $2\sin\frac{x}{2}\cos\frac{x}{2}$

$$= \int \frac{x}{2\cos^2\frac{x}{2}} dx + \int \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}} dx$$

$$= \frac{1}{2}\int x\sec^2\frac{x}{2} + \int \tan\frac{x}{2} dx$$

Using integration by parts,

$$= \frac{1}{2}\left[x\int \sec^2\frac{x}{2} - \int \frac{d}{dx}x \int \sec^2\frac{x}{2} dx\right] + \int \tan\frac{x}{2} dx$$

$$= \frac{1}{2}\left[2x\tan\frac{x}{2} - 2\int \tan\frac{x}{2} dx\right] + \int \tan\frac{x}{2} dx$$

$$= x\tan\frac{x}{2} - \int \tan\frac{x}{2} dx + \int \tan\frac{x}{2} dx$$

$$= x\tan\frac{x}{2} + c$$

25. Question

Evaluate the following integrals:

$\int \log_{10} x\, dx$

Answer

Let $I = \int \log_{10} x\, dx$

$$= \int \frac{\log x}{\log 10}\, dx$$

$$= \frac{1}{\log 10} \int 1 \times \log x\, dx$$

Using integration by parts,

$$= \frac{1}{\log 10}\left(\log x \int dx - \int \frac{d}{dx} \log x \int 1\, dx\right)$$

We know that $\frac{d}{dx} \log x = \frac{1}{x}$

$$= \frac{1}{\log 10}\left(x \log x - \int \frac{1}{x} \times x\, dx\right)$$

$$= \frac{1}{\log 10}\left(x \log x - \int dx\right)$$

$$= \frac{1}{\log 10}(x \log x - x) + c$$

$$= \frac{x}{\log 10}(1 - \log x) + c$$

26. Question

Evaluate the following integrals:

$\int \cos \sqrt{x}\, dx$

Answer

Let $I = \int \cos\sqrt{x} dx$

$\sqrt{x} = t; x = t^2$

dx=2tdt

$$= \int 2t \cos t\, dt$$

$$I = 2 \int t \cos t\, dt$$

Using integration by parts,

$$I = 2\left(t \int \cos t\, dt - \int \frac{d}{dt} t \int \cos t\, dt\right)$$

$$= 2\left(t \times \sin t - \int \sin t\, dt\right)$$

$$= 2(t \sin t + \cos t) + c$$

Replace the value of t, $I = 2\left(\sqrt{x} \sin \sqrt{x} + \cos \sqrt{x}\right) + c$

27. Question

Evaluate the following integrals:

$$\int \frac{x\cos^{-1}x}{\sqrt{1-x^2}}dx$$

Answer

Let $I = \int \frac{x\cos^{-1}x}{\sqrt{1-x^2}}dx$

Let $t = \cos^{-1}x$

$$dt = \frac{1}{\sqrt{1-x^2}}dx$$

Also,

cos t =x

Thus,

$$I = -\int t\cos t\,dt$$

Now let us solve this by 'by parts' method

Using integration by parts,

$$I = -t\left(\int \cos t\,dt - \int \frac{d}{dt}t\int \cos t\,dt\right)$$

Let

U=t; du=dt

$$\int \cos t\,dt = v;\ \sin t = dv$$

Thus,

$$I = -\left[t\sin t - \int \sin t\,dt\right]$$

$$I = -[t\sin t + \cos t] + c$$

Substituting

$$t = \cos^{-1}x$$

$$I = -[\cos^{-1}x\sin t + x] + c$$

$$I = -\left[\cos^{-1}x\sqrt{1-x^2} + x\right] + c$$

28. Question

Evaluate the following integrals:

$$\int \frac{\log x}{(x+1)^2}dx$$

Answer

We know that integration by parts is given by:

$$\int UV = U\int Vdv - \int \frac{d}{dx}U\int Vdv$$

Choosing log x as first function and $\frac{1}{(x+1)^2}$ as second function we get,

$$\int \frac{\log x}{(x+1)^2}dx = \log x \int \left(\frac{1}{(x+1)^2}\right)dx - \int (\frac{d}{dx}(\log x)\int \frac{1}{(x+1)^2}dx)\ dx$$

$$\int \frac{\log x}{(x+1)^2}dx = \log x \left(-\frac{1}{x+1}\right) + \int \frac{1}{x}\left(\frac{1}{x+1}\right)dx$$

$$\int \frac{\log x}{(x+1)^2}dx = -\frac{\log x}{x+1} + \int \frac{(x+1)-(x)}{x(x+1)}dx$$

$$\int \frac{\log x}{(x+1)^2}dx = -\frac{\log x}{x+1} + \int \left(\frac{1}{x} - \frac{1}{x+1}\right)dx$$

$$\int \frac{\log x}{(x+1)^2}dx = -\frac{\log x}{x+1} + \log x - \log(x+1) + c$$

$$\int \frac{\log x}{(x+1)^2}dx = -\frac{\log x}{x+1} + \log\left(\frac{x}{x+1}\right) + c$$

29. Question

Evaluate the following integrals:

$\int \operatorname{cosec}^3 x\ dx$

Answer

Let $I = \int \operatorname{cosec}^3 x\, dx$

$$= \int \operatorname{cosec} x \times \operatorname{cosec}^2 x\, dx$$

Using integration by parts,

$$= \operatorname{cosec} x \int \operatorname{cosec}^2 x\, dx - \int \frac{d}{dx}\operatorname{cosec} x \int \operatorname{cosec}^2 x\, dx$$

We know that, $\int \operatorname{cosec}^2 x\, dx = -\cot x$ and $\frac{d}{dx}\operatorname{cosec} x = \operatorname{cosec} x \cot x$

$$= \operatorname{cosec} x \times -\cot x + \int \operatorname{cosec} x \cot x \times -\cot x\ dx$$

$$= -\operatorname{cosec} x \cot x + \int \operatorname{cosec} x \cot^2 x\, dx$$

Using integration by parts,

$$= -\operatorname{cosec} x \cot x + \int \operatorname{cosec} x(\operatorname{cosec}^2 x - 1)\ dx$$

$$= -\operatorname{cosec} x \cot x + \int \operatorname{cosec}^3 x dx - \int \operatorname{cosec} x dx$$

$$I = -\operatorname{cosec} x \cot x - I + \log\left|\tan\frac{x}{2}\right| + c_1$$

$$2I = -\operatorname{cosec} x \cot x + \log\left|\tan\frac{x}{2}\right| + c_1$$

$$I = -\frac{1}{2}\operatorname{cosec} x \cot x + \frac{1}{2}\log\left|\tan\frac{x}{2}\right| + c_1$$

30. Question

Evaluate the following integrals:

$\int \sec^{-1} \sqrt{x}\, dx$

Answer

Let $I = \int \sec^{-1}\sqrt{x}\ dx$

$\sqrt{x} = t; x = t^2$

dx=2tdt

$I = \int 2t\sec^{-1}t\ dt$

Using integration by parts,

$= 2\left[\sec^{-1}t\int tdt - \int \frac{d}{dt}\sec^{-1}t\int tdt\right]$

We know that, $\frac{d}{dt}\sec^{-1}t = \frac{1}{t\sqrt{t^2-1}}$

$= 2\left[\frac{t^2}{2}\sec^{-1}t - \int \frac{1}{t\sqrt{t^2-1}}\int tdt\right]$

$= 2\left[\frac{t^2}{2}\sec^{-1}t - \int \frac{t^2}{2t\sqrt{t^2-1}}dt\right]$

$= t^2\sec^{-1}t - \int \frac{t}{t\sqrt{t^2-1}}dt$

$= t^2\sec^{-1}t - \frac{1}{2}\int \frac{2t}{\sqrt{t^2-1}}dt$

$= t^2\sec^{-1}t - \frac{1}{2} \times 2\sqrt{t^2-1} + c$

Substitute value for t,

$I = x\sec^{-1}\sqrt{x} - \sqrt{x-1} + c$

31. Question

Evaluate the following integrals:

$\int \sin^{-1} \sqrt{x}\, dx$

Answer

Let $I = \sin^{-1}\sqrt{x}\,dx$

$\sqrt{x} = t; x = t^2$

dx=2tdt

$= \sin^{-1}t\,2t\,dt$

Using integration by parts,

$= \sin^{-1}t\int 2tdt - \int \frac{d}{dt}\sin^{-1}t\int 2tdt$

We know that, $\frac{d}{dt}\sin^{-1}t = \frac{t}{\sqrt{1-t^2}}$

$$= t^2\sin^{-1}t - 2\int \frac{t^2}{\sqrt{1-t^2}}\,dt$$

let us solve, $\int \frac{t^2}{\sqrt{1-t^2}}\,dt$

$$= \int \frac{t^2-1+1}{\sqrt{1-t^2}}dt = \int \frac{t^2-1}{\sqrt{1-t^2}}dt + \int \frac{1}{\sqrt{1-t^2}}dt$$

$$\int \frac{1}{\sqrt{1-t^2}}dt = \sin^{-1}t$$

$$\int \frac{t^2-1}{\sqrt{1-t^2}}dt = \int -\sqrt{1-t^2}\,dt$$

t=sin u;dt=cos u du

$$\int -\sqrt{1-t^2}\,dt = \int -\cos^2u\,du = -\int \left[\frac{1+\cos 2u}{2}\right]du$$

$$= -\frac{u}{2} - \frac{\sin 2u}{4}$$

$u = \sin^{-1}t$ and $t = \sqrt{x}$

$$= -\frac{\sin^{-1}t}{2} - \frac{\sin(2\sin^{-1}t)}{4}$$

There fore, $\int \sin^{-1}\sqrt{x}\,dx = x\sin^{-1}\sqrt{x} - \frac{\sin^{-1}\sqrt{x}}{2} - \frac{\sin(2\sin^{-1}t)}{4}$

$$= x\sin^{-1}\sqrt{x} - \frac{\sin^{-1}\sqrt{x}}{2} - \frac{\sqrt{x(1-x)}}{2}$$

32. Question

Evaluate the following integrals:

$\int x\tan^2 x\,dx$

Answer

Let $I = \int x\tan^2x\,dx$

$$= \int x(\sec^2x - 1)\,dx$$

$$= \int x\sec^2x dx - \int x\,dx$$

Using integration by parts,

$$= x\int \sec^2x dx - \int \frac{d}{dx}x\int \sec^2x dx - \frac{x^2}{2}$$

We know that, $\int \sec^2x dx = \tan x$

$$= x\tan x - \int \tan x\,dx - \frac{x^2}{2}$$

$$= x\tan x - \log|\sec x| - \frac{x^2}{2} + c$$

33. Question

Evaluate the following integrals:

$$\int x\left(\frac{\sec 2x-1}{\sec 2x+1}\right)dx$$

Answer

Let $I=\int x\left(\frac{\sec 2x-1}{\sec 2x+1}\right)dx$ it can be written n terms of cos x

$$=\int x\left(\frac{1-\cos 2x}{1+\cos 2x}\right)dx$$

$$=\int x\left(\frac{\sec^2 x}{\cos^2 x}\right)dx$$

$$=\int x\tan^2 x\,dx$$

$$=\int x\left(\sec^2 x-1\right)dx$$

$$=\int x\sec^2 x-\int x\,dx$$

Using integration by parts,

$$=x\int \sec^2 x dx-\int \frac{d}{dx}x\int \sec^2 x dx-\frac{x^2}{2}$$

$$=x\tan x-\int \tan x\,dx-\frac{x^2}{2}$$

$$=x\tan x-\log|\sec x|-\frac{x^2}{2}+c$$

34. Question

Evaluate the following integrals:

$\int (x+1)e^x \log(xe^x)\,dx$

Answer

Let $I=\int (x+1)e^x\log(xe^x)\,dx$

$xe^x=t$

$(1\times e^x+xe^x)dx=dt$

$(x+1)e^x dx=dt$

$$I=\int \log t\,dt$$

$$=\int 1\times\log t\,dt$$

Using integration by parts,

$$=\log t\int dt-\int \frac{d}{dt}\log t\int dt$$

$$=t\log t-\int \frac{1}{t}t\,dt$$

$= t\log t - t + c$

$= t(\log t - 1) + c$

Substitute value for t,

$I = xe^x(\log xe^x - 1) + c$

35. Question

Evaluate the following integrals:

$\int \sin^{-1}(3x - 4x^3)\,dx$

Answer

Let $\int \sin^{-1}(3x - 4x^3)\,dx$

$x = \sin\theta$

$dx = \cos\theta\, d\theta$

$$= \int \sin^{-1}(3\sin\theta - 4\sin^3\theta)\cos\theta\, d\theta$$

We know that $3\sin\theta - 4\sin^3\theta = \sin 3\theta$

$$= \int \sin^{-1}(\sin 3\theta)\cos\theta d\theta$$

We know that, $\int \sin^{-1}(\sin 3\theta) = 3\theta$

$$= \int 3\theta\cos\theta d\theta$$

$$= 3\int \theta\cos\theta d\theta$$

Using integration by parts,

$$= 3\left(\theta\int \cos\theta\, d\theta - \int \frac{d}{d\theta}\theta \int \cos\theta\, d\theta\right)$$

$$= 3\left(\theta \times \sin\theta - \int \sin\theta\ d\theta\right)$$

$$= 3(\theta\sin\theta + \cos\theta) + c$$

$$I = 3\left[x\sin^{-1}x + \sqrt{1 - x^2}\right] + c$$

36. Question

Evaluate the following integrals:

$$\int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$$

Answer

Let $I = \int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

$x = \tan\theta \Rightarrow dx = \sec^2\theta d\theta$

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta$$

$$\int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = \int 2\theta \sec^2\theta d\theta$$

Using integration by parts,

$$= 2\left(\theta\int \sec^2\theta d\theta - \int \frac{d}{d\theta}\theta \int \sec^2\theta d\theta\right)$$

$$= 2\left(\theta \tan\theta - \int \tan\theta\, d\theta\right)$$

We know that, $\int \tan\theta\, d\theta = \log|\cos\theta|$

$$= 2(\theta\tan\theta - \log|\cos\theta|) + c$$

$$= 2\left[x\tan^{-1}x + \log\left|\frac{1}{\sqrt{1+x^2}}\right|\right] + c$$

$$= 2x\tan^{-1}x + 2\log\left|(1+x^2)^{\frac{1}{2}}\right| + c$$

$$= 2x\tan^{-1}x + 2\left[\frac{1}{2}\log(1+x)^2\right] + c$$

$$= 2x\tan^{-1}x + \log(1+x)^2 + c$$

37. Question

Evaluate the following integrals:

$$\int \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)dx$$

Answer

Let $I = \int \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)dx$

$x = \tan\theta \Rightarrow dx = \sec^2\theta d\theta$

We know that, $\frac{3\tan\theta - \tan\theta^3}{1-3\tan\theta^2} = \tan 3\theta$

$$I = \int \tan^{-1}\left(\frac{3\tan\theta - \tan\theta^3}{1-3\tan\theta^2}\right)\sec^2\theta d\theta$$

We know that, $\tan^{-1}(\tan 3\theta) = 3\theta$

$$= \int \tan^{-1}(\tan 3\theta)\sec^2\theta d\theta$$

$$= \int 3\theta \sec^2\theta d\theta$$

Using integration by parts,

$$= 3\left(\theta\int \sec^2\theta d\theta - \int \frac{d}{d\theta}\theta \int \sec^2\theta d\theta\right)$$

$$= 3\left(\theta\tan\theta - \int \tan\theta\, d\theta\right)$$

$$= 3(\theta\tan\theta - \log|\sec\theta|) + c$$

$$= 3\left[x\tan^{-1}x + \log\left|\sqrt{1+x^2}\right|\right] + c$$

$= 3x\tan^{-1}x + \frac{3}{2}\log|1 + x^2| + c$

38. Question

Evaluate the following integrals:

$\int x^2 \sin^{-1} x \, dx$

Answer

Let $I = \int x^2 \sin^{-1}x \, dx$

Using integration by parts,

$$I = \sin^{-1}x \int x^2 dx - \int \frac{d}{dx}\sin^{-1}x \int x^2 dx$$

$$= \frac{x^3}{3}\sin^{-1}x - \int \frac{x^3}{3\sqrt{1-x^2}} dx$$

$$I = \frac{x^3}{3}\sin^{-1}x - \int I_1 + C \text{ --------------- } (1)$$

$$I_1 = -\int \frac{x^3}{3\sqrt{1-x^2}} dx$$

Let $1\text{-}x^2=t^2$

$-2x \, dx=2t \, dt$

$-x \, dx=t \, dt$

$$I_1 = -\int \frac{(1-t^2)t dt}{t}$$

$$I_1 = \int (t^2 - 1) dt$$

$$= \frac{t^3}{3} - t + c_2$$

$$= \frac{(1-x^2)^{\frac{3}{2}}}{3} - (1-x^2)^{\frac{1}{2}} + c_2$$

$$= \frac{x^3}{3}\sin^{-1}x - \frac{(1-x^2)^{\frac{3}{2}}}{9} + \frac{1}{3}(1-x^2)^{\frac{1}{2}} + c$$

39. Question

Evaluate the following integrals:

$$\int \frac{\sin^{-1} x}{x^2} dx$$

Answer

Let $I = \int \frac{\sin^{-1}x}{x^2} dx$

$$= \int \frac{1}{x^2}\sin^{-1}x \, dx$$

Using integration by parts,

$$I = \left[\sin^{-1}x \times \int \frac{1}{x^2} - \int \left(\frac{1}{\sqrt{1-x^2}}\right) \int \frac{1}{x^2} dx\right] dx$$

$$= \sin^{-1}x\left(\frac{-1}{x}\right) - \int \frac{1}{\sqrt{1-x^2}}\left(\frac{-1}{x}\right) dx$$

$$I = \frac{-1}{x}\sin^{-1}x + \int \frac{1}{x\sqrt{1-x^2}} dx$$

$$I = \frac{-1}{x}\sin^{-1}x + I_1 \text{-------------}(1)$$

Where,

$$I_1 = \int \frac{1}{x\sqrt{1-x^2}}$$

$$1 - x^2 = t^2$$

-2xdx=2tdt

$$I_1 = \int \frac{tdt}{(1-t^2)\sqrt{t}}$$

$$= \frac{1}{2}\log\left|\frac{t-1}{t+1}\right|$$

$$= \frac{1}{2}\log\left|\frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}+1}\right| + c_1$$

$$I = \frac{-1}{x}\sin^{-1}x + \frac{1}{2}\log\left|\frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}+1}\right| + c$$

$$= \frac{-1}{x}\sin^{-1}x + \frac{1}{2}\log\left(\frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}+1}\right)\left(\frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}-1}\right) + c$$

$$= \frac{-1}{x}\sin^{-1}x + \frac{1}{2}\log\left(\frac{(\sqrt{1-x^2}-1^2)}{-x^2}\right) + c$$

$$= \frac{-1}{x}\sin^{-1}x + \log\left|\frac{1-\sqrt{1-x^2}}{x}\right| + c$$

40. Question

Evaluate the following integrals:

Answer

Let $I = \int \frac{x^2 \tan^{-1}x}{1+x^2} dx$

$\tan^{-1}x = t\,;\, x = \text{tant} \int \frac{x^2 \tan^{-1}x}{1+x^2} dx$

$$\frac{1}{1+x^2} dx = dt$$

$$I = \int t\tan^2 t\, dt$$

We know that, $\tan^2 t = \sec^2 t - 1$

$$= \int t(\sec^2 t - 1)dt$$

$$= \int t\sec^2 t\, dt - \int tdt$$

Using integration by parts,

$$= \left(t\int \sec^2 tdt - \int \frac{d}{dt}t\int \sec^2 tdt\right) - \frac{t^2}{2}$$

$$= \left(t\tan t - \int \tan t\, dt\right) - \frac{t^2}{2}$$

$$= (t\tan t - \log|\sec t|) - \frac{t^2}{2} + c$$

$$= \left[x\tan^{-1}x + \log\left|\sqrt{1+x^2}\right|\right] - \frac{\tan^2 x}{2} + c$$

$$= x\tan^{-1}x + \frac{1}{2}\log|1+x^2| - \frac{\tan^2 x}{2} + c$$

41. Question

Evaluate the following integrals:

$\int \cos^{-1}(4x^3 - 3x)\, dx$

Answer

Let $I = \int \cos^{-1}(4x^3 - 3x)dx$

$x = \cos\theta \Rightarrow dx = -\sin\theta d\theta$

$$I = -\int \cos^{-1}(4\cos^3\theta - 3\cos\theta)\sin\theta d\theta$$

We know that, $(4\cos^3\theta - 3\cos\theta) = \cos 3\theta$

$$= -\int \cos^{-1}(\cos 3\theta)\sin\theta d\theta$$

$$= -\int 3\theta\sin\theta d\theta$$

Using integration by parts,

$$= -3\left[\theta\int \sin\theta d\theta - \int \frac{d}{d\theta}\theta\int \sin\theta d\theta\right]$$

$$= 3[-\theta\cos\theta + \int \cos\theta d\theta$$

$$= 3\theta\cos\theta - 3\sin\theta + c$$

$$I = 3x\cos^{-1}x - 3\sqrt{1-x^2} + c$$

42. Question

Evaluate the following integrals:

$$\int \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)dx$$

Answer

Let $I = \int \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) dx$

Let x=tan t

dx=sec^{2}t dt

$$I = \int \cos^{-1}\left(\frac{1-\tan^2 t}{1+\tan^2 t}\right) \sec^2 t\, dt$$

We know that $\frac{1-\tan^2 t}{1+\tan^2 t} = \cos 2t$

$$= \int \cos^{-1}(\cos 2t) \sec^2 t\, dt$$

$$= \int 2t \sec^2 t\, dt$$

Using integration by parts,

$$= 2[t \int \sec^2 t\, dt - \int \frac{d}{dt} t \int \sec^2 t\, dt]$$

$$= 2[t \tan t - \int \tan t\, dt]$$

$$= 2[t \tan t - \log \sec t] + c$$

$$= 2[x \tan^{-1} x - \log|\sqrt{1+x^2}|] + c$$

$$= 2x \tan^{-1} x - \log|1+x^2| + c$$

43. Question

Evaluate the following integrals:

$$\int \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$$

Answer

Let $I = \int \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$

$x = \tan\theta \Rightarrow dx = \sec^2\theta d\theta$

$$I = \int \tan^{-1}\left(\frac{2\tan\theta}{1-2\tan\theta^2}\right) \sec^2\theta d\theta$$

We know that, $\frac{2\tan\theta}{1-2\tan\theta^2} = \tan 2\theta$

$$= \int \tan^{-1}(\tan 2\theta) \sec^2\theta d\theta$$

$$\int 2\theta \sec^2\theta d\theta$$

Using integration by parts,

$$= 2\left(\theta \int \sec^2\theta d\theta - \int \frac{d}{d\theta}\theta \int \sec^2\theta d\theta\right)$$

$$= 2\left(\theta \tan\theta - \int \tan\theta\, d\theta\right)$$

$= 2(\theta\tan\theta - \log|\sec\theta|) + c$

$= 2\left[x\tan^{-1}x + \log\left|\sqrt{1+x^2}\right|\right] + c$

$= 2x\tan^{-1}x + \log|1 + x^2| + c$

44. Question

Evaluate the following integrals:

∫ (x + 1) log x dx

Answer

Let $I = \int (x+1)\log x\, dx$

Using integration by parts,

$= \log x\int(x+1)dx - \int\frac{d}{dx}\log x\int(x+1)dx$

We know that, $\frac{d}{dx}\log x = \frac{1}{x}$

$= \log x\left(\frac{x^2}{2} + x\right) - \int\frac{1}{x}\left(\frac{x^2}{2} + x\right)dx$

$= \left(\frac{x^2}{2} + x\right)\log x - \int\frac{x}{2}dx - \int dx$

$= \left(\frac{x^2}{2} + x\right)\log x - \frac{x^2}{4} - x + c$

$= \left(\frac{x^2}{2} + x\right)\log x - (\frac{x^2}{4} + x) + c$

45. Question

Evaluate the following integrals:

∫ x^2 $\tan^{-1}$ x dx

Answer

Let $I = \int x^2\tan^{-1}x\, dx$

Using integration by parts,

Taking inverse function as first function and algebraic function as second function,

$= \tan^{-1}x\int x^2dx - \int\left(\frac{1}{1+x^2}\right)\int x^2dx$

$= \tan^{-1}x\frac{x^3}{3} - \frac{1}{3}\int\frac{x^3}{1+x^2}dx$

$= \tan^{-1}x\frac{x^3}{3} - \frac{1}{3}\int x - \frac{x}{1+x^2}dx$

$= \tan^{-1}x\frac{x^3}{3} - \frac{1}{3}\times\frac{x^2}{2} + \int\frac{x}{1+x^2}dx$

$= \frac{1}{3}x^3\tan^{-1}x - \frac{x^2}{6} + \frac{1}{6}\log|1 + x^2| + c$

46. Question

Evaluate the following integrals:

$\int (e^{\log x} + \sin x) \cos x \, dx$

Answer

Let $I = \int (e^{\log x} + \sin x) \cos x \, dx$

$$= \int (x + \sin x) \cos x \, dx$$

$$= \int x \cos x \, dx + \int \sin x \cos x \, dx$$

Using integration by parts,

$$= x \int \cos x \, dx - \int \frac{d}{dx} x \int \cos x \, dx + \frac{1}{2} \int \sin 2x \, dx$$

$$= x \times \sin x - \int \sin x \, dx + \frac{1}{2}\left(\frac{-\cos 2x}{2}\right) + c$$

$$= x\sin x + \cos x - \frac{1}{4} \cos 2x + c$$

$$= x\sin x + \cos x - \frac{1}{4} [1 - 2\sin^2 x] + c$$

$$I = x\sin x + \cos x - \frac{1}{4} + \frac{1}{2}\sin^2 x + c$$

$$I = x\sin x + \cos x + \frac{1}{2}\sin^2 x + c - \frac{1}{4}$$

$$I = x\sin x + \cos x + \frac{1}{2}\sin^2 x + k \text{ where, } k = c - \frac{1}{4}$$

47. Question

Evaluate the following integrals:

$$\int \frac{\left(x \tan^{-1} x\right)}{\left(1 + x^2\right)^{3/2}} dx$$

Answer

Let $I = \int \frac{x\tan^{-1} x}{(1+x^2)^{\frac{3}{2}}} dx$

$$\tan^{-1} x = t$$

$$\frac{1}{1 + x^2} dx = dt$$

$$I = \int \frac{t\tan t}{\sqrt{1 + \tan^2 t}} dt$$

We know that, $\sqrt{1 + \tan^2 t} = \sec t$

$$= \int \frac{t \tan t}{\sec t} dt$$

$$= \int t \frac{\sin t}{\cos t} \cos t \, dt$$

$$= \int t \sin t \, dt$$

Using integration by parts,

$$= t \int \sin t \, dt - \int \frac{d}{dt} t \int \sin t \, dt$$

$$= -t \cos t + \int \cos t \, dt$$

$$= -t \cos t + \sin t + c$$

Substitute value for t

$$I = \frac{\tan^{-1} x}{\sqrt{1 + x^2}} + \frac{x}{\sqrt{1 + x^2}} + c$$

48. Question

Evaluate the following integrals:

$\int \tan^{-1} (\sqrt{x}) \, dx$

Answer

Let $I = \int \tan^{-1}(\sqrt{x}) dx$

$x=t^2$

$dx=2tdt$

$$I = \int 2t \tan^{-1} t \, dt$$

Using integration by parts,

$$= 2\left(\tan^{-1} t \int t dt - \int \frac{d}{dt} \tan^{-1} t \int t \, dt\right)$$

We know that,

$$\frac{d}{dt} \tan^{-1} t = \frac{1}{2(1 + t^2)}$$

$$= 2\left[\frac{t^2}{2} \tan^{-1} t - \int \frac{t^2}{2(1 + t^2)} dt\right]$$

$$= t^2 \tan^{-1} t - \int \frac{t^2 + 1 - 1}{1 + t^2} dt$$

$$= t^2 \tan^{-1} t - \int \left(1 - \frac{1}{1 + t^2}\right) dt$$

$$= t^2 \tan^{-1} t - t + \tan^{-1} t + c$$

$$= (t^2 + 1) \tan^{-1} t - t + c$$

$$= (x + 1) \tan^{-1} \sqrt{x} - \sqrt{x} + c$$

49. Question

Evaluate the following integrals:

$\int x^3 \tan^{-1} x \, dx$

Answer

Let $I = \int x^3 \tan^{-1} x\, dx$

Using integration by parts,

We know that,

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{2(1+x^2)}$$

$$= \tan^{-1}x \int x^3 dx - \int \left(\frac{1}{1+x^2}\right) \int x^3\, dx$$

$$= \tan^{-1}x \frac{x^4}{4} - \frac{1}{4}\int \frac{x^4}{1+x^2} dx$$

$$\frac{1}{4}\int \frac{x^4}{1+x^2} dx = \frac{1}{4}\left[\int \frac{1}{1+x^2} dx + (x^2-1)dx\right] = \frac{1}{4}\left[\tan^{-1}x + \frac{x^3}{3} - x\right]$$

$$= \frac{x^4}{4}\tan^{-1}x - \frac{1}{4}\left[\tan^{-1}x + \frac{x^3}{3} - x\right] + c$$

50. Question

Evaluate the following integrals:

$\int x \sin x \cos 2x\, dx$

Answer

Let $I = \int x \sin x \cos 2x\, dx = \frac{1}{2}\int x \times 2 \sin x \cos 2x\, dx$

Using integration by parts,

$$= \frac{1}{2}\int x(\sin(x+2x) - \sin(2x-x))dx$$

$$= \frac{1}{2}\int x(\sin 3x - \sin x)dx$$

Using integration by parts,

$$= \frac{1}{2}(x \int (\sin 3x - \sin x)dx - \int \frac{d}{dx}x \int (\sin 3x - \sin x)dx)dx$$

$$= \frac{1}{2}\left[x\left(\frac{-\cos 3x}{3} + \cos x\right) - \int -\left(\frac{\cos 3x}{3} + \cos x\right)dx\right)$$

$$I = \frac{1}{2}\left[-x\frac{\cos 3x}{3} + x\cos x + \frac{1}{9}\sin 3x - \sin x\right] + c$$

51. Question

Evaluate the following integrals:

$\int (\tan^{-1} x^2)\, x\, dx$

Answer

Let $I = \int (\tan^{-1}x^2) x\, dx$

$X^2=t$

$2xdx=dt$

$$I = \frac{1}{2}\int (\tan^{-1}t)\, dt$$

Using integration by parts,

$$= \frac{1}{2}\left(\tan^{-1}t \int dt - \int \frac{d}{dt}\tan^{-1}t \int dt\right)$$

We know that,

$$\frac{d}{dt}\tan^{-1}t = \frac{1}{2(1+t^2)}$$

$$= \frac{1}{2}\left[t\tan^{-1}t - \int \frac{t}{(1+t^2)}dt\right]$$

$$= \frac{t}{2}\tan^{-1}t - \frac{1}{4}\int \frac{2t}{1+t^2}dt$$

$$= \frac{t}{2}\tan^{-1}t - \frac{1}{4}\log|1+t^2| + c$$

$$= \frac{x^2}{2}\tan^{-1}x^2 - \frac{1}{4}\log|1+x^4| + c$$

52. Question

Evaluate the following integrals:

$$\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$$

Answer

Let $I = \int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

We are splitting this in to two functions

First we find the integral of:

$$\int \frac{x}{\sqrt{1-x^2}}\,dx$$

Put $1-x^2=t$

$-2xdx=dt$

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int \frac{dt}{\sqrt{t}} = -\sqrt{t} = -\sqrt{1-x^2}$$

$$I = \int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx$$

Using integration by parts,

$$= (\sin^{-1}x) \times -\sqrt{1-x^2} - \int \frac{1}{\sqrt{1-x^2}}(-\sqrt{1-x^2})\,dx$$

$$= (\sin^{-1}x) \times -\sqrt{1-x^2} - \int dx$$

$$= (\sin^{-1}x) \times -\sqrt{1-x^2} + x + c$$

$$= x - \sqrt{1-x^2}(\sin^{-1}x) + c$$

53. Question

Evaluate the following integrals:

$\int \sin^3 \sqrt{x}\, dx$

Answer

Let

$\sqrt{x} = t$

$x = t^2$

dx=2tdt

$I = 2\int t\sin^3 t dt$

$= 2\int t\left(\frac{3\sin t - \sin 3t}{4}\right) dt$

$= \frac{1}{2}\int t(3\sin t - \sin 3t) dt$

Using integration by parts,

$= \frac{1}{2}\left[t\left(-3\cos t + \frac{1}{3}\cos 3t\right) - \int\left(-3\cos t + \frac{\cos 3t}{3}\right) dt\right]$

$= \frac{1}{2}\left[\frac{-9t\cos t + t\cos 3t}{3} - \left\{-3\sin t + \frac{\sin 3t}{9}\right\}\right] + c$

$= \frac{1}{2}\left[\frac{-9\cos t + t\cos 3t}{3} + \frac{27\sin t - 3\sin 3t}{9}\right] + c$

$= \frac{1}{18}[-27\cos t + 3t\cos 3t + 27\sin t - 3\sin 3t] + c$

$I = \frac{1}{18}\left[3\sqrt{x}\cos 3\sqrt{x} + 27\sin\sqrt{x} - 27\sqrt{x}\cos\sqrt{x} - 3\sin 3\sqrt{x}\right] + c$

54. Question

Evaluate the following integrals:

$\int x \sin^3 x\, dx$

Answer

Let $I = \int x\sin^3 x\, dx$

We know that, $\sin^3 x = \frac{3\sin x - \sin 3x}{4}$

$= \int x\left(\frac{3\sin x - \sin 3x}{4}\right) dx$

$= \frac{1}{4}\int x(3\sin x - \sin 3x) dx$

Using integration by parts,

$I = \frac{1}{4}\left[x\int(3\sin x - \sin 3x)dx - \int 1\int(3\sin x - \sin 3x)dx\right]$

$= \frac{1}{4}\left[x\left(-3\cos x + \frac{\cos 3x}{3}\right) - \int\left(-3\cos x + \frac{\cos 3x}{3}\right) dx\right]$

$= \frac{1}{4}\left[-3x\cos x + \frac{x\cos 3x}{3} + 3\sin x - \frac{\sin 3x}{9}\right] + c$

$I = \frac{1}{36}[3x\cos 3x - 27x\cos x + 27\sin x - \sin 3x] + c$

55. Question

Evaluate the following integrals:

$\int \cos^3 \sqrt{x}\, dx$

Answer

Let

$\sqrt{x} = t$

$x = t^2$

dx=2tdt

let $I = 2\int t\cos^3 t dt$

we know that, $\cos^3 t dt = \frac{3\cos t + \cos 3t}{4}$

$= 2\int t\left(\frac{3\cos t + \cos 3t}{4}\right)dt$

$= \frac{1}{2}\int t(3\cos t - \cos 3t)dt$

Using integration by parts,

$= \frac{1}{2}\left[t\left(3\sin t + \frac{1}{3}\sin 3t\right) + \int\left(3\sin t + \frac{\sin.3t}{3}\right)dt\right]$

$= \frac{1}{2}\left[\frac{9t\sin t + t\sin 3t}{3} + \left\{3\cos t + \frac{\cos 3t}{9}\right\}\right] + c$

$= \frac{1}{18}[27\, t\sin t + 3t\sin 3t + 9\cos t + \cos 3t] + c$

$I = \frac{1}{18}[27\sqrt{x}\sin\sqrt{x} + 3\sqrt{x}\,\sin 3\sqrt{x} + 9\cos\sqrt{x} + \cos 3\sqrt{x}] + c$

56. Question

Evaluate the following integrals:

$\int x\cos^3 x\, dx$

Answer

Let $I = \int x\cos^3 x\, dx$

we know that, $\cos^3 t dt = \frac{3\cos t + \cos 3t}{4}$

$= \int x\left(\frac{3\cos x + \cos 3x}{4}\right)dx$

$= \frac{1}{4}\int x(3\cos x + \cos 3x)dx$

Using integration by parts,

$I = \frac{1}{4}\left[x\int(3\cos x + \cos 3x)dx - \int 1\int(3\cos x + \cos 3x)dx\right]$

$$= \frac{1}{4}\left[x\left(3\sin x + \frac{\sin 3x}{3}\right) - \int\left(3\sin x + \frac{\sin 3x}{3}\right)dx\right]$$

$$= \frac{1}{4}\left[3x\sin x + \frac{x\sin 3x}{3} + 3\cos x + \frac{\cos 3x}{9}\right] + c$$

$$I = \frac{3x\sin x}{4} + \frac{x\sin x}{12} + \frac{3\cos x}{4} + \frac{\cos 3x}{36} + c$$

57. Question

Evaluate the following integrals:

$$\int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx$$

Answer

Let $I = \int \tan^{-1}\sqrt{\frac{1-x}{1+x}}dx$

$x = \cos\theta\,; dx = -\sin\theta d\theta$

$$I = \int \tan^{-1}(\tan\frac{\theta}{2}) - \sin\theta d\theta$$

$$= -\frac{1}{2}\int \theta\sin\theta d\theta$$

Using integration by parts,

$$= -\frac{1}{2}\left[\theta\int \sin\theta d\theta - \int \frac{d}{d\theta}\theta\int \sin\theta d\theta\right]$$

$$= \frac{1}{2}[-\theta\cos\theta + \int \cos\theta d\theta]$$

$$= \frac{1}{2}[-\theta\cos\theta + \sin\theta] + c$$

$$I = \frac{1}{2}[-x\cos^{-1}x + \sqrt{1-x^2}] + c$$

58. Question

Evaluate the following integrals:

$$\int \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$$

Answer

Let $I = \int \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$

Let $x = a\tan^2\theta$

$dx = 2a\tan^2\theta\sec^2\theta$

$$I = \int\left(\sin^{-1}\sqrt{\frac{a\tan^2\theta}{a + a\tan^2\theta}}\right)2a\tan^2\theta\sec^2\theta d\theta$$

$$= \int \sin^{-1}(\sin\theta)2a\tan^2\theta\sec^2\theta d\theta$$

$$= \int 2\theta a \tan^2\theta \sec^2\theta d\theta$$

$$= 2a \int \theta \tan^2\theta \sec^2\theta d\theta$$

Using integration by parts,

$$= 2a \left(\theta \int \tan^2\theta \sec^2\theta d\theta - \int 1 \int \tan^2\theta \sec^2\theta d\theta\right)$$

$$= 2a\left[\theta \frac{\tan^2\theta}{2} - \int \frac{\tan^2\theta}{2} d\theta\right]$$

$$= a\theta\tan^2\theta - \frac{2a}{2}\int (\sec^2\theta - 1)d\theta$$

$$= a\theta\tan^2\theta - a\tan\theta + a\theta + c$$

$$= a\left(\tan^{-1}\sqrt{\frac{x}{a}}\right)\frac{x}{a} - a\sqrt{\frac{x}{a}} + a\tan^{-1}\sqrt{\frac{x}{a}} + c$$

$$= x\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax} + a\tan^{-1}\sqrt{\frac{x}{a}} + c$$

59. Question

Evaluate the following integrals:

$$\int \frac{x^3 \sin^{-1} x^2}{\sqrt{1-x^4}} dx$$

Answer

Let $I = \int \frac{x^3 \sin^{-1} x^2}{\sqrt{1-x^4}} dx$

$\sin^{-1}x^2 = t$

$$\frac{1}{\sqrt{1-x^4}} 2x dx = dt$$

$$I = \int \frac{x^2 \sin^{-1} x^2}{\sqrt{1-x^4}} x dx$$

$$= \int (\sin t)\, t \frac{dt}{2}$$

Using integration by parts,

$$= \frac{1}{2}\left[t \int \sin t dt - \int \frac{d}{dt} t \int \sin t dt\right]$$

$$= \frac{1}{2}\left[-t\cos t - \int -\cos t dt\right]$$

$$= \frac{1}{2}[-t\cos t + \sin t] + c$$

$$= \frac{1}{2}\left[x^2 - \sqrt{1-x^4}\sin^{-1}x^2\right] + c$$

60. Question

Evaluate the following integrals:

$$\int \frac{x^2 \sin^{-1} x}{\left(1-x^2\right)^{3/2}} dx$$

Answer

Let $I = \int \frac{x^2 \sin^{-1} x}{(1-x^2)^{3/2}} dx$

$\sin^{-1} x = t$

$$\frac{1}{\sqrt{1-x^2}} dx = dt$$

$$I = \int \frac{\sin^2 t \times t dt}{1-\sin^2 t}$$

$$= \int \frac{t\sin^2 t}{\cos^2 t} dt$$

$$= \int t\tan^2 t \, dt$$

$$= \int t(\sec^2 t - 1) dt$$

Using integration by parts,

$$= \int t\sec^2 t \, dt - \int t dt$$

$$= t \int \sec^2 t dt - \int \frac{d}{dt} t \int \sec^2 t dt - \frac{t^2}{2}$$

We know that, $\int \sec^2 t \, dt = \tan t$

$$= t\tan t - \int \tan t \, dt - \frac{t^2}{2}$$

$$= t\tan t - \log|\sec t| - \frac{t^2}{2} + c$$

$$I = \frac{x}{\sqrt{1-x^2}} \sin^{-1} x + \log|1-x^2| - \frac{1}{2}(\sin^{-1} x)^2 + c$$

Exercise 19.26

1. Question

Evaluate the following integrals:

$\int e^x (\cos x - \sin x) \, dx$

Answer

Let $I = \int e^x (\cos x - \sin x) dx$

Using integration by parts,

$$= \int e^x \cos x \, dx - \int e^x \sin x \, dx$$

We know that, $\frac{d}{dx} \cos x = -\sin x$

$$= \cos x \int e^x - \int \frac{d}{dx}\cos x \int e^x - \int e^x \sin x\, dx$$

$$= e^x \cos x + \int e^x \sin x\, dx - \int e^x \sin x\, dx$$

$$= e^x \cos x + c$$

2. Question

Evaluate the following integrals:

$$\int e^x \left(\frac{1}{x^2} - \frac{2}{x^3}\right) dx$$

Answer

Let $I = \int e^x \left(\frac{1}{x^2} - \frac{2}{x^3}\right) dx$

$$= \int e^x x^{-2} dx - 2\int e^x x^{-3} dx$$

Integrating by parts

$$= x^{-2} \int e^x dx - \int \frac{d}{dx} x^{-2} \int e^x dx - 2\int e^x x^{-3} dx$$

We know that,

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$= e^x x^{-2} + 2\int e^x x^{-3} dx - 2\int e^x x^{-3} dx$$

$$= \frac{e^x}{x^2} + c$$

3. Question

Evaluate the following integrals:

$$\int e^x \left(\frac{1 + \sin x}{1 + \cos x}\right) dx$$

Answer

Let $I = \int e^x \left(\frac{1+\sin x}{1+\cos x}\right) dx$

We know that, $\sin^2 x + \cos^2 x = 1$ and $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

$$= e^x \left(\frac{\sin^2\frac{x}{2} + \cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)$$

$$= \frac{e^x \left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2}{2\cos^2\frac{x}{2}}$$

$$= \frac{1}{2} e^x \left(\frac{\sin\frac{x}{2} + \cos\frac{x}{2}}{2\cos\frac{x}{2}}\right)^2$$

$= \frac{1}{2} e^x \left[\tan\frac{x}{2} + 1\right]^2$

$= \frac{1}{2} e^x \left[1 + \tan\frac{x}{2}\right]^2$

$= \frac{1}{2} e^x \left[1 + \tan^2\frac{x}{2} + 2\tan\frac{x}{2}\right]$

$= \frac{1}{2} e^x \left[\sec^2\frac{x}{2} + 2\tan\frac{x}{2}\right]$

$= e^x \left[\frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2}\right] \ldots\ldots(1)$

Let $\tan\frac{x}{2} = f(x)$

$f'(x) = \frac{1}{2}\sec^2\frac{x}{2}$

We know that,

$\int e^x\{f(x) + f'(x)\}dx = e^x f(x) + c$

From equation(1), we obtain

$\int e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx = e^x \tan\frac{x}{2} + c$

4. Question

Evaluate the following integrals:

$\int e^x (\cot x - \mathrm{cosec}^2 x)\, dx$

Answer

Let $I = \int e^x(\cot x - \mathrm{cosec}^2 x)dx$

$= \int e^x \cot x dx - \int e^x \mathrm{cosec}^2 x dx$

Integrating by parts,

$= \cot x \int e^x dx - \int \frac{d}{dx}\cot x \int e^x dx - \int e^x \mathrm{cosec}^2 x dx$

$= \cot x\, e^x + \int e^x \mathrm{cosec}^2 x dx - \int e^x \mathrm{cosec}^2 x dx$

$= e^x \cot x + c$

5. Question

Evaluate the following integrals:

$\int e^x \left(\frac{x-1}{2x^2}\right) dx$

Answer

$\int e^x \left(\frac{x-1}{2x^2}\right) dx$

Let $I = \int e^x \frac{1}{2x} dx - \int e^x \frac{1}{2x^2} dx$

Integrating by parts,

$$= \frac{e^x}{2x} - \int e^x\left(\frac{d}{dx}\left(\frac{1}{2x}\right)\right)dx - \int \frac{e^x}{2x^2}dx$$

$$= \frac{e^x}{2x} + \int \frac{e^x}{2x^2}dx - \int \frac{e^x}{2x^2}dx$$

$$= \frac{e^x}{2x} + c$$

6. Question

Evaluate the following integrals:

$\int e^x \sec x\ (1 + \tan x)\ dx$

Answer

Let $I = \int e^x \sec x(1 + \tan x)dx$

$$= \int e^x \sec x dx + \int e^x \sec x \tan x dx$$

Integrating by parts,

$$= e^x \sec x dx - \int e^x \frac{d}{dx} \sec x dx + \int e^x \sec x \tan x dx$$

$$= e^x \sec x dx - \int e^x \sec x \tan x dx + \int e^x \sec x \tan x dx$$

$$= e^x \sec x dx + c$$

7. Question

Evaluate the following integrals:

$\int e^x (\tan x - \log \cos x)\ dx$

Answer

Let $I = \int e^x(\tan x - \log\cos x)dx$

$$I = \int e^x \tan x dx - \int e^x \log\cos x dx$$

Integrating by parts,

$$= \int e^x \tan x dx - \{e^x \log\cos x - \int e^x\left(\frac{d}{dx}\log\cos x\right)dx$$

$$= \int e^x \tan x dx - e^x \log\cos x dx - \int e^x \tan x dx$$

$$= -e^x \log\cos x dx + c$$

$$= e^x \log\sec x + c$$

8. Question

Evaluate the following integrals:

$\int e^x [\sec x + \log (\sec x + \tan x)]\ dx$

Answer

Let $I = \int e^x[\sec x + \log(\sec x + \tan x)]dx$

$$I = \int e^x \sec x dx + \int \log(\sec x + \tan x) dx$$

Integrating by parts

$$= \int e^x \sec x \, dx + e^x \log(\sec x + \tan x) - \int e^x \sec x \, dx$$

$$= e^x \log(\sec x + \tan x) + c$$

9. Question

Evaluate the following integrals:

$\int e^x (\cot x + \log \sin x) \, dx$

Answer

Let $I = \int e^x (\cot x + \log \sin x) dx$

$$= \int e^x \cot x \, dx + \int e^x \log \sin x \, dx$$

Integrating by parts

$$= \int e^x \log \sin x \, dx + \int e^x \cot x \, dx$$

$$= (\log \sin x) e^x - \int e^x \frac{d}{dx} \log \sin x dx + \int e^x \cot x \, dx + c$$

$$= (\log \sin x) e^x - \int e^x \cot x \, dx + \int e^x \cot x \, dx + c$$

$$= (\log \sin x) e^x + c$$

10. Question

Evaluate the following integrals:

$$\int e^x \frac{x-1}{(x+1)^3} dx$$

Answer

Let $I = \int e^x \frac{x+1-2}{(x+1)^3} dx$

$$= \int e^x \left\{ \frac{1}{(x+1)^2} + \frac{-2}{(x+1)^2} \right\} dx$$

$$= \int e^x \frac{1}{(x+1)^2} dx + \int e^x \frac{-2}{(x+1)^2} dx$$

Integrating by parts

$$= e^x \frac{1}{(x+1)^2} - \int e^x \frac{-2}{(x+1)^2} + \int e^x \frac{-2}{(x+1)^2}$$

$$= e^x \frac{1}{(x+1)^2} + c$$

11. Question

Evaluate the following integrals:

$$\int e^{x}\left(\frac{\sin 4x-4}{1-\cos 4x}\right)dx$$

Answer

Let $I=\int e^{x}\left(\frac{\sin 4x-4}{1-\cos 4x}\right)dx$

$$=\int e^{x}\left\{\frac{2\sin 2x\cos 2x}{2\sin^{2}x}-\frac{4}{2\sin^{2}x}\right\}dx$$

$$=\int e^{x}\{\cot 2x-2\,\mathrm{cosec}^{2}2x\}dx$$

$$=\int e^{x}\cot 2x dx-\int e^{x}\,2\,\mathrm{cosec}^{2}2x\}dx$$

Integrating by parts,

$$=e^{x}\cot 2x-\int e^{x}\frac{d}{dx}\cot 2x\,dx-2\int e^{x}\,\mathrm{cosec}^{2}2x\,dx$$

$$=e^{x}\cot 2x+2\int e^{x}\,\mathrm{cosec}^{2}2x-2\int e^{x}\,\mathrm{cosec}^{2}2x$$

$$=e^{x}\cot 2x+c$$

12. Question

Evaluate the following integrals:

$$\int\frac{2-x}{(1-x)^{2}}e^{x}\,dx$$

Answer

Let $I=\int\frac{2-x}{(1-x)^{2}}e^{x}dx$

$$=\int e^{x}\left\{\frac{(1-x)+1}{(1-x)^{2}}\right\}dx$$

$$=\int e^{x}\left\{\frac{1}{1-x}+\frac{1}{(1-x)^{2}}\right\}$$

$$\frac{1}{1-x}=f(x)\,\frac{1}{(1-x)^{2}}=f'(x)$$

We know that, $\int e^{x}\{f(x)+f'(x)\}=e^{x}f(x)+c$

$$=e^{x}\frac{1}{1-x}+c$$

13. Question

Evaluate the following integrals:

$$\int e^{x}\frac{1+x}{(2+x)^{2}}dx$$

Answer

Let $I=\int\frac{1+x}{(2+x)^{2}}e^{x}dx$

$$= \int e^x \left\{\frac{(x+2)-1}{(x+2)^2}\right\} dx$$

$$= \int e^x \left\{\frac{1}{x+2} - \frac{1}{(x+2)^2}\right\}$$

$$= \int e^x \frac{1}{x+2} dx - \int e^x \frac{1}{(x+2)^2} dx$$

Using integration by parts,

$$= \frac{e^x}{x+2} + \int e^x \frac{1}{(x+2)^2} dx - \int e^x \frac{1}{(x+2)^2} dx$$

$$= e^x \frac{1}{x+2} + c$$

14. Question

Evaluate the following integrals:

$$\int \frac{\sqrt{1-\sin x}}{1+\cos x} e^{-x/2} dx$$

Answer

Let $I = \int \frac{\sqrt{1-\sin x}}{1+\cos x} e^{-x/2} dx$

put $\frac{x}{2} = t \Rightarrow x = 2t \Rightarrow dx = 2dt$

$$\int \frac{\sqrt{1-\sin x}}{1+\cos x} e^{-x/2} dx = 2\int \frac{\sqrt{1-\sin 2t}}{1+\cos 2t} e^{-t} dt$$

$$= 2\int \frac{\sqrt{\sin^2 t + \cos^2 t - 2\sin t\cos t}}{1+\cos 2t} e^{-t} dt$$

$$= 2\int \frac{\sqrt{(\cos t - \sin t)^2}}{2\cos^2 t} e^{-t} dt$$

$$= \int (\sec t - \tan t \sec t) e^{-t} dt$$

$$= \int \sec t \, e^{-t} dt - \int \tan t \sec t \, e^{-t} dt$$

Integrating by parts

$$= e^{-t} \sec t + \int \tan t \sec t \, e^{-t} dt - \int \tan t \sec t \, e^{-t} dt$$

$$= e^{-t} \sec t + c$$

$$= e^{-\frac{x}{2}} \sec \frac{x}{2} + c$$

15. Question

Evaluate the following integrals:

$$\int e^x \left(\log x + \frac{1}{x}\right) dx$$

Answer

Let $I = \int e^x\left(\log x + \frac{1}{x}\right)dx$

We know that

$$\int e^x\{f(x) + f'(x)\} = e^x f(x) + c$$

Here,

$$f(x) = \log x; f'(x) = \frac{1}{x}$$

$$\int e^x\left(\log x + \frac{1}{x}\right)dx = e^x \log x + c$$

16. Question

Evaluate the following integrals:

$$\int e^x\left(\log x + \frac{1}{x^2}\right)dx$$

Answer

Let $I = \int e^x\left(\log x + \frac{1}{x^2}\right)dx$

$$= \int e^x\left(\log x + \frac{1}{x} - \frac{1}{x} + \frac{1}{x^2}\right)dx$$

$$= \int e^x\left(\log x - \frac{1}{x}\right)dx + \int e^x\left(\frac{1}{x} + \frac{1}{x^2}\right)dx$$

Using integration by parts,

$$= e^x\left(\log x - \frac{1}{x}\right) - \int e^x \frac{d}{dx}\left(\log x - \frac{1}{x}\right)dx + \int e^x\left(\frac{1}{x} + \frac{1}{x^2}\right)dx$$

$$= e^x\left(\log x - \frac{1}{x}\right) - \int e^x\left(\frac{1}{x} + \frac{1}{x^2}\right)dx + \int e^x\left(\frac{1}{x} + \frac{1}{x^2}\right)dx$$

$$= e^x\left(\log x - \frac{1}{x}\right) + c$$

17. Question

Evaluate the following integrals:

$$\int \frac{e^x}{x}\left\{x(\log x)^2 + 2\log x\right\}dx$$

Answer

Let $I = \int \frac{e^x}{x}\{x(\log x)^2 + 2\log x\}dx$

$$= \int e^x(\log x)^2\,dx + 2\int \frac{e^x}{x}\log x\,dx$$

Using integration by parts,

$$= e^x(\log x)^2 - \int e^x \frac{d}{dx}(\log x)^2 + 2\int \frac{e^x}{x}\log x\,dx$$

$$= e^x(\log x)^2 - 2\int \frac{e^x}{x}\log x\,dx + 2\int \frac{e^x}{x}\log x\,dx$$

$= e^x(\log x)^2 + c$

18. Question

Evaluate the following integrals:

$$\int e^x.\frac{\sqrt{1-x^2}\sin^{-1}x+1}{\sqrt{1-x^2}}dx$$

Answer

Let $I = \int e^x \frac{\sqrt{1-x^2}\sin^{-1}x+1}{\sqrt{1-x^2}}dx$

$$I = \int e^x \sin^{-1}x + \int e^x \frac{1}{\sqrt{1-x^2}}dx$$

Integrating by parts

$$= e^x \sin^{-1}x - \int e^x\left(\frac{d}{dx}(\sin^{-1}x)\right)dx + \int e^x \frac{1}{\sqrt{1-x^2}}dx$$

$$= e^x \sin^{-1}x - \int e^x \frac{1}{\sqrt{1-x^2}}dx + \int e^x \frac{1}{\sqrt{1-x^2}}dx$$

$$= e^x \sin^{-1}x + c$$

19. Question

Evaluate the following integrals:

$\int e^{2x}$ (– sin x + 2 cos x) dx

Answer

Let $I = \int e^{2x}(-\sin x + 2\cos x)dx$

$$I = \int e^{2x} - \sin x dx + 2\int e^{2x}\cos x\, dx$$

Applying by parts in the second integral,

$$I = -\int e^{2x}\sin x\, dx + 2\left\{\frac{1}{2}e^{2x}\cos x + \int \frac{1}{2}e^{2x}\sin x\, dx\right\}$$

$$= -\int e^{2x}\sin x dx + e^{2x}\cos x + \int e^{2x}\sin x dx + c$$

$$= e^{2x}\cos x + c$$

20. Question

Evaluate the following integrals:

$$\int e^x\left(\tan^{-1}x + \frac{1}{1+x^2}\right)dx$$

Answer

Let $I = \int e^x\left(\tan^{-1}x + \frac{1}{1+x^2}\right)dx$

here, $f(x) = \tan^{-1}x$ and $f'(x) = \frac{1}{1+x^2}$

and we know that,

$$\int e^x\{f(x)+f'(x)\}=e^xf(x)+c$$

$$\int e^x\left(\tan^{-1}x+\frac{1}{1+x^2}\right)dx=e^x\tan^{-1}x+c$$

21. Question

Evaluate the following integrals:

$$\int e^x\left(\frac{\sin x\cos x-1}{\sin^2 x}\right)dx$$

Answer

Let $I=\int e^x\left(\frac{\sin x\cos x-1}{\sin^2 x}\right)dx$

$$=\int e^x(\cot x-\operatorname{cosec}^2 x)dx$$

$$=\int e^x(\cot x+-\operatorname{cosec}^2 x)dx$$

We know that, $\int e^x\{f(x)+f'(x)\}=e^xf(x)+c$

let $f(x)=\cot x\,; f'(x)=-\operatorname{cosec}^2 x$

$$\int e^x\left(\frac{\sin x\cos x-1}{\sin^2 x}\right)dx=e^x\cot x+c$$

22. Question

Evaluate the following integrals:

$\int \{\tan(\log x)+\sec^2(\log x)\}\,dx$

Answer

Let $I=\int[\tan(\log x)+\sec^2(\log x)]dx$

$\log x=z\ \Rightarrow x=e^z\Rightarrow dx=e^z dz$

$$I=\int(\tan z+\sec^2 z)e^z dz$$

$f(z)=\tan z; f'(z)=\sec^2 z$

We know that, $\int e^x\{f(x)+f'(x)\}=e^xf(x)+c$

$I=x\tan(\log x)+c$

23. Question

Evaluate the following integrals:

$$\int e^x\frac{(x-4)}{(x-2)^3}dx$$

Answer

Let $I=\int e^x\frac{(x-4)}{(x-2)^3}dx$

$$=\int e^x\frac{(x-2)-2)}{(x-2)^3}dx$$

$= \int e^x\left\{\frac{1}{(x-2)^2} - \frac{2}{(x-2)^2}\right\}dx$

Let $f(x) = \frac{1}{(x-2)^2}$ and $f'(x) = \frac{2}{(x-2)^2}$

We know that, $\int e^x\{f(x) + f'(x)\} = e^x f(x) + c$

$I = \frac{e^x}{(x-2)^2} + c$

24. Question

Evaluate the following integrals:

$$\int e^{2x}\left(\frac{1-\sin 2x}{1-\cos 2x}\right)dx$$

Answer

Let $I = \int e^{2x}\left(\frac{1-\sin 2x}{1-\cos 2x}\right)dx$

We have,

$\cos 2x = 1 - 2\sin^2 x$

$I = e^{2x}\left(\frac{1-\sin 2x}{1-(1-2\sin^2 x)}\right)dx$

$= \int e^{2x}\left(\frac{1-\sin 2x}{2\sin^2 x}\right)dx$

$= \int e^{2x}\left(\frac{\operatorname{cosec}^2 x}{2} - \frac{2\sin x\cos x}{2\sin^2 x}\right)dx$

$= \int e^{2x}\left(\frac{\operatorname{cosec}^2 x}{2} - \frac{\cos x}{\sin x}\right)dx$

$= \int e^{2x}\left(\frac{\operatorname{cosec}^2 x}{2} - \cot x\right)dx$

Using integration by parts,

$= \frac{1}{2}\int e^{2x}\operatorname{cosec}^2 x\,dx - \int e^{2x}\cot x\,dx$

That is,

$I = I_1 + I_2$

$I_1 = \frac{1}{2}\int e^{2x}\operatorname{cosec}^2 x\,dx$

$I_2 = \int e^{2x}\cot x\,dx$

Consider

$I_1 = \frac{1}{2}\int e^{2x}\operatorname{cosec}^2 x\,dx$

take e^{2x} as first function and $\operatorname{cosec}^2 x$ as second function

$u = e^{2x}; du = 2e^{2x}dx$

$$\int \mathrm{cosec}^2 x \, dx = \int dv$$

Let $v = -\cot x$

$$I_1 = \frac{1}{2}\left[e^{2x}(-\cot x) - \int (-\cot x) 2e^{2x} dx\right]$$

$$I_1 = \frac{1}{2}\left[e^{2x}(-\cot x) - 2\int \cot x e^{2x} dx\right]$$

$$I_1 = \frac{1}{2}(e^{2x}(-\cot x)) + \int \cot x e^{2x} dx$$

Thus,

$$I = \frac{1}{2}(e^{2x}(-\cot x)) + \int \cot x e^{2x} dx - \int e^{2x} \cot x dx$$

$$I = \frac{1}{2}[e^{2x}(-\cot x)] + c$$

Exercise 19.27

1. Question

Evaluate the following integrals:

$\int e^{ax} \cos bx \, dx$

Answer

Let $I = e^{ax} \cos bx \, dx$

Integrating by parts,

$$I = e^{ax}\frac{\sin bx}{b} - a\int e^{ax}\frac{\sin bx}{b} dx$$

$$= \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\int e^{ax}\sin bx \, dx$$

$$= \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\left[-e^{ax}\frac{\cos bx}{b} - a\int e^{ax}\frac{\cos bx}{b} dx\right]$$

$$= \frac{1}{b}e^{ax}\sin bx - \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}\int e^{ax}\cos bx \, dx$$

$$I = \frac{e^{ax}}{b^2}[b \sin bx + a\cos bx] - \frac{a^2}{b^2} I + c$$

$$= \frac{e^{ax}}{a^2 + b^2}[b \cos bx + a\cos bx] + c$$

2. Question

Evaluate the following integrals:

$\int e^{ax} \sin (bx + c) \, dx$

Answer

Let $I = \int e^{ax} \sin(bx + c) \, dx$

$$= -e^{ax}\frac{\cos(bx+c)}{b} + \int ae^{ax}\frac{\cos(bx+c)}{b} dx$$

$$= -\frac{1}{b}e^{ax}\cos(bx+c) + \frac{a}{b}\int e^{ax}\cos(bx+c)$$

$$I = \frac{e^{ax}}{b^2}\{a\sin(bx+c) - b\cos(bx+c)\} - \frac{a^2}{b^2}I + c_1$$

$$I = \left\{\frac{a^2+b^2}{b^2}\right\} - \frac{e^{ax}}{b^2}\{a\sin(bx+c) - b\cos(bx+c)\} + c_1$$

$$= \frac{e^{ax}}{a^2+b^2}\{a\sin(bx+c) - b\cos(bx+c)\}$$

3. Question

Evaluate the following integrals:

$\int \cos(\log x)\,dx$

Answer

Let $I = \int \cos(\log x)\,dx$

Let log x=t

$$\frac{1}{x}dx = dt$$

dx=xdt

$$= \int e^t \cos t\,dt$$

We know that, $\int \cos(\log x)\,dx = \frac{e^{ax}}{a^2+b^2}\{a\sin(bx+c) - b\cos(bx+c)\}$

Hence, a=1, b=1

So ,$I = \frac{e^t}{2}[\cos t + \sin t] + c$

Hence,

$$\int \cos(\log x)\,dx = \frac{e^{\log x}}{2}\{\cos(\log x) + \sin(\log x)\} + c$$

$$I = \frac{x}{2}\{\cos(\log x) + \sin(\log x)\} + c$$

4. Question

Evaluate the following integrals:

$\int e^{2x}\cos(3x+4)\,dx$

Answer

Let $I = \int e^{2x}\cos(3x+4)\,dx$

Integrating by parts

$$I = e^{2x}\frac{\sin(3x+4)}{3} - \int 2e^{2x}\frac{\sin(3x+4)}{3}dx$$

$$= \frac{1}{3}e^{2x}\sin(3x+4) - \frac{2}{3}\int e^{2x}\sin(3x+4)\,dx$$

$$= \frac{1}{3}e^{2x}\sin(3x+4) - \frac{2}{3}\left\{-e^{2x}\frac{\cos(3x+4)}{3} + \int 2e^{2x}\frac{\cos(3x+4)}{3}dx\right\}$$

$$I = \frac{e^{2x}}{9}[2\cos(3x+4) + 3\sin(3x+4)] + c$$

Hence,

$$I = \frac{e^{2x}}{9}[2\cos(3x+4) + 3\sin(3x+4)] + c$$

5. Question

Evaluate the following integrals:

$\int e^{2x} \sin x \cos x \, dx$

Answer

Let $I = \int e^{2x} \sin x \cos x dx$

$$= \frac{1}{2}\int e^{2x} 2 \sin x \cos x dx$$

$$= \frac{1}{2}\int e^{2x} \sin 2x \, dx$$

We know that,

$$\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2}\{a \sin bx - b \cos bx\} + c$$

$$= \frac{e^{2x}}{8}\{2 \sin 2x - 2 \cos 2x\} + c$$

$$I = \frac{1}{2}\frac{e^{2x}}{8}\{2 \sin 2x - 2 \cos 2x\} + c$$

$$I = \frac{e^{2x}}{8}\{\sin 2x - \cos 2x\} + c$$

6. Question

Evaluate the following integrals:

$e^{2x} \sin x \, dx$

Answer

Let $I = \int e^{2x} \sin x \, dx$

Integrating by parts,

$$I = \sin x \int e^{2x} \, dx - \int \frac{d}{dx}\sin x \int e^{2x} dx$$

$$I = \sin x \frac{e^{2x}}{2} - \int \cos x \frac{e^{2x}}{2} dx$$

$$I = \sin x \frac{e^{2x}}{2} - \frac{1}{2}\int e^{2x} \cos x dx$$

Again integrating by parts,

$$I = \sin x \frac{e^{2x}}{2} - \frac{1}{2}\left\{\cos x \int e^{2x} dx - \int \frac{d}{dx}\cos x \int e^{2x} dx\right\}$$

$$I = \sin x \frac{e^{2x}}{2} - \frac{1}{2}\left[\cos x \frac{e^{2x}}{2} - \int (-\sin x)\frac{e^{2x}}{2} dx\right]$$

$$I = \sin x\frac{e^{2x}}{2} - \frac{1}{2}\left[\cos x\frac{e^{2x}}{2} + \frac{1}{2}\int \sin x e^{2x} dx\right]$$

$$I = \sin x\frac{e^{2x}}{2} - \frac{1}{2}\cos x\frac{e^{2x}}{2} - \frac{1}{4}I$$

$$I + \frac{I}{4} = \sin x\frac{e^{2x}}{2} - \frac{1}{2}\cos x\frac{e^{2x}}{2}$$

$$\frac{5}{4}I = \frac{e^{2x}\sin x}{2} - \frac{e^{2x}\cos x}{4}$$

$$I = \frac{4}{5}\left[\frac{e^{2x}\sin x}{2} - \frac{e^{2x}\cos x}{4}\right] + c$$

$$I = \frac{e^{2x}}{5}[2\sin x - \cos x] + c$$

7. Question

Evaluate the following integrals:

$\int e^{2x} \sin (3x + 1)\, dx$

Answer

Let I = $\int e^{2x} \sin (3x + 1)\, dx$

Now Integrating by parts choosing sin (3x + 1) as first function and e^{2x} as second function we get,

$$I = \sin(3x+1)\int e^{2x}dx - \int\left(\frac{d}{dx}\sin(3x+1)\int e^{2x}dx\right)dx$$

$$I = \frac{e^{2x}}{2}\sin(3x+1) - \int \frac{3e^{2x}}{2}\cos(3x+1)\, dx$$

Now again integrating by parts by taking cos(3x + 1) as first function and e^{2x} as second function we get,

$$I = \frac{e^{2x}}{2}\sin(3x+1) - [\cos(3x+1)\int \frac{3e^{2x}}{2}dx - \int \frac{3}{2}\left(\frac{d}{dx}\cos(3x+1)\int e^{2x}dx\right) dx$$

$$I = \frac{e^{2x}}{2}\sin(3x+1) - \frac{3}{4}e^{2x}\cos(3x+1) - \frac{9}{4}\int e^{2x}\sin(3x+1)\, dx$$

$$\int e^{2x}\sin(3x+1)\, dx = I$$

Therefore,

$$I = \frac{e^{2x}}{2}\sin(3x+1) - \frac{3}{4}e^{2x}\cos(3x+1) - \frac{9}{4}I$$

$$I + \frac{9}{4}I = \frac{e^{2x}}{2}\sin(3x+1) - \frac{3}{4}e^{2x}\cos(3x+1)$$

$$\frac{13I}{4} = \frac{e^{2x}}{2}\sin(3x+1) - \frac{3}{4}e^{2x}\cos(3x+1)$$

$$I = \frac{e^{2x}}{13}\{2\sin(3x+1) - 3\cos(3x+1)\} + c$$

8. Question

Evaluate the following integrals:

$\int e^x \sin^2 x\, dx$

Answer

Let $I = \int e^x \sin^2 x\, dx$

$I = \frac{1}{2}\int e^x 2\sin^2 x\, dx$

$= \frac{1}{2}\int e^x(1 - \cos 2x)dx$

Using integration by parts,

$= \frac{1}{2}\int e^x dx - \frac{1}{2}\int e^x \cos 2x dx$

We know that, $\int e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}\{a\cos bx - b\sin bx\} + c$

$I = \frac{1}{2}\left[e^x - \frac{e^x}{5}(\cos 2x + 2\sin 2x)\right] + c$

$= \frac{e^x}{2} - \frac{e^x}{10}(\cos 2x + 2\sin 2x) + c$

9. Question

Evaluate the following integrals:

$\int \frac{1}{x^3}\sin(\log x)dx$

Answer

Let $I = \int \frac{1}{x^3}\sin(\log x)dx$

$\text{let} \log x = t \Rightarrow \frac{1}{x}dx = dt \Rightarrow dx = e^x dt$

We know that

$\int e^{ax}\sin bx dx = \frac{e^{ax}}{a^2 + b^2}\{a\sin bx - b\cos bx\} + c$

$\int e^{-2t}\sin t\, dt = \frac{e^{-2t}}{5}\{-2\sin t - \cos t\} + c$

$I = \frac{x^{-2}}{5}\{-2\sin(\log x) - \cos(\log x)\} + c$

$= \frac{-1}{5x^2}\{2\sin(\log x) + \cos(\log x)\} + c$

10. Question

Evaluate the following integrals:

$\int e^{2x} \cos^2 x\, dx$

Answer

Let $I = \int e^{2x}\cos^2 x\, dx$

$= \frac{1}{2}\int e^{2x} 2\cos^2 x\, dx$

$$= \frac{1}{2}\int e^{2x}(1 + \cos 2x)\, dx$$

$$= \frac{1}{2}\int e^{2x}\, dx + \frac{1}{2}\int e^{2x} \cos 2x dx$$

We know that, $\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2+b^2}\{a\cos bx - b\sin bx\} + c$

$$I = \frac{1}{2}\left[\frac{e^{2x}}{2} - \frac{e^{2x}}{8}(2\cos 2x + 2\sin 2x)\right] + c$$

$$= \frac{e^{2x}}{4} + \frac{e^{2x}}{16}(2\cos 2x + 2\sin 2x) + c$$

$$= \frac{e^{2x}}{4} + \frac{e^{2x}}{8}(\cos 2x + \sin 2x) + c$$

11. Question

Evaluate the following integrals:

$\int e^{-2x} \sin x\, dx$

Answer

Let $I = \int e^{-2x} \sin x\, dx$

We know that, $\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2+b^2}\{a\sin bx - b\cos bx\} + c$

$$= \frac{e^{-2x}}{5}\{-2\sin x - \cos x\} + c$$

12. Question

Evaluate the following integrals:

$$\int x^2 e^{x^3} \cos x^3\, dx$$

Answer

Let $I = \int x^2 e^{x^3} \cos x^3\, dx$

$x^3 = t$

$3x^2 dx = dt$

$$I = \frac{1}{3}\int e^t \cos t\, dt$$

We know that, $\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2+b^2}\{a\cos bx - b\sin bx\} + c$

$$I = \frac{1}{3}\left[\frac{e^t}{2}(\cos t + \sin t)\right] + c$$

$$I = \frac{1}{3}\left[\frac{e^{x^3}}{2}(\cos x^3 + \sin x^3)\right] + c$$

Exercise 19.28

1. Question

Evaluate the integral:

$$\int \sqrt{3+2x-x^2}\, dx$$

Answer

Key points to solve the problem:

• Such problems require the use of method of substitution along with method of integration by parts. By method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx)\, dx$

• To solve the integrals of the form: $\int \sqrt{ax^2+bx+c}\, dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$$

$$\int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$$

Let, I = $\int \sqrt{3+2x-x^2}\, dx$

∴ I = $\int \sqrt{3-(x^2-2(1)x)}dx = \int \sqrt{3-(x^2-2(1)x+1)+1}\, dx$

Using $a^2 - 2ab + b^2 = (a-b)^2$

We have:

I = $\int \sqrt{4-(x-1)^2}\, dx = \int \sqrt{2^2-(x-1)^2}dx$

As I match with the form: $\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

∴ I = $\frac{x-1}{2}\sqrt{4-(x-1)^2} + \frac{4}{2}\sin^{-1}\left(\frac{x-1}{2}\right) + C$

⇒ I = $\frac{1}{2}(x-1)\sqrt{3+2x-x^2} + 2\sin^{-1}\left(\frac{x-1}{2}\right) + C$

2. Question

Evaluate the integral:

$$\int \sqrt{x^2+x+1}\, dx$$

Answer

Key points to solve the problem:

• Such problems require the use of the method of substitution along with a method of integration by parts. By the method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx)\, dx$

• To solve the integrals of the form: $\int \sqrt{ax^2+bx+c}\, dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$$

$$\int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$$

Let, I = $\int \sqrt{(x^2+x+1)}\,dx$

$\therefore$ I = $\int \sqrt{x^2+2\left(\frac{1}{2}\right)x+\left(\frac{1}{2}\right)^2+1-\left(\frac{1}{2}\right)^2}\,dx$

Using $a^2 + 2ab + b^2 = (a + b)^2$

We have:

I = $\int \sqrt{\left(x+\frac{1}{2}\right)^2+1-\frac{1}{4}}\,dx = \int \sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx$

As I match with the form:

$$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$$

$\therefore$ I = $\frac{\left(x+\frac{1}{2}\right)}{2}\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right|+C$

$\Rightarrow$ I = $\frac{1}{4}(2x+1)\sqrt{x^2+x+1}+\frac{3}{8}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right|+C$

$\Rightarrow$ I = $\frac{1}{4}(2x+1)\sqrt{x^2+x+1}+\frac{3}{8}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right|+C$

3. Question

Evaluate the integral:

$$\int \sqrt{x-x^2}\,dx$$

Answer

Key points to solve the problem:

• Such problems require the use of the method of substitution along with a method of integration by parts. By the method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)\left(\int g(x)dx\right)dx$

• To solve the integrals of the form: $\int \sqrt{ax^2+bx+c}\,dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+C$$

$$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$$

$$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$$

Let, I = $\int \sqrt{x-x^2}\,dx$

$\therefore$ I = $\int \sqrt{-\left(x^2-2\left(\frac{1}{2}\right)x\right)}\,dx = \int \sqrt{\frac{1}{4}-\left(x^2-2\left(\frac{1}{2}\right)x+\left(\frac{1}{2}\right)^2\right)}\,dx$

Using $a^2 - 2ab + b^2 = (a - b)^2$

We have:

I = $\int \sqrt{\frac{1}{4}-\left(x-\frac{1}{2}\right)^2}\,dx = \int \sqrt{\left(\frac{1}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}\,dx$

As I match with the form: $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

$\therefore I = \frac{x - \frac{1}{2}}{2}\sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2} + \frac{\frac{1}{4}}{2}\sin^{-1}\left(\frac{x - \frac{1}{2}}{\frac{1}{2}}\right) + C$

$\Rightarrow I = \frac{1}{4}(2x - 1)\sqrt{x - x^2} + \frac{1}{8}\sin^{-1}(2x - 1) + C$

4. Question

Evaluate the integral:

$$\int \sqrt{1 + x - 2x^2}\,dx$$

Answer

Key points to solve the problem:

• Such problems require the use of method of substitution along with method of integration by parts. By method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx)\,dx$

• To solve the integrals of the form: $\int \sqrt{ax^2 + bx + c}\,dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$$

$$\int \sqrt{x^2 + a^2}\,dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C$$

Let, $I = \int \sqrt{1 + x - 2x^2}\,dx$

$\therefore I = \int \sqrt{1 - 2\left(x^2 - 2\left(\frac{1}{4}\right)x\right)}\,dx = \int \sqrt{1 - 2\left(x^2 - 2\left(\frac{1}{4}\right)x + \left(\frac{1}{4}\right)^2\right) + 2\left(\frac{1}{4}\right)^2}\,dx$

Using $a^2 - 2ab + b^2 = (a - b)^2$

We have:

$I = \int \sqrt{\frac{9}{8} - 2\left(x - \frac{1}{4}\right)^2}\,dx = \int \sqrt{2}\sqrt{\left(\frac{3}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2}\,dx$

As I match with the form: $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

$\therefore I = \sqrt{2}\left\{\frac{x - \frac{1}{4}}{2}\sqrt{\left(\frac{3}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2} + \frac{\frac{9}{16}}{2}\sin^{-1}\left(\frac{x - \frac{1}{4}}{\frac{3}{4}}\right)\right\} + C$

$\Rightarrow I = \frac{1}{8}(4x - 1)\sqrt{2\left\{\left(\frac{3}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2\right\}} + \frac{9\sqrt{2}}{32}\sin^{-1}\left(\frac{4x - 1}{3}\right) + C$

$\Rightarrow I = \frac{1}{8}(4x - 1)\sqrt{1 + x - 2x^2} + \frac{9\sqrt{2}}{32}\sin^{-1}\left(\frac{4x - 1}{3}\right) + C$

5. Question

Evaluate the integral:

$$\int \cos x\sqrt{4 - \sin^2 x}\,dx$$

Answer

Key points to solve the problem:

• Such problems require the use of the method of substitution along with a method of integration by parts. By the method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx)\,dx$

• To solve the integrals of the form: $\int \sqrt{ax^2+bx+c}\,dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$$

$$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$$

Let, I = $\int \cos x\sqrt{4-\sin^2 x}\,dx$

Let, sin x = t

Differentiating both sides:

⇒ cos x dx = dt

Substituting sin x with t, we have:

∴ I = $\int \sqrt{4-t^2}dt = \int \sqrt{2^2-t^2}dt$

As I match with the form: $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

∴ I = $\frac{t}{2}\sqrt{4-(t)^2} + \frac{4}{2}\sin^{-1}(\frac{t}{2}) + C$

Putting the value of t i.e. t = sin x

⇒ I = $\frac{1}{2}\sin x\sqrt{4-\sin^2 x} + 2\sin^{-1}\left(\frac{\sin x}{2}\right) + C$

6. Question

Evaluate the integral:

$$\int e^x\sqrt{e^{2x}+1}\,dx$$

Answer

Key points to solve the problem:

• Such problems require the use of method of substitution along with method of integration by parts. By method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx)\,dx$

• To solve the integrals of the form: $\int \sqrt{ax^2+bx+c}\,dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$$

$$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$$

Let, I = $\int e^x\sqrt{e^{2x}+1}\,dx$

Let, $e^x = t$

Differentiating both sides:

$\Rightarrow e^x\, dx = dt$

Substituting e^x with t, we have:

We have:

I = $\int \sqrt{t^2+1}\, dt = \int \sqrt{t^2+1^2}\, dt$

As I match with the form:

$$\int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$$

$\therefore$ I = $\frac{t}{2}\sqrt{t^2+1} + \frac{1}{2}\log|t+\sqrt{t^2+1}|$

$\Rightarrow$ I = $\frac{t}{2}\sqrt{t^2+1} + \frac{1}{2}\log|t+\sqrt{t^2+1}| + C$

Putting the value of t back:

$\Rightarrow$ I = $\frac{e^x}{2}\sqrt{e^{2x}+1} + \frac{1}{2}\log|e^x+\sqrt{e^{2x}+1}| + C$

7. Question

Evaluate the integral:

$$\int \sqrt{9-x^2}\, dx$$

Answer

Key points to solve the problem:

• Such problems require the use of the method of substitution along with a method of integration by parts. By the method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx)\, dx$

• To solve the integrals of the form: $\int \sqrt{ax^2+bx+c}\, dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$$

$$\int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$$

Let, I = $\int \sqrt{9-x^2}\, dx$

$\therefore$ I = $\int \sqrt{9-x^2}\, dx = \int \sqrt{3^2-x^2}\, dx$

As I match with the form: $\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

$\therefore$ I = $\frac{x}{2}\sqrt{9-(x)^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + C$

8. Question

Evaluate the integral:

$$\int \sqrt{16x^2+25}\, dx$$

Answer

Key points to solve the problem:

• Such problems require the use of the method of substitution along with a method of integration by parts. By the method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx)\,dx$

• To solve the integrals of the form: $\int \sqrt{ax^2+bx+c}\,dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$$

$$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$$

Let, I = $\int \sqrt{16x^2+25}\,dx$

We have:

I = $\int \sqrt{16x^2+25}\,dx = \int \sqrt{(4x)^2+5^2}\,dx$

⇒ I = $\int 4\sqrt{x^2+\left(\frac{5}{4}\right)^2}\,dx$

As I match with the form:

$$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$$

∴ I = $4\left\{\frac{x}{2}\sqrt{x^2+\left(\frac{5}{4}\right)^2} + \frac{\frac{25}{16}}{2}\log\left|x+\sqrt{x^2+\left(\frac{5}{4}\right)^2}\right|\right\}$

⇒ I = $\frac{x}{2}\sqrt{16x^2+25} + \frac{25}{8}\log\left|x+\sqrt{x^2+\frac{25}{16}}\right| + C$

9. Question

Evaluate the integral:

$$\int \sqrt{4x^2-5}\,dx$$

Answer

Key points to solve the problem:

• Such problems require the use of method of substitution along with method of integration by parts. By method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx)\,dx$

• To solve the integrals of the form: $\int \sqrt{ax^2+bx+c}\,dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$$

$$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$$

Let, I = $\int \sqrt{4x^2 - 5}\, dx$

We have:

I = $\int \sqrt{4x^2 - 5}\, dx = \int 2\sqrt{x^2 - \frac{5}{4}}\, dx$

$\Rightarrow$ I = $2\int \sqrt{x^2 - \left(\frac{\sqrt{5}}{2}\right)^2}\, dx$

As I match with the form:

$$\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$$

$\therefore$ I = $2\left\{\frac{x}{2}\sqrt{x^2 - \left(\frac{\sqrt{5}}{2}\right)^2} - \frac{\frac{5}{4}}{2}\log\left|x + \sqrt{x^2 - \left(\frac{\sqrt{5}}{2}\right)^2}\right|\right\}$

$\Rightarrow$ I = $x\sqrt{x^2 - \frac{5}{4}} - \frac{5}{4}\log\left|x + \sqrt{x^2 - \frac{5}{4}}\right| + C$

10. Question

Evaluate the integral:

$$\int \sqrt{2x^2 + 3x + 4}\, dx$$

Answer

Key points to solve the problem:

• Such problems require the use of method of substitution along with method of integration by parts. By method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx)\, dx$

• To solve the integrals of the form: $\int \sqrt{ax^2 + bx + c}\, dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$$

$$\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C$$

Let, I = $\int \sqrt{(2x^2 + 3x + 4)}\, dx$

$\therefore$ I = $\int \sqrt{2\left\{x^2 + 2\left(\frac{3}{4}\right)x + \left(\frac{3}{4}\right)^2 + 2 - \left(\frac{3}{4}\right)^2\right\}}\, dx$

Using $a^2 + 2ab + b^2 = (a + b)^2$

We have:

I = $\sqrt{2}\int \sqrt{\left(x + \frac{3}{4}\right)^2 + 2 - \frac{9}{16}}\, dx = \int \sqrt{\left(x + \frac{3}{4}\right)^2 + \left(\frac{\sqrt{23}}{4}\right)^2}\, dx$

As I match with the form:

$$\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C$$

$$\therefore I = \sqrt{2}\left\{\frac{\left(x+\frac{3}{4}\right)}{2}\sqrt{\left(x+\frac{3}{4}\right)^2+\left(\frac{\sqrt{23}}{4}\right)^2}+\frac{\left(\frac{\sqrt{23}}{4}\right)^2}{2}\log\left|\left(x+\frac{3}{4}\right)+\sqrt{\left(x+\frac{3}{4}\right)^2+\left(\frac{\sqrt{23}}{4}\right)^2}\right|\right\}+C$$

$$\Rightarrow I = \frac{1}{8}(4x+3)\sqrt{2\left\{\left(x+\frac{3}{4}\right)^2+\left(\frac{\sqrt{23}}{4}\right)^2\right\}}+\frac{23\sqrt{2}}{32}\log\left|\left(x+\frac{3}{4}\right)+\sqrt{\left(x+\frac{3}{4}\right)^2+\left(\frac{\sqrt{23}}{4}\right)^2}\right|+C$$

$$\Rightarrow I = \frac{1}{8}(4x+3)\sqrt{2x^2+3x+4}+\frac{23\sqrt{2}}{32}\log\left|\left(x+\frac{3}{4}\right)+\sqrt{x^2+\frac{3}{2}x+2}\right|+C$$

11. Question

Evaluate the integral:

$$\int\sqrt{3-2x-2x^2}\,dx$$

Answer

Key points to solve the problem:

• Such problems require the use of method of substitution along with method of integration by parts. By method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx)\,dx$

• To solve the integrals of the form: $\int\sqrt{ax^2+bx+c}\,dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+C$$

$$\int\sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$$

$$\int\sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$$

Let, $I = \int\sqrt{3-2x-2x^2}\,dx$

$$\therefore I = \int\sqrt{3-2\left(x^2+2\left(\frac{1}{2}\right)x\right)}\,dx = \int\sqrt{3-2\left(x^2+2\left(\frac{1}{2}\right)x+\left(\frac{1}{2}\right)^2\right)+2\left(\frac{1}{2}\right)^2}\,dx$$

Using $a^2 + 2ab + b^2 = (a + b)^2$

We have:

$$I = \int\sqrt{\frac{7}{4}-2\left(x+\frac{1}{2}\right)^2}\,dx = \int\sqrt{2}\sqrt{\left(\frac{\sqrt{7}}{2}\right)^2-\left(x+\frac{1}{2}\right)^2}\,dx$$

As I match with the form: $\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+C$

$$\therefore I = \sqrt{2}\left\{\frac{x+\frac{1}{2}}{2}\sqrt{\left(\frac{\sqrt{7}}{2}\right)^2-\left(x+\frac{1}{2}\right)^2}+\frac{\frac{7}{4}}{2}\sin^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{7}}{2}}\right)\right\}+C$$

$$\Rightarrow I = \frac{1}{4}(2x+1)\sqrt{2\left\{\left(\frac{\sqrt{7}}{2}\right)^2-\left(x+\frac{1}{2}\right)^2\right\}}+\frac{7\sqrt{2}}{8}\sin^{-1}\left(\frac{2x+1}{\sqrt{7}}\right)+C$$

$$\Rightarrow I = \frac{1}{4}(2x+1)\sqrt{3-2x-2x^2}+\frac{7\sqrt{2}}{8}\sin^{-1}\left(\frac{2x+1}{\sqrt{7}}\right)+C$$

12. Question

Evaluate the integral:

$$\int x\sqrt{x^4+1}\ dx$$

Answer

Key points to solve the problem:

• Such problems require the use of method of substitution along with method of integration by parts. By method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx)\,dx$

• To solve the integrals of the form: $\int \sqrt{ax^2+bx+c}\,dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$$

$$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$$

Let, I = $\int x\sqrt{x^4+1}\,dx$ = $\int x\sqrt{(x^2)^2+1}\,dx$

Let, $x^2 = t$

Differentiating both sides:

⇒ 2x dx = dt ⇒ x dx = 1/2 dt

Substituting x^2 with t, we have:

We have:

$I = \frac{1}{2}\int \sqrt{t^2+1}\ dt = \frac{1}{2}\int \sqrt{t^2+1^2}dt$

As I match with the form:

$$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$$

$\therefore I = \frac{1}{2}\left\{\frac{t}{2}\sqrt{t^2+1} + \frac{1}{2}\log\left|t+\sqrt{t^2+1}\right|\right\} + C$

$\Rightarrow I = \frac{t}{4}\sqrt{t^2+1} + \frac{1}{4}\log\left|t+\sqrt{t^2+1}\right| + C$

Putting the value of t back:

$\Rightarrow I = \frac{x^2}{4}\sqrt{(x^2)^2+1} + \frac{1}{4}\log\left|x^2+\sqrt{(x^2)^2+1}\right| + C$

$\Rightarrow I = \frac{x^2}{4}\sqrt{x^4+1} + \frac{1}{4}\log\left|x^2+\sqrt{x^4+1}\right| + C$

13. Question

Evaluate the integral:

$$\int x^2\sqrt{a^6-x^6}\,dx$$

Answer

Key points to solve the problem:

• Such problems require the use of method of substitution along with method of integration by parts. By method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx)\,dx$

• To solve the integrals of the form: $\int \sqrt{ax^2+bx+c}\,dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$$

$$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$$

Let, I = $\int x^2\sqrt{a^6-x^6}\,dx = \int x^2\sqrt{a^6-(x^3)^2}\,dx$

Let, $x^3 = t$

Differentiating both sides:

⇒ $3x^2\,dx = dt$

⇒ $x^2\,dx = 1/3\,dt$

Substituting x^3 with t, we have:

$\therefore I = \frac{1}{3}\int \sqrt{(a^3)^2-t^2}\,dt = \int \sqrt{(a^3)^2-t^2}\,dt$

As I match with the form: $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

$\therefore I = \frac{1}{3}\left\{\frac{t}{2}\sqrt{a^6-(t)^2} + \frac{a^6}{2}\sin^{-1}\left(\frac{t}{a^3}\right) + c\right\}$

Putting the value of t i.e. $t = x^3$

$\Rightarrow I = \frac{x^3}{6}\sqrt{a^6-x^6} + \frac{a^6}{6}\sin^{-1}\left(\frac{x^3}{a^3}\right) + C$

14. Question

Evaluate the integral:

$$\int \frac{\sqrt{16+(\log x)^2}}{x}\,dx$$

Answer

Key points to solve the problem:

• Such problems require the use of method of substitution along with method of integration by parts. By method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx)\,dx$

• To solve the integrals of the form: $\int \sqrt{ax^2+bx+c}\,dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$$

$$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$$

Let, I = $\int \frac{1}{x}\sqrt{16+(\log x)^2}\,dx$

Let, log x = t

Differentiating both sides:

$\Rightarrow \frac{1}{x}\,dx = dt$

Substituting (log x) with t, we have:

We have:

I = $\int \sqrt{t^2+16}\,dt = \int \sqrt{t^2+4^2}\,dt$

As I match with the form:

$$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$$

$\therefore$ I = $\left\{\frac{t}{2}\sqrt{t^2+16} + \frac{16}{2}\log\left|t+\sqrt{t^2+16}\right|\right\} + C$

Putting the value of t back:

$\Rightarrow$ I = $\frac{\log x}{2}\sqrt{(\log x)^2+16} + 8\log\left|\log x+\sqrt{(\log x)^2+16}\right| + C$

15. Question

Evaluate the integral:

$$\int \sqrt{2ax - x^2}\,dx$$

Answer

Key points to solve the problem:

• Such problems require the use of method of substitution along with method of integration by parts. By method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)\left(\int g(x)dx\right)dx$

• To solve the integrals of the form: $\int \sqrt{ax^2+bx+c}\,dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$$

$$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$$

Let, I = $\int \sqrt{2ax-x^2}\,dx$

$\therefore$ I = $\int \sqrt{-(x^2-2(a)x)}\,dx = \int \sqrt{a^2-(x^2-2(a)x+(a)^2)}\,dx$

Using $a^2 - 2ab + b^2 = (a - b)^2$

We have:

I = $\int \sqrt{a^2-(x-a)^2}\,dx = \int \sqrt{(a)^2-(x-a)^2}\,dx$

As I match with the form: $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

$\therefore$ I = $\frac{x-a}{2}\sqrt{(a)^2-(x-a)^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + C$

$\Rightarrow I = \frac{1}{2}(x-a)\sqrt{2ax-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + C$

16. Question

Evaluate the integral:

$$\int \sqrt{3-x^2}\ dx$$

Answer

Key points to solve the problem:

• Such problems require the use of method of substitution along with method of integration by parts. By method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx)\ dx$

• To solve the integrals of the form: $\int \sqrt{ax^2+bx+c}\, dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$$

$$\int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$$

Let, $I = \int \sqrt{3-x^2}\, dx$

$\therefore I = \int \sqrt{3-x^2}dx = \int \sqrt{(\sqrt{3})^2 - x^2}dx$

As I match with the form: $\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

$\therefore I = \frac{x}{2}\sqrt{3-x^2} + \frac{3}{2}\sin^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$

17. Question

Evaluate the integral:

$$\int \sqrt{x^2-2x}\ dx$$

Answer

Key points to solve the problem:

• Such problems require the use of method of substitution along with method of integration by parts. By method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx)\ dx$

• To solve the integrals of the form: $\int \sqrt{ax^2+bx+c}\, dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$$

$$\int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$$

Let, $I = \int \sqrt{x^2-2x}\, dx$

We have:

I = $\int \sqrt{x^2-2x}\,dx = \int \sqrt{x^2-2(1)x+1^2-1^2}\,dx$

Using $a^2 - 2ab + b^2 = (a-b)^2$

I = $\int \sqrt{(x-1)^2-1^2}\,dx$

As I match with the form:

$$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$$

$\therefore$ I = $\frac{x-1}{2}\sqrt{(x-1)^2-1} - \frac{1}{2}\log\left|x-1+\sqrt{(x-1)^2-1}\right| + C$

$\Rightarrow$ I = $\frac{x-1}{2}\sqrt{x^2-2x} - \frac{1}{2}\log\left|x-1+\sqrt{x^2-2x}\right| + C$

18. Question

Evaluate the integral:

$$\int \sqrt{2x-x^2}\,dx$$

Answer

Key points to solve the problem:

• Such problems require the use of method of substitution along with method of integration by parts. By method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx)\,dx$

• To solve the integrals of the form: $\int \sqrt{ax^2+bx+c}\,dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$$

$$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$$

Let, I = $\int \sqrt{2x-x^2}\,dx$

$\therefore$ I = $\int \sqrt{-(x^2-2(1)x)}\,dx = \int \sqrt{1^2-(x^2-2(1)x+(1)^2)}\,dx$

Using $a^2 - 2ab + b^2 = (a - b)^2$

We have:

I = $\int \sqrt{1^2-(x-a)^2}\,dx = \int \sqrt{(1)^2-(x-1)^2}\,dx$

As I match with the form: $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

$\therefore$ I = $\frac{x-1}{2}\sqrt{(1)^2-(x-1)^2} + \frac{1^2}{2}\sin^{-1}\left(\frac{x-1}{1}\right) + C$

$\Rightarrow$ I = $\frac{1}{2}(x-1)\sqrt{2x-x^2} + \frac{1}{2}\sin^{-1}(x-1) + C$

Exercise 19.29

1. Question

Evaluate the following integrals –

$$\int (x+1)\sqrt{x^2 - x + 1}\, dx$$

Answer

Let $I = \int (x+1)\sqrt{x^2 - x + 1}dx$

Let us assume $x + 1 = \lambda \frac{d}{dx}(x^2 - x + 1) + \mu$

$$\Rightarrow x + 1 = \lambda\left[\frac{d}{dx}(x^2) - \frac{d}{dx}(x) + \frac{d}{dx}(1)\right] + \mu$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$\Rightarrow x + 1 = \lambda(2x^{2-1} - 1 + 0) + \mu$

$\Rightarrow x + 1 = \lambda(2x - 1) + \mu$

$\Rightarrow x + 1 = 2\lambda x + \mu - \lambda$

Comparing the coefficient of x on both sides, we get

$2\lambda = 1 \Rightarrow \lambda = \frac{1}{2}$

Comparing the constant on both sides, we get

$\mu - \lambda = 1$

$$\Rightarrow \mu - \frac{1}{2} = 1$$

$$\therefore \mu = \frac{3}{2}$$

Hence, we have $x + 1 = \frac{1}{2}(2x - 1) + \frac{3}{2}$

Substituting this value in I, we can write the integral as

$$I = \int \left[\frac{1}{2}(2x - 1) + \frac{3}{2}\right]\sqrt{x^2 - x + 1}dx$$

$$\Rightarrow I = \int \left[\frac{1}{2}(2x - 1)\sqrt{x^2 - x + 1} + \frac{3}{2}\sqrt{x^2 - x + 1}\right]dx$$

$$\Rightarrow I = \int \frac{1}{2}(2x - 1)\sqrt{x^2 - x + 1}dx + \int \frac{3}{2}\sqrt{x^2 - x + 1}dx$$

$$\Rightarrow I = \frac{1}{2}\int (2x - 1)\sqrt{x^2 - x + 1}dx + \frac{3}{2}\int \sqrt{x^2 - x + 1}dx$$

Let $I_1 = \frac{1}{2}\int (2x - 1)\sqrt{x^2 - x + 1}dx$

Now, put $x^2 - x + 1 = t$

$\Rightarrow (2x - 1)dx = dt$ (Differentiating both sides)

Substituting this value in I_1, we can write

$$I_1 = \frac{1}{2}\int \sqrt{t}dt$$

$$\Rightarrow I_1 = \frac{1}{2}\int t^{\frac{1}{2}}dt$$

Recall $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow I_1 = \frac{1}{2}\left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) + c$$

$$\Rightarrow I_1 = \frac{1}{2}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + c$$

$$\Rightarrow I_1 = \frac{1}{2} \times \frac{2}{3} t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = \frac{1}{3} t^{\frac{3}{2}} + c$$

$$\therefore I_1 = \frac{1}{3}(x^2 - x + 1)^{\frac{3}{2}} + c$$

Let $I_2 = \frac{3}{2}\int \sqrt{x^2 - x + 1}\, dx$

We can write $x^2 - x + 1 = x^2 - 2(x)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1$

$$\Rightarrow x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} + 1$$

$$\Rightarrow x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$$

$$\Rightarrow x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

Hence, we can write I_2 as

$$I_2 = \frac{3}{2}\int \sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\, dx$$

Recall $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\ln\left|x + \sqrt{x^2 + a^2}\right| + c$

$$\Rightarrow I_2 = \frac{3}{2}\left[\frac{\left(x - \frac{1}{2}\right)}{2}\sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2}\ln\left|\left(x - \frac{1}{2}\right) + \sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right|\right] + c$$

$$\Rightarrow I_2 = \frac{3}{2}\left[\frac{2x - 1}{4}\sqrt{x^2 - x + 1} + \frac{3}{8}\ln\left|x - \frac{1}{2} + \sqrt{x^2 - x + 1}\right|\right] + c$$

$$\therefore I_2 = \frac{3}{8}(2x - 1)\sqrt{x^2 - x + 1} + \frac{9}{16}\ln\left|x - \frac{1}{2} + \sqrt{x^2 - x + 1}\right| + c$$

Substituting I_1 and I_2 in I, we get

$$I = \frac{1}{3}(x^2 - x + 1)^{\frac{3}{2}} + \frac{3}{8}(2x - 1)\sqrt{x^2 - x + 1} + \frac{9}{16}\ln\left|x - \frac{1}{2} + \sqrt{x^2 - x + 1}\right| + c$$

Thus, $\int(x+1)\sqrt{x^2-x+1}dx = \frac{1}{3}(x^2-x+1)^{\frac{3}{2}} + \frac{3}{8}(2x-1)\sqrt{x^2-x+1} + \frac{9}{16}\ln\left|x-\frac{1}{2}+\sqrt{x^2-x+1}\right| + c$

2. Question

Evaluate the following integrals –

$$\int(x+1)\sqrt{2x^2+3}\ dx$$

Answer

Let $I = \int(x+1)\sqrt{2x^2+3}dx$

Let us assume $x+1 = \lambda\frac{d}{dx}(2x^2+3)+\mu$

$$\Rightarrow x+1 = \lambda\left[\frac{d}{dx}(2x^2)+\frac{d}{dx}(1)\right]+\mu$$

$$\Rightarrow x+1 = \lambda\left[2\frac{d}{dx}(x^2)+\frac{d}{dx}(1)\right]+\mu$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$\Rightarrow x + 1 = \lambda(2 \times 2x^{2-1} + 0) + \mu$

$\Rightarrow x + 1 = \lambda(4x) + \mu$

$\Rightarrow x + 1 = 4\lambda x + \mu$

Comparing the coefficient of x on both sides, we get

$4\lambda = 1 \Rightarrow \lambda = \frac{1}{4}$

Comparing the constant on both sides, we get

$\mu = 1$

Hence, we have $x+1 = \frac{1}{4}(4x)+1$

Substituting this value in I, we can write the integral as

$$I = \int\left[\frac{1}{4}(4x)+1\right]\sqrt{2x^2+3}dx$$

$$\Rightarrow I = \int\left[\frac{1}{2}(4x)\sqrt{2x^2+3}+\sqrt{2x^2+3}\right]dx$$

$$\Rightarrow I = \int\frac{1}{4}(4x)\sqrt{2x^2+3}dx + \int\sqrt{2x^2+3}dx$$

$$\Rightarrow I = \frac{1}{4}\int(4x)\sqrt{2x^2+3}dx + \int\sqrt{2x^2+3}dx$$

Let $I_1 = \frac{1}{4}\int(4x)\sqrt{2x^2+3}dx$

Now, put $2x^2 + 3 = t$

$\Rightarrow (4x)dx = dt$ (Differentiating both sides)

Substituting this value in I_1, we can write

$$I_1 = \frac{1}{4}\int\sqrt{t}dt$$

$$\Rightarrow I_1 = \frac{1}{4}\int t^{\frac{1}{2}}dt$$

Recall $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow I_1 = \frac{1}{4}\left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) + c$$

$$\Rightarrow I_1 = \frac{1}{4}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + c$$

$$\Rightarrow I_1 = \frac{1}{4} \times \frac{2}{3} t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = \frac{1}{6} t^{\frac{3}{2}} + c$$

$$\therefore I_1 = \frac{1}{6}(2x^2 + 3)^{\frac{3}{2}} + c$$

Let $I_2 = \int \sqrt{2x^2 + 3}dx$

We can write $2x^2 + 3 = 2\left(x^2 + \frac{3}{2}\right)$

$$\Rightarrow 2x^2 + 3 = 2\left[x^2 + \left(\sqrt{\frac{3}{2}}\right)^2\right]$$

Hence, we can write I_2 as

$$I_2 = \int \sqrt{2\left[x^2 + \left(\sqrt{\frac{3}{2}}\right)^2\right]}dx$$

$$\Rightarrow I_2 = \sqrt{2}\int \sqrt{x^2 + \left(\sqrt{\frac{3}{2}}\right)^2}dx$$

Recall $\int \sqrt{x^2 + a^2}dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\ln\left|x + \sqrt{x^2 + a^2}\right| + c$

$$\Rightarrow I_2 = \sqrt{2}\left[\frac{x}{2}\sqrt{x^2 + \left(\sqrt{\frac{3}{2}}\right)^2} + \frac{\left(\sqrt{\frac{3}{2}}\right)^2}{2}\ln\left|x + \sqrt{x^2 + \left(\sqrt{\frac{3}{2}}\right)^2}\right|\right] + c$$

$$\Rightarrow I_2 = \sqrt{2}\left[\frac{x}{2}\sqrt{x^2 + \frac{3}{2}} + \frac{3}{4}\ln\left|x + \sqrt{x^2 + \frac{3}{2}}\right|\right] + c$$

$$\Rightarrow I_2 = \sqrt{2}\left[\frac{x}{2\sqrt{2}}\sqrt{2x^2 + 3} + \frac{3}{2 \times 2}\ln\left|x + \sqrt{x^2 + \frac{3}{2}}\right|\right] + c$$

$$\therefore I_2 = \frac{x}{2}\sqrt{2x^2+3} + \frac{3}{2\sqrt{2}}\ln\left|x + \sqrt{x^2+\frac{3}{2}}\right| + c$$

Substituting I_1 and I_2 in I, we get

$$I = \frac{1}{6}(2x^2+3)^{\frac{3}{2}} + \frac{x}{2}\sqrt{2x^2+3} + \frac{3}{2\sqrt{2}}\ln\left|x + \sqrt{x^2+\frac{3}{2}}\right| + c$$

Thus, $\int (x+1)\sqrt{2x^2+3}dx = \frac{1}{6}(2x^2+3)^{\frac{3}{2}} + \frac{x}{2}\sqrt{2x^2+3} + \frac{3}{2\sqrt{2}}\ln\left|x + \sqrt{x^2+\frac{3}{2}}\right| + c$

3. Question

Evaluate the following integrals –

$$\int (2x-5)\sqrt{2+3x-x^2}\ dx$$

Answer

Let $I = \int (2x-5)\sqrt{2+3x-x^2}dx$

Let us assume $2x - 5 = \lambda\frac{d}{dx}(2+3x-x^2) + \mu$

$$\Rightarrow 2x - 5 = \lambda\left[\frac{d}{dx}(2) + \frac{d}{dx}(3x) - \frac{d}{dx}(x^2)\right] + \mu$$

$$\Rightarrow 2x - 5 = \lambda\left[\frac{d}{dx}(2) + 3\frac{d}{dx}(x) - \frac{d}{dx}(x^2)\right] + \mu$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$\Rightarrow 2x - 5 = \lambda(0 + 3 - 2x^{2-1}) + \mu$

$\Rightarrow 2x - 5 = \lambda(3 - 2x) + \mu$

$\Rightarrow 2x - 5 = -2\lambda x + 3\lambda + \mu$

Comparing the coefficient of x on both sides, we get

$-2\lambda = 2 \Rightarrow \lambda = -1$

Comparing the constant on both sides, we get

$3\lambda + \mu = -5$

$\Rightarrow 3(-1) + \mu = -5$

$\Rightarrow -3 + \mu = -5$

$\therefore \mu = -2$

Hence, we have $2x - 5 = -(3 - 2x) - 2$

Substituting this value in I, we can write the integral as

$$I = \int [-(3-2x) - 2]\sqrt{2+3x-x^2}dx$$

$$\Rightarrow I = \int \left[-(3-2x)\sqrt{2+3x-x^2} - 2\sqrt{2+3x-x^2}\right]dx$$

$$\Rightarrow I = -\int (3-2x)\sqrt{2+3x-x^2}dx - \int 2\sqrt{2+3x-x^2}dx$$

$$\Rightarrow I = -\int (3-2x)\sqrt{2+3x-x^2}dx - 2\int \sqrt{2+3x-x^2}dx$$

Let $I_1 = -\int (3-2x)\sqrt{2+3x-x^2}dx$

Now, put $2 + 3x - x^2 = t$

$\Rightarrow (3 - 2x)dx = dt$ (Differentiating both sides)

Substituting this value in I_1, we can write

$$I_1 = -\int \sqrt{t}dt$$

$$\Rightarrow I_1 = -\int t^{\frac{1}{2}}dt$$

Recall $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow I_1 = -\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$

$$\Rightarrow I_1 = -\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = -\frac{2}{3}t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = -\frac{2}{3}t^{\frac{3}{2}} + c$$

$$\therefore I_1 = -\frac{2}{3}(2+3x-x^2)^{\frac{3}{2}} + c$$

Let $I_2 = -2\int \sqrt{2+3x-x^2}dx$

We can write $2 + 3x - x^2 = -(x^2 - 3x - 2)$

$$\Rightarrow 2+3x-x^2 = -\left[x^2 - 2(x)\left(\frac{3}{2}\right) + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 - 2\right]$$

$$\Rightarrow 2+3x-x^2 = -\left[\left(x-\frac{3}{2}\right)^2 - \frac{9}{4} - 2\right]$$

$$\Rightarrow 2+3x-x^2 = -\left[\left(x-\frac{3}{2}\right)^2 - \frac{17}{4}\right]$$

$$\Rightarrow 2+3x-x^2 = \frac{17}{4} - \left(x-\frac{3}{2}\right)^2$$

$$\Rightarrow 2+3x-x^2 = \left(\frac{\sqrt{17}}{2}\right)^2 - \left(x-\frac{3}{2}\right)^2$$

Hence, we can write I_2 as

$$I_2 = -2\int \sqrt{\left(\frac{\sqrt{17}}{2}\right)^2 - \left(x-\frac{3}{2}\right)^2}\,dx$$

Recall $\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

$$\Rightarrow I_2 = -2\left[\frac{\left(x-\frac{3}{2}\right)}{2}\sqrt{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}+\frac{\left(\frac{\sqrt{17}}{2}\right)^2}{2}\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{17}}{2}}\right)\right]+c$$

$$\Rightarrow I_2 = -2\left[\frac{2x-3}{4}\sqrt{2+3x-x^2}+\frac{17}{8}\sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right)\right]+c$$

$$\therefore I_2 = -\frac{1}{2}(2x-3)\sqrt{2+3x-x^2}-\frac{17}{4}\sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right)+c$$

Substituting I_1 and I_2 in I, we get

$$I = -\frac{2}{3}(2+3x-x^2)^{\frac{3}{2}}-\frac{1}{2}(2x-3)\sqrt{2+3x-x^2}-\frac{17}{4}\sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right)+c$$

Thus, $\int(2x-5)\sqrt{2+3x-x^2}dx = -\frac{2}{3}(2+3x-x^2)^{\frac{3}{2}}-\frac{1}{2}(2x-3)\sqrt{2+3x-x^2}-\frac{17}{4}\sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right)+c$

4. Question

Evaluate the following integrals –

$$\int(x+2)\sqrt{x^2+x+1}\,dx$$

Answer

Let $I = \int(x+2)\sqrt{x^2+x+1}dx$

Let us assume $x+2 = \lambda\frac{d}{dx}(x^2+x+1)+\mu$

$$\Rightarrow x+2 = \lambda\left[\frac{d}{dx}(x^2)+\frac{d}{dx}(x)+\frac{d}{dx}(1)\right]+\mu$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$\Rightarrow x + 2 = \lambda(2x^{2-1} + 1 + 0) + \mu$

$\Rightarrow x + 2 = \lambda(2x + 1) + \mu$

$\Rightarrow x + 2 = 2\lambda x + \lambda + \mu$

Comparing the coefficient of x on both sides, we get

$2\lambda = 1 \Rightarrow \lambda = \frac{1}{2}$

Comparing the constant on both sides, we get

$\lambda + \mu = 2$

$$\Rightarrow \frac{1}{2}+\mu = 2$$

$$\therefore \mu = \frac{3}{2}$$

Hence, we have $x+2 = \frac{1}{2}(2x+1)+\frac{3}{2}$

Substituting this value in I, we can write the integral as

$$I = \int\left[\frac{1}{2}(2x+1)+\frac{3}{2}\right]\sqrt{x^2+x+1}dx$$

$$\Rightarrow I = \int\left[\frac{1}{2}(2x+1)\sqrt{x^2+x+1}+\frac{3}{2}\sqrt{x^2+x+1}\right]dx$$

$$\Rightarrow I = \int \frac{1}{2}(2x+1)\sqrt{x^2+x+1}dx + \int \frac{3}{2}\sqrt{x^2+x+1}dx$$

$$\Rightarrow I = \frac{1}{2}\int (2x+1)\sqrt{x^2+x+1}dx + \frac{3}{2}\int \sqrt{x^2+x+1}dx$$

Let $I_1 = \frac{1}{2}\int (2x+1)\sqrt{x^2-x+1}dx$

Now, put $x^2 + x + 1 = t$

$\Rightarrow (2x + 1)dx = dt$ (Differentiating both sides)

Substituting this value in I_1, we can write

$$I_1 = \frac{1}{2}\int \sqrt{t}dt$$

$$\Rightarrow I_1 = \frac{1}{2}\int t^{\frac{1}{2}}dt$$

Recall $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow I_1 = \frac{1}{2}\left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) + c$$

$$\Rightarrow I_1 = \frac{1}{2}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + c$$

$$\Rightarrow I_1 = \frac{1}{2}\times\frac{2}{3}t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = \frac{1}{3}t^{\frac{3}{2}} + c$$

$$\therefore I_1 = \frac{1}{3}(x^2+x+1)^{\frac{3}{2}} + c$$

Let $I_2 = \frac{3}{2}\int \sqrt{x^2+x+1}dx$

We can write $x^2+x+1 = x^2 + 2(x)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1$

$$\Rightarrow x^2+x+1 = \left(x+\frac{1}{2}\right)^2 - \frac{1}{4} + 1$$

$$\Rightarrow x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4}$$

$$\Rightarrow x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

Hence, we can write I_2 as

$$I_2 = \frac{3}{2}\int \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

Recall $\int \sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + c$

$$\Rightarrow I_2 = \frac{3}{2}\left[\frac{\left(x+\frac{1}{2}\right)}{2}\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} + \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2}\ln\left|\left(x+\frac{1}{2}\right)+\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right|\right]+c$$

$$\Rightarrow I_2 = \frac{3}{2}\left[\frac{2x+1}{4}\sqrt{x^2+x+1}+\frac{3}{8}\ln\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right|\right]+c$$

$$\therefore I_2 = \frac{3}{8}(2x+1)\sqrt{x^2+x+1}+\frac{9}{16}\ln\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right|+c$$

Substituting I_1 and I_2 in I, we get

$$I = \frac{1}{3}(x^2+x+1)^{\frac{3}{2}}+\frac{3}{8}(2x+1)\sqrt{x^2+x+1}+\frac{9}{16}\ln\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right|+c$$

Thus, $\int(x+2)\sqrt{x^2+x+1}dx = \frac{1}{3}(x^2+x+1)^{\frac{3}{2}}+\frac{3}{8}(2x+1)\sqrt{x^2+x+1}+\frac{9}{16}\ln\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right|+c$

5. Question

Evaluate the following integrals –

$$\int(4x+1)\sqrt{x^2-x-2x}\ dx$$

Answer

Let $I = \int(4x+1)\sqrt{x^2-x-2}dx$

Let us assume $4x+1 = \lambda\frac{d}{dx}(x^2-x-2)+\mu$

$$\Rightarrow 4x+1 = \lambda\left[\frac{d}{dx}(x^2)-\frac{d}{dx}(x)-\frac{d}{dx}(2)\right]+\mu$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$\Rightarrow 4x + 1 = \lambda(2x^{2-1} - 1 - 0) + \mu$

$\Rightarrow 4x + 1 = \lambda(2x - 1) + \mu$

$\Rightarrow 4x + 1 = 2\lambda x + \mu - \lambda$

Comparing the coefficient of x on both sides, we get

$2\lambda = 4 \Rightarrow \lambda = \frac{4}{2} = 2$

Comparing the constant on both sides, we get

$\mu - \lambda = 1$

$\Rightarrow \mu - 2 = 1$

$\therefore \mu = 3$

Hence, we have $4x + 1 = 2(2x - 1) + 3$

Substituting this value in I, we can write the integral as

$$I = \int[2(2x-1)+3]\sqrt{x^2-x-2}dx$$

$$\Rightarrow I = \int \left[2(2x-1)\sqrt{x^2-x-2} + 3\sqrt{x^2-x-2}\right] dx$$

$$\Rightarrow I = \int 2(2x-1)\sqrt{x^2-x-2}dx + \int 3\sqrt{x^2-x-2}dx$$

$$\Rightarrow I = 2\int (2x-1)\sqrt{x^2-x-2}dx + 3\int \sqrt{x^2-x-2}dx$$

Let $I_1 = 2\int(2x-1)\sqrt{x^2-x-2}dx$

Now, put $x^2 - x - 2 = t$

$\Rightarrow (2x - 1)dx = dt$ (Differentiating both sides)

Substituting this value in I_1, we can write

$$I_1 = 2\int \sqrt{t}dt$$

$$\Rightarrow I_1 = 2\int t^{\frac{1}{2}}dt$$

Recall $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow I_1 = 2\left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) + c$$

$$\Rightarrow I_1 = 2\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + c$$

$$\Rightarrow I_1 = 2 \times \frac{2}{3}t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = \frac{4}{3}t^{\frac{3}{2}} + c$$

$$\therefore I_1 = \frac{4}{3}(x^2-x-2)^{\frac{3}{2}} + c$$

Let $I_2 = 3\int \sqrt{x^2-x-2}dx$

We can write $x^2 - x - 2 = x^2 - 2(x)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 - 2$

$$\Rightarrow x^2 - x - 2 = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} - 2$$

$$\Rightarrow x^2 - x - 2 = \left(x - \frac{1}{2}\right)^2 - \frac{9}{4}$$

$$\Rightarrow x^2 - x - 2 = \left(x - \frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2$$

Hence, we can write I_2 as

$$I_2 = 3\int \sqrt{\left(x-\frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2}\, dx$$

Recall $\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x + \sqrt{x^2-a^2}\right| + c$

$$\Rightarrow I_2 = 3\left[\frac{\left(x-\frac{1}{2}\right)}{2}\sqrt{\left(x-\frac{1}{2}\right)^2-\left(\frac{3}{2}\right)^2}-\frac{\left(\frac{3}{2}\right)^2}{2}\ln\left|\left(x-\frac{1}{2}\right)+\sqrt{\left(x-\frac{1}{2}\right)^2-\left(\frac{3}{2}\right)^2}\right|\right] + c$$

$$\Rightarrow I_2 = 3\left[\frac{2x-1}{4}\sqrt{x^2-x-2}-\frac{9}{8}\ln\left|x-\frac{1}{2}+\sqrt{x^2-x-2}\right|\right]+c$$

$$\therefore I_2 = \frac{3}{4}(2x-1)\sqrt{x^2-x-2}-\frac{27}{8}\ln\left|x-\frac{1}{2}+\sqrt{x^2-x-2}\right|+c$$

Substituting I_1 and I_2 in I, we get

$$I = \frac{4}{3}(x^2-x-2)^{\frac{3}{2}}+\frac{3}{4}(2x-1)\sqrt{x^2-x-2}-\frac{27}{8}\ln\left|x-\frac{1}{2}+\sqrt{x^2-x-2}\right|+c$$

Thus, $\int(4x+1)\sqrt{x^2-x-2}dx = \frac{4}{3}(x^2-x-2)^{\frac{3}{2}}+\frac{3}{4}(2x-1)\sqrt{x^2-x-2}-\frac{27}{8}\ln\left|x-\frac{1}{2}+\sqrt{x^2-x-2}\right|+c$

6. Question

Evaluate the following integrals –

$$\int(x-2)\sqrt{2x^2-6x+5}\,dx$$

Answer

Let $I = \int(x-2)\sqrt{2x^2-6x+5}dx$

Let us assume $x-2 = \lambda\frac{d}{dx}(2x^2-6x+5)+\mu$

$$\Rightarrow x-2 = \lambda\left[\frac{d}{dx}(2x^2)-\frac{d}{dx}(6x)-\frac{d}{dx}(5)\right]+\mu$$

$$\Rightarrow x-2 = \lambda\left[2\frac{d}{dx}(x^2)-6\frac{d}{dx}(x)-\frac{d}{dx}(5)\right]+\mu$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$\Rightarrow$ x – 2 = $\lambda(2 \times 2x^{2-1} - 6 - 0) + \mu$

$\Rightarrow$ x – 2 = $\lambda(4x - 6) + \mu$

$\Rightarrow$ x – 2 = $4\lambda x + \mu - 6\lambda$

Comparing the coefficient of x on both sides, we get

$4\lambda = 1 \Rightarrow \lambda = \frac{1}{4}$

Comparing the constant on both sides, we get

$\mu - 6\lambda = -2$

$$\Rightarrow \mu - 6\left(\frac{1}{4}\right) = -2$$

$$\Rightarrow \mu - \frac{3}{2} = -2$$

$$\therefore \mu = -\frac{1}{2}$$

Hence, we have $x-2 = \frac{1}{4}(4x-6)-\frac{1}{2}$

Substituting this value in I, we can write the integral as

$$I = \int \left[\frac{1}{4}(4x-6) - \frac{1}{2}\right]\sqrt{2x^2-6x+5}dx$$

$$\Rightarrow I = \int \left[\frac{1}{4}(4x-6)\sqrt{2x^2-6x+5} - \frac{1}{2}\sqrt{2x^2-6x+5}\right] dx$$

$$\Rightarrow I = \int \frac{1}{4}(4x-6)\sqrt{2x^2-6x+5}dx - \int \frac{1}{2}\sqrt{2x^2-6x+5}dx$$

$$\Rightarrow I = \frac{1}{4}\int (4x-6)\sqrt{2x^2-6x+5}dx - \frac{1}{2}\int \sqrt{2x^2-6x+5}dx$$

Let $I_1 = \frac{1}{4}\int (4x-6)\sqrt{2x^2-6x+5}dx$

Now, put $2x^2 - 6x + 5 = t$

$\Rightarrow (4x - 6)dx = dt$ (Differentiating both sides)

Substituting this value in I_1, we can write

$$I_1 = \frac{1}{4}\int \sqrt{t}dt$$

$$\Rightarrow I_1 = \frac{1}{4}\int t^{\frac{1}{2}}dt$$

Recall $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow I_1 = \frac{1}{4}\left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) + c$$

$$\Rightarrow I_1 = \frac{1}{4}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + c$$

$$\Rightarrow I_1 = \frac{1}{4}\times\frac{2}{3}t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = \frac{1}{6}t^{\frac{3}{2}} + c$$

$$\therefore I_1 = \frac{1}{6}(2x^2-6x+5)^{\frac{3}{2}} + c$$

Let $I_2 = -\frac{1}{2}\int \sqrt{2x^2-6x+5}dx$

We can write $2x^2 - 6x + 5 = 2\left(x^2 - 3x + \frac{5}{2}\right)$

$$\Rightarrow 2x^2-6x+5 = 2\left[x^2 - 2(x)\left(\frac{3}{2}\right) + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + \frac{5}{2}\right]$$

$$\Rightarrow 2x^2-6x+5 = 2\left[\left(x-\frac{3}{2}\right)^2 - \frac{9}{4} + \frac{5}{2}\right]$$

$$\Rightarrow 2x^2-6x+5 = 2\left[\left(x-\frac{3}{2}\right)^2 + \frac{1}{4}\right]$$

$$\Rightarrow 2x^2-6x+5 = 2\left[\left(x-\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right]$$

Hence, we can write I_2 as

$$I_2 = -\frac{1}{2}\int \sqrt{2\left[\left(x-\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right]}dx$$

$$\Rightarrow I_2 = -\frac{\sqrt{2}}{2}\int \sqrt{\left(x-\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2}\,dx$$

$$\Rightarrow I_2 = -\frac{1}{\sqrt{2}}\int \sqrt{\left(x-\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2}\,dx$$

Recall $\int \sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + c$

$$\Rightarrow I_2 = -\frac{1}{\sqrt{2}}\left[\frac{\left(x-\frac{3}{2}\right)}{2}\sqrt{\left(x-\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} + \frac{\left(\frac{1}{2}\right)^2}{2}\ln\left|\left(x-\frac{3}{2}\right) + \sqrt{\left(x-\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2}\right|\right] + c$$

$$\Rightarrow I_2 = -\frac{1}{\sqrt{2}}\left[\frac{2x-3}{4}\sqrt{x^2-3x+\frac{5}{2}} + \frac{1}{8}\ln\left|x-\frac{3}{2}+\sqrt{x^2-3x+\frac{5}{2}}\right|\right] + c$$

$$\Rightarrow I_2 = -\frac{1}{\sqrt{2}}\left[\frac{2x-3}{4\sqrt{2}}\sqrt{2x^2-6x+5} + \frac{1}{8}\ln\left|x-\frac{3}{2}+\sqrt{x^2-3x+\frac{5}{2}}\right|\right] + c$$

$$\therefore I_2 = -\frac{1}{8}(2x-3)\sqrt{2x^2-6x+5} - \frac{1}{8\sqrt{2}}\ln\left|x-\frac{3}{2}+\sqrt{x^2-3x+\frac{5}{2}}\right| + c$$

Substituting I_1 and I_2 in I, we get

$$I = \frac{1}{6}(2x^2-6x+5)^{\frac{3}{2}} - \frac{1}{8}(2x-3)\sqrt{2x^2-6x+5} - \frac{1}{8\sqrt{2}}\ln\left|x-\frac{3}{2}+\sqrt{x^2-3x+\frac{5}{2}}\right| + c$$

Thus, $\int (x-2)\sqrt{2x^2-6x+5}dx = \frac{1}{6}(2x^2-6x+5)^{\frac{3}{2}} - \frac{1}{8}(2x-3)\sqrt{2x^2-6x+5} - \frac{1}{8\sqrt{2}}\ln\left|x-\frac{3}{2}+\sqrt{x^2-3x+\frac{5}{2}}\right| + c$

7. Question

Evaluate the following integrals –

$$\int (x+1)\sqrt{x^2+x+1}\,dx$$

Answer

Let $I = \int (x+1)\sqrt{x^2+x+1}dx$

Let us assume $x+1 = \lambda\frac{d}{dx}(x^2+x+1) + \mu$

$\Rightarrow x+1=\lambda\left[\frac{d}{dx}(x^2)+\frac{d}{dx}(x)+\frac{d}{dx}(1)\right]+\mu$

We know $\frac{d}{dx}(x^n)=nx^{n-1}$ and derivative of a constant is 0.

$\Rightarrow x + 1 = \lambda(2x^{2-1} + 1 + 0) + \mu$

$\Rightarrow x + 1 = \lambda(2x + 1) + \mu$

$\Rightarrow x + 1 = 2\lambda x + \lambda + \mu$

Comparing the coefficient of x on both sides, we get

$2\lambda = 1 \Rightarrow \lambda=\frac{1}{2}$

Comparing the constant on both sides, we get

$\lambda + \mu = 1$

$\Rightarrow \frac{1}{2}+\mu=1$

$\therefore \mu=\frac{1}{2}$

Hence, we have $x+1=\frac{1}{2}(2x+1)+\frac{1}{2}$

Substituting this value in I, we can write the integral as

$I=\int\left[\frac{1}{2}(2x+1)+\frac{1}{2}\right]\sqrt{x^2+x+1}dx$

$\Rightarrow I=\int\left[\frac{1}{2}(2x+1)\sqrt{x^2+x+1}+\frac{1}{2}\sqrt{x^2+x+1}\right]dx$

$\Rightarrow I=\int\frac{1}{2}(2x+1)\sqrt{x^2+x+1}dx+\int\frac{1}{2}\sqrt{x^2+x+1}dx$

$\Rightarrow I=\frac{1}{2}\int(2x+1)\sqrt{x^2+x+1}dx+\frac{1}{2}\int\sqrt{x^2+x+1}dx$

Let $I_1=\frac{1}{2}\int(2x+1)\sqrt{x^2+x+1}dx$

Now, put $x^2 + x + 1 = t$

$\Rightarrow (2x + 1)dx = dt$ (Differentiating both sides)

Substituting this value in I_1, we can write

$I_1=\frac{1}{2}\int\sqrt{t}dt$

$\Rightarrow I_1=\frac{1}{2}\int t^{\frac{1}{2}}dt$

Recall $\int x^n dx=\frac{x^{n+1}}{n+1}+c$

$\Rightarrow I_1=\frac{1}{2}\left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right)+c$

$\Rightarrow I_1=\frac{1}{2}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right)+c$

$\Rightarrow I_1 = \frac{1}{2} \times \frac{2}{3} t^{\frac{3}{2}} + c$

$\Rightarrow I_1 = \frac{1}{3} t^{\frac{3}{2}} + c$

$\therefore I_1 = \frac{1}{3}(x^2 + x + 1)^{\frac{3}{2}} + c$

Let $I_2 = \frac{1}{2}\int \sqrt{x^2 + x + 1}dx$

We can write $x^2 + x + 1 = x^2 + 2(x)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1$

$\Rightarrow x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 - \frac{1}{4} + 1$

$\Rightarrow x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$

$\Rightarrow x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$

Hence, we can write I_2 as

$$I_2 = \frac{1}{2}\int \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\, dx$$

Recall $\int \sqrt{x^2 + a^2}dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\ln\left|x + \sqrt{x^2 + a^2}\right| + c$

$$\Rightarrow I_2 = \frac{1}{2}\left[\frac{\left(x + \frac{1}{2}\right)}{2}\sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2}\ln\left|\left(x + \frac{1}{2}\right) + \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right|\right] + c$$

$$\Rightarrow I_2 = \frac{1}{2}\left[\frac{2x + 1}{4}\sqrt{x^2 + x + 1} + \frac{3}{8}\ln\left|x + \frac{1}{2} + \sqrt{x^2 + x + 1}\right|\right] + c$$

$$\therefore I_2 = \frac{1}{8}(2x + 1)\sqrt{x^2 + x + 1} + \frac{3}{16}\ln\left|x + \frac{1}{2} + \sqrt{x^2 + x + 1}\right| + c$$

Substituting I_1 and I_2 in I, we get

$$I = \frac{1}{3}(x^2 + x + 1)^{\frac{3}{2}} + \frac{1}{8}(2x + 1)\sqrt{x^2 + x + 1} + \frac{3}{16}\ln\left|x + \frac{1}{2} + \sqrt{x^2 + x + 1}\right| + c$$

Thus, $\int (x + 1)\sqrt{x^2 + x + 1}dx = \frac{1}{3}(x^2 + x + 1)^{\frac{3}{2}} + \frac{1}{8}(2x + 1)\sqrt{x^2 + x + 1} + \frac{3}{16}\ln\left|x + \frac{1}{2} + \sqrt{x^2 + x + 1}\right| + c$

8. Question

Evaluate the following integrals –

$$\int (2x + 3)\sqrt{x^2 + 4x + 3}\, dx$$

Answer

Let $I = \int (2x+3)\sqrt{x^2+4x+3}dx$

Let us assume $2x + 3 = \lambda \frac{d}{dx}(x^2 + 4x + 3) + \mu$

$\Rightarrow 2x + 3 = \lambda\left[\frac{d}{dx}(x^2) + \frac{d}{dx}(4x) + \frac{d}{dx}(3)\right] + \mu$

$\Rightarrow 2x + 3 = \lambda\left[\frac{d}{dx}(x^2) + 4\frac{d}{dx}(x) + \frac{d}{dx}(3)\right] + \mu$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$\Rightarrow 2x + 3 = \lambda(2x^{2-1} + 4 + 0) + \mu$

$\Rightarrow 2x + 3 = \lambda(2x + 4) + \mu$

$\Rightarrow 2x + 3 = 2\lambda x + 4\lambda + \mu$

Comparing the coefficient of x on both sides, we get

$2\lambda = 2 \Rightarrow \lambda = 1$

Comparing the constant on both sides, we get

$4\lambda + \mu = 3$

$\Rightarrow 4(1) + \mu = 3$

$\Rightarrow 4 + \mu = 3$

$\therefore \mu = -1$

Hence, we have $2x + 3 = (2x + 4) - 1$

Substituting this value in I, we can write the integral as

$I = \int [(2x + 4) - 1]\sqrt{x^2 + 4x + 3}dx$

$\Rightarrow I = \int \left[(2x + 4)\sqrt{x^2 + 4x + 3} - \sqrt{x^2 + 4x + 3}\right] dx$

$\Rightarrow I = \int (2x + 4)\sqrt{x^2 + 4x + 3}dx - \int \sqrt{x^2 + 4x + 3}dx$

Let $I_1 = \int (2x + 4)\sqrt{x^2 + 4x + 3}dx$

Now, put $x^2 + 4x + 3 = t$

$\Rightarrow (2x + 4)dx = dt$ (Differentiating both sides)

Substituting this value in I_1, we can write

$I_1 = \int \sqrt{t}dt$

$\Rightarrow I_1 = \int t^{\frac{1}{2}}dt$

Recall $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$\Rightarrow I_1 = \frac{t^{\frac{1}{2}+1}}{\frac{1}{2} + 1} + c$

$$\Rightarrow I_1 = \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = \frac{2}{3}t^{\frac{3}{2}} + c$$

$$\therefore I_1 = \frac{2}{3}(x^2 + 4x + 3)^{\frac{3}{2}} + c$$

Let $I_2 = -\int \sqrt{x^2 + 4x + 3}dx$

We can write $x^2 + 4x + 3 = x^2 + 2(x)(2) + 2^2 - 2^2 + 3$

$\Rightarrow x^2 + 4x + 3 = (x + 2)^2 - 4 + 3$

$\Rightarrow x^2 + 4x + 3 = (x + 2)^2 - 1$

$\Rightarrow x^2 + 4x + 3 = (x + 2)^2 - 1^2$

Hence, we can write I_2 as

$$I_2 = -\int \sqrt{(x+2)^2 - 1^2}dx$$

Recall $\int \sqrt{x^2 - a^2}dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ln\left|x + \sqrt{x^2 - a^2}\right| + c$

$$\Rightarrow I_2 = -\left[\frac{(x+2)}{2}\sqrt{(x+2)^2 - 1^2} - \frac{1^2}{2}\ln\left|(x+2) + \sqrt{(x+2)^2 - 1^2}\right|\right] + c$$

$$\Rightarrow I_2 = -\left[\frac{(x+2)}{2}\sqrt{x^2 + 4x + 3} - \frac{1}{2}\ln\left|x + 2 + \sqrt{x^2 + 4x + 3}\right|\right] + c$$

$$\therefore I_2 = -\frac{1}{2}(x+2)\sqrt{x^2 + 4x + 3} + \frac{1}{2}\ln\left|x + 2 + \sqrt{x^2 + 4x + 3}\right| + c$$

Substituting I_1 and I_2 in I, we get

$$I = \frac{2}{3}(x^2 + 4x + 3)^{\frac{3}{2}} - \frac{1}{2}(x+2)\sqrt{x^2 + 4x + 3} + \frac{1}{2}\ln\left|x + 2 + \sqrt{x^2 + 4x + 3}\right| + c$$

Thus, $\int (2x+3)\sqrt{x^2 + 4x + 3}dx = \frac{2}{3}(x^2 + 4x + 3)^{\frac{3}{2}} - \frac{1}{2}(x+2)\sqrt{x^2 + 4x + 3} + \frac{1}{2}\ln\left|x + 2 + \sqrt{x^2 + 4x + 3}\right| + c$

9. Question

Evaluate the following integrals –

$$\int (2x-4)\sqrt{x^2 - 4x + 3}\ dx$$

Answer

Let $I = \int (2x-5)\sqrt{x^2 - 4x + 3}dx$

Let us assume $2x - 5 = \lambda\frac{d}{dx}(x^2 - 4x + 3) + \mu$

$$\Rightarrow 2x - 5 = \lambda\left[\frac{d}{dx}(x^2) - \frac{d}{dx}(4x) + \frac{d}{dx}(3)\right] + \mu$$

$$\Rightarrow 2x - 5 = \lambda\left[\frac{d}{dx}(x^2) - 4\frac{d}{dx}(x) + \frac{d}{dx}(3)\right] + \mu$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$\Rightarrow 2x - 5 = \lambda(2x^{2-1} - 4 + 0) + \mu$

$\Rightarrow 2x - 5 = \lambda(2x - 4) + \mu$

$\Rightarrow 2x - 5 = 2\lambda x + \mu - 4\lambda$

Comparing the coefficient of x on both sides, we get

$2\lambda = 2 \Rightarrow \lambda = 1$

Comparing the constant on both sides, we get

$\mu - 4\lambda = -5$

$\Rightarrow \mu - 4(1) = -5$

$\Rightarrow \mu - 4 = -5$

$\therefore \mu = -1$

Hence, we have $2x - 5 = (2x - 4) - 1$

Substituting this value in I, we can write the integral as

$$I = \int [(2x - 4) - 1]\sqrt{x^2 - 4x + 3}dx$$

$$\Rightarrow I = \int \left[(2x - 4)\sqrt{x^2 - 4x + 3} - \sqrt{x^2 - 4x + 3}\right] dx$$

$$\Rightarrow I = \int (2x - 4)\sqrt{x^2 - 4x + 3}dx - \int \sqrt{x^2 - 4x + 3}dx$$

Let $I_1 = \int (2x - 4)\sqrt{x^2 - 4x + 3}dx$

Now, put $x^2 - 4x + 3 = t$

$\Rightarrow (2x - 4)dx = dt$ (Differentiating both sides)

Substituting this value in I_1, we can write

$$I_1 = \int \sqrt{t}dt$$

$$\Rightarrow I_1 = \int t^{\frac{1}{2}}dt$$

Recall $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow I_1 = \frac{t^{\frac{1}{2}+1}}{\frac{1}{2} + 1} + c$$

$$\Rightarrow I_1 = \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = \frac{2}{3}t^{\frac{3}{2}} + c$$

$$\therefore I_1 = \frac{2}{3}(x^2 - 4x + 3)^{\frac{3}{2}} + c$$

Let $I_2 = -\int \sqrt{x^2 - 4x + 3}dx$

We can write $x^2 - 4x + 3 = x^2 - 2(x)(2) + 2^2 - 2^2 + 3$

$\Rightarrow x^2 - 4x + 3 = (x - 2)^2 - 4 + 3$

$\Rightarrow x^2 - 4x + 3 = (x - 2)^2 - 1$

$\Rightarrow x^2 - 4x + 3 = (x - 2)^2 - 1^2$

Hence, we can write I_2 as

$$I_2 = -\int \sqrt{(x-2)^2 - 1^2}dx$$

Recall $\int \sqrt{x^2 - a^2}dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ln\left|x + \sqrt{x^2 - a^2}\right| + c$

$$\Rightarrow I_2 = -\left[\frac{(x-2)}{2}\sqrt{(x-2)^2 - 1^2} - \frac{1^2}{2}\ln\left|(x-2) + \sqrt{(x-2)^2 - 1^2}\right|\right] + c$$

$$\Rightarrow I_2 = -\left[\frac{(x-2)}{2}\sqrt{x^2 - 4x + 3} - \frac{1}{2}\ln\left|x - 2 + \sqrt{x^2 - 4x + 3}\right|\right] + c$$

$$\therefore I_2 = -\frac{1}{2}(x-2)\sqrt{x^2 - 4x + 3} + \frac{1}{2}\ln\left|x - 2 + \sqrt{x^2 - 4x + 3}\right| + c$$

Substituting I_1 and I_2 in I, we get

$$I = \frac{2}{3}(x^2 - 4x + 3)^{\frac{3}{2}} - \frac{1}{2}(x-2)\sqrt{x^2 - 4x + 3} + \frac{1}{2}\ln\left|x - 2 + \sqrt{x^2 - 4x + 3}\right| + c$$

Thus, $\int (2x-5)\sqrt{x^2 - 4x + 3}dx = \frac{2}{3}(x^2 - 4x + 3)^{\frac{3}{2}} - \frac{1}{2}(x-2)\sqrt{x^2 - 4x + 3} + \frac{1}{2}\ln\left|x - 2 + \sqrt{x^2 - 4x + 3}\right| + c$

10. Question

Evaluate the following integrals –

$$\int x\sqrt{x^2 + x}\ dx$$

Answer

Let $I = \int x\sqrt{x^2 + x}dx$

Let us assume $x = \lambda\frac{d}{dx}(x^2 + x) + \mu$

$$\Rightarrow x = \lambda\left[\frac{d}{dx}(x^2) + \frac{d}{dx}(x)\right] + \mu$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$\Rightarrow x = \lambda(2x^{2-1} + 1) + \mu$

$\Rightarrow x = \lambda(2x + 1) + \mu$

$\Rightarrow x = 2\lambda x + \lambda + \mu$

Comparing the coefficient of x on both sides, we get

$2\lambda = 1 \Rightarrow \lambda = \frac{1}{2}$

Comparing the constant on both sides, we get

$\lambda + \mu = 0$

$$\Rightarrow \frac{1}{2} + \mu = 0$$

$\therefore \mu = -\frac{1}{2}$

Hence, we have $x = \frac{1}{2}(2x+1) - \frac{1}{2}$

Substituting this value in I, we can write the integral as

$$I = \int \left[\frac{1}{2}(2x+1) - \frac{1}{2}\right]\sqrt{x^2+x}dx$$

$$\Rightarrow I = \int \left[\frac{1}{2}(2x+1)\sqrt{x^2+x} - \frac{1}{2}\sqrt{x^2+x}\right]dx$$

$$\Rightarrow I = \int \frac{1}{2}(2x+1)\sqrt{x^2+x}dx - \int \frac{1}{2}\sqrt{x^2+x}dx$$

$$\Rightarrow I = \frac{1}{2}\int (2x+1)\sqrt{x^2+x}dx - \frac{1}{2}\int \sqrt{x^2+x}dx$$

Let $I_1 = \frac{1}{2}\int (2x+1)\sqrt{x^2+x}dx$

Now, put $x^2 + x = t$

$\Rightarrow (2x + 1)dx = dt$ (Differentiating both sides)

Substituting this value in I_1, we can write

$$I_1 = \frac{1}{2}\int \sqrt{t}dt$$

$$\Rightarrow I_1 = \frac{1}{2}\int t^{\frac{1}{2}}dt$$

Recall $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow I_1 = \frac{1}{2}\left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) + c$$

$$\Rightarrow I_1 = \frac{1}{2}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + c$$

$$\Rightarrow I_1 = \frac{1}{2} \times \frac{2}{3}t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = \frac{1}{3}t^{\frac{3}{2}} + c$$

$$\therefore I_1 = \frac{1}{3}(x^2+x)^{\frac{3}{2}} + c$$

Let $I_2 = -\frac{1}{2}\int \sqrt{x^2+x}dx$

We can write $x^2 + x = x^2 + 2(x)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2$

$$\Rightarrow x^2 + x = \left(x + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$

Hence, we can write I_2 as

$$I_2 = -\frac{1}{2}\int \sqrt{\left(x+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}dx$$

Recall $\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + c$

$$\Rightarrow I_2 = -\frac{1}{2}\left[\frac{\left(x+\frac{1}{2}\right)}{2}\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2} - \frac{\left(\frac{1}{2}\right)^2}{2}\ln\left|\left(x+\frac{1}{2}\right)+\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right|\right] + c$$

$$\Rightarrow I_2 = -\frac{1}{2}\left[\frac{2x+1}{4}\sqrt{x^2+x} - \frac{1}{8}\ln\left|x+\frac{1}{2}+\sqrt{x^2+x}\right|\right] + c$$

$$\therefore I_2 = -\frac{1}{8}(2x+1)\sqrt{x^2+x} + \frac{1}{16}\ln\left|x+\frac{1}{2}+\sqrt{x^2+x}\right| + c$$

Substituting I_1 and I_2 in I, we get

$$I = \frac{1}{3}(x^2+x)^{\frac{3}{2}} - \frac{1}{8}(2x+1)\sqrt{x^2+x} + \frac{1}{16}\ln\left|x+\frac{1}{2}+\sqrt{x^2+x}\right| + c$$

Thus, $\int x\sqrt{x^2+x}dx = \frac{1}{3}(x^2+x)^{\frac{3}{2}} - \frac{1}{8}(2x+1)\sqrt{x^2+x} + \frac{1}{16}\ln\left|x+\frac{1}{2}+\sqrt{x^2+x}\right| + c$

11. Question

Evaluate the following integrals –

$$\int (x-3)\sqrt{x^2+3x-18}\, dx$$

Answer

Let $I = \int (x-3)\sqrt{x^2+3x-18}dx$

Let us assume $x-3 = \lambda\frac{d}{dx}(x^2+3x-18)+\mu$

$$\Rightarrow x-3 = \lambda\left[\frac{d}{dx}(x^2)+\frac{d}{dx}(3x)-\frac{d}{dx}(18)\right]+\mu$$

$$\Rightarrow x-3 = \lambda\left[\frac{d}{dx}(x^2)+3\frac{d}{dx}(x)-\frac{d}{dx}(18)\right]+\mu$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$\Rightarrow x - 3 = \lambda(2x^{2-1} + 3 + 0) + \mu$

$\Rightarrow x - 3 = \lambda(2x + 3) + \mu$

$\Rightarrow x - 3 = 2\lambda x + 3\lambda + \mu$

Comparing the coefficient of x on both sides, we get

$2\lambda = 1 \Rightarrow \lambda = \frac{1}{2}$

Comparing the constant on both sides, we get

$3\lambda + \mu = -3$

$$\Rightarrow 3\left(\frac{1}{2}\right)+\mu = -3$$

$$\Rightarrow \frac{3}{2}+\mu = -3$$

$\therefore \mu = -\frac{9}{2}$

Hence, we have $x - 3 = \frac{1}{2}(2x + 3) - \frac{9}{2}$

Substituting this value in I, we can write the integral as

$$I = \int \left[\frac{1}{2}(2x + 3) - \frac{9}{2}\right] \sqrt{x^2 + 3x - 18}dx$$

$$\Rightarrow I = \int \left[\frac{1}{2}(2x + 3)\sqrt{x^2 + 3x - 18} - \frac{9}{2}\sqrt{x^2 + 3x - 18}\right] dx$$

$$\Rightarrow I = \int \frac{1}{2}(2x + 3)\sqrt{x^2 + 3x - 18}dx - \int \frac{9}{2}\sqrt{x^2 + 3x - 18}dx$$

$$\Rightarrow I = \frac{1}{2}\int (2x + 3)\sqrt{x^2 + 3x - 18}dx - \frac{9}{2}\int \sqrt{x^2 + 3x - 18}dx$$

Let $I_1 = \frac{1}{2}\int (2x + 3)\sqrt{x^2 + 3x - 18}dx$

Now, put $x^2 + 3x - 18 = t$

$\Rightarrow (2x + 3)dx = dt$ (Differentiating both sides)

Substituting this value in I_1, we can write

$$I_1 = \frac{1}{2}\int \sqrt{t}dt$$

$$\Rightarrow I_1 = \frac{1}{2}\int t^{\frac{1}{2}}dt$$

Recall $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow I_1 = \frac{1}{2}\left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2} + 1}\right) + c$$

$$\Rightarrow I_1 = \frac{1}{2}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + c$$

$$\Rightarrow I_1 = \frac{1}{2} \times \frac{2}{3}t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = \frac{1}{3}t^{\frac{3}{2}} +$$

$$\therefore I_1 = \frac{1}{3}(x^2 + 3x - 18)^{\frac{3}{2}} + c$$

Let $I_2 = -\frac{9}{2}\int \sqrt{x^2 + 3x - 18}dx$

We can write $x^2 + 3x - 18 = x^2 + 2(x)\left(\frac{3}{2}\right) + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 - 18$

$$\Rightarrow x^2 + 3x - 18 = \left(x + \frac{3}{2}\right)^2 - \frac{9}{4} - 18$$

$$\Rightarrow x^2 + 3x - 18 = \left(x + \frac{3}{2}\right)^2 - \frac{81}{4}$$

$$\Rightarrow x^2 + 3x - 18 = \left(x + \frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2$$

Hence, we can write I_2 as

$$I_2 = -\frac{9}{2}\int \sqrt{\left(x+\frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2}\,dx$$

Recall $\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + c$

$$\Rightarrow I_2 = -\frac{9}{2}\left[\frac{\left(x+\frac{3}{2}\right)}{2}\sqrt{\left(x+\frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2} - \frac{\left(\frac{9}{2}\right)^2}{2}\ln\left|\left(x+\frac{3}{2}\right) + \sqrt{\left(x+\frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2}\right|\right] + c$$

$$\Rightarrow I_2 = -\frac{9}{2}\left[\frac{(2x+3)}{4}\sqrt{x^2+3x-18} - \frac{81}{8}\ln\left|x+\frac{3}{2}+\sqrt{x^2+3x-18}\right|\right] + c$$

$$\therefore I_2 = -\frac{9}{8}(2x+3)\sqrt{x^2+3x-18} + \frac{729}{16}\ln\left|x+\frac{3}{2}+\sqrt{x^2+3x-18}\right| + c$$

Substituting I_1 and I_2 in I, we get

$$I = \frac{1}{3}(x^2+3x-18)^{\frac{3}{2}} - \frac{9}{8}(2x+3)\sqrt{x^2+3x-18} + \frac{729}{16}\ln\left|x+\frac{3}{2}+\sqrt{x^2+3x-18}\right| + c$$

Thus, $\int (x-3)\sqrt{x^2+3x-18}dx = \frac{1}{3}(x^2+3x-18)^{\frac{3}{2}} - \frac{9}{8}(2x+3)\sqrt{x^2+3x-18} + \frac{729}{16}\ln\left|x+\frac{3}{2}+\sqrt{x^2+3x-18}\right| + c$

12. Question

Evaluate the following integrals –

$$\int (x+3)\sqrt{3-4x-x^2}\,dx$$

Answer

Let $I = \int (x+3)\sqrt{3-4x-x^2}dx$

Let us assume $x+3 = \lambda\frac{d}{dx}(3-4x-x^2) + \mu$

$$\Rightarrow x+3 = \lambda\left[\frac{d}{dx}(3) - \frac{d}{dx}(4x) - \frac{d}{dx}(x^2)\right] + \mu$$

$$\Rightarrow x+3 = \lambda\left[\frac{d}{dx}(3) - 4\frac{d}{dx}(x) - \frac{d}{dx}(x^2)\right] + \mu$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$\Rightarrow x + 3 = \lambda(0 - 4 - 2x^{2-1}) + \mu$

$\Rightarrow x + 3 = \lambda(-4 - 2x) + \mu$

$\Rightarrow x + 3 = -2\lambda x + \mu - 4\lambda$

Comparing the coefficient of x on both sides, we get

$-2\lambda = 1 \Rightarrow \lambda = -\frac{1}{2}$

Comparing the constant on both sides, we get

$\mu - 4\lambda = 3$

$\Rightarrow \mu - 4\left(-\frac{1}{2}\right) = 3$

$\Rightarrow \mu + 2 = 3$

$\therefore \mu = 1$

Hence, we have $x + 3 = -\frac{1}{2}(-4 - 2x) + 1$

Substituting this value in I, we can write the integral as

$I = \int \left[-\frac{1}{2}(-4 - 2x) + 1\right]\sqrt{3 - 4x - x^2}dx$

$\Rightarrow I = \int \left[-\frac{1}{2}(-4 - 2x)\sqrt{3 - 4x - x^2} + \sqrt{3 - 4x - x^2}\right]dx$

$\Rightarrow I = -\int \frac{1}{2}(-4 - 2x)\sqrt{3 - 4x - x^2}dx + \int \sqrt{3 - 4x - x^2}dx$

$\Rightarrow I = -\frac{1}{2}\int (-4 - 2x)\sqrt{3 - 4x - x^2}dx + \int \sqrt{3 - 4x - x^2}dx$

Let $I_1 = -\frac{1}{2}\int(-4 - 2x)\sqrt{3 - 4x - x^2}dx$

Now, put $3 - 4x - x^2 = t$

$\Rightarrow (-4 - 2x)dx = dt$ (Differentiating both sides)

Substituting this value in I_1, we can write

$I_1 = -\frac{1}{2}\int \sqrt{t}dt$

$\Rightarrow I_1 = -\frac{1}{2}\int t^{\frac{1}{2}}dt$

Recall $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$\Rightarrow I_1 = -\frac{1}{2}\left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2} + 1}\right) + c$

$\Rightarrow I_1 = -\frac{1}{2}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + c$

$\Rightarrow I_1 = -\frac{1}{2} \times \frac{2}{3}t^{\frac{3}{2}} + c$

$\Rightarrow I_1 = -\frac{1}{3}t^{\frac{3}{2}} + c$

$\therefore I_1 = -\frac{1}{3}(3 - 4x - x^2)^{\frac{3}{2}} + c$

Let $I_2 = \int \sqrt{3 - 4x - x^2}dx$

We can write $3 - 4x - x^2 = -(x^2 + 4x - 3)$

$\Rightarrow 3 - 4x - x^2 = -[x^2 + 2(x)(2) + 2^2 - 2^2 - 3]$

$\Rightarrow 3 - 4x - x^2 = -[(x + 2)^2 - 4 - 3]$

$\Rightarrow 3 - 4x - x^2 = -[(x + 2)^2 - 7]$

$\Rightarrow 3 - 4x - x^2 = 7 - (x + 2)^2$

$\Rightarrow 3 - 4x - x^2 = \left(\sqrt{7}\right)^2 - (x+2)^2$

Hence, we can write I_2 as

$$I_2 = \int \sqrt{\left(\sqrt{7}\right)^2 - (x+2)^2}\,dx$$

Recall $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

$$\Rightarrow I_2 = \frac{(x+2)}{2}\sqrt{\left(\sqrt{7}\right)^2 - (x+2)^2} + \frac{\left(\sqrt{7}\right)^2}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{7}}\right) + c$$

$$\therefore I_2 = \frac{1}{2}(x+2)\sqrt{3 - 4x - x^2} + \frac{7}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{7}}\right) + c$$

Substituting I_1 and I_2 in I, we get

$$I = -\frac{1}{3}(3 - 4x - x^2)^{\frac{3}{2}} + \frac{1}{2}(x+2)\sqrt{3 - 4x - x^2} + \frac{7}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{7}}\right) + c$$

Thus, $\int (x+3)\sqrt{3 - 4x - x^2}\,dx = -\frac{1}{3}(3 - 4x - x^2)^{\frac{3}{2}} + \frac{1}{2}(x+2)\sqrt{3 - 4x - x^2} + \frac{7}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{7}}\right) + c$

13. Question

Evaluate the following integrals –

$$\int (3x+1)\sqrt{4 - 3x - 2x^2}\,dx$$

Answer

Let $I = \int (3x+1)\sqrt{4 - 3x - 2x^2}\,dx$

Let us assume $3x + 1 = \lambda\frac{d}{dx}(4 - 3x - 2x^2) + \mu$

$$\Rightarrow 3x + 1 = \lambda\left[\frac{d}{dx}(4) - \frac{d}{dx}(3x) - \frac{d}{dx}(2x^2)\right] + \mu$$

$$\Rightarrow 3x + 1 = \lambda\left[\frac{d}{dx}(4) - 3\frac{d}{dx}(x) - 2\frac{d}{dx}(x^2)\right] + \mu$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$\Rightarrow 3x + 1 = \lambda(0 - 3 - 2\times 2x^{2-1}) + \mu$

$\Rightarrow 3x + 1 = \lambda(-3 - 4x) + \mu$

$\Rightarrow 3x + 1 = -4\lambda x + \mu - 3\lambda$

Comparing the coefficient of x on both sides, we get

$-4\lambda = 3 \Rightarrow \lambda = -\frac{3}{4}$

Comparing the constant on both sides, we get

$\mu - 3\lambda = 1$

$$\Rightarrow \mu - 3\left(-\frac{3}{4}\right) = 1$$

$\Rightarrow \mu + \frac{9}{4} = 1$

$\therefore \mu = -\frac{5}{4}$

Hence, we have $3x + 1 = -\frac{3}{4}(-3 - 4x) - \frac{5}{4}$

Substituting this value in I, we can write the integral as

$$I = \int \left[-\frac{3}{4}(-3 - 4x) - \frac{5}{4}\right]\sqrt{4 - 3x - 2x^2}dx$$

$$\Rightarrow I = \int \left[-\frac{3}{4}(-3 - 4x)\sqrt{4 - 3x - 2x^2} - \frac{5}{4}\sqrt{4 - 3x - 2x^2}\right]dx$$

$$\Rightarrow I = -\int \frac{3}{4}(-3 - 4x)\sqrt{4 - 3x - 2x^2}dx - \int \frac{5}{4}\sqrt{4 - 3x - 2x^2}dx$$

$$\Rightarrow I = -\frac{3}{4}\int (-3 - 4x)\sqrt{4 - 3x - 2x^2}dx - \frac{5}{4}\int \sqrt{4 - 3x - 2x^2}dx$$

Let $I_1 = -\frac{3}{4}\int(-3 - 4x)\sqrt{4 - 3x - 2x^2}dx$

Now, put 4 – 3x – $2x^2$ = t

⇒ (–3 – 4x)dx = dt (Differentiating both sides)

Substituting this value in I_1, we can write

$$I_1 = -\frac{3}{4}\int \sqrt{t}dt$$

$$\Rightarrow I_1 = -\frac{3}{4}\int t^{\frac{1}{2}}dt$$

Recall $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow I_1 = -\frac{3}{4}\left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) + c$$

$$\Rightarrow I_1 = -\frac{3}{4}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + c$$

$$\Rightarrow I_1 = -\frac{3}{4} \times \frac{2}{3}t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = -\frac{1}{2}t^{\frac{3}{2}} + c$$

$$\therefore I_1 = -\frac{1}{2}(4 - 3x - 2x^2)^{\frac{3}{2}} + c$$

Let $I_2 = -\frac{5}{4}\int \sqrt{4 - 3x - 2x^2}dx$

We can write 4 – 3x – $2x^2$ = –($2x^2$ + 3x – 4)

$$\Rightarrow 4 - 3x - 2x^2 = -2\left[x^2 + \frac{3}{2}x - 2\right]$$

$$\Rightarrow 4 - 3x - 2x^2 = -2\left[x^2 + 2(x)\left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^2 - 2\right]$$

$$\Rightarrow 4-3x-2x^2=-2\left[\left(x+\frac{3}{4}\right)^2-\frac{9}{16}-2\right]$$

$$\Rightarrow 4-3x-2x^2=-2\left[\left(x+\frac{3}{4}\right)^2-\frac{41}{16}\right]$$

$$\Rightarrow 4-3x-2x^2=2\left[\frac{41}{16}-\left(x+\frac{3}{4}\right)^2\right]$$

$$\Rightarrow 4-3x-2x^2=2\left[\left(\frac{\sqrt{41}}{4}\right)^2-\left(x+\frac{3}{4}\right)^2\right]$$

Hence, we can write I_2 as

$$I_2=-\frac{5}{4}\int\sqrt{2\left[\left(\frac{\sqrt{41}}{4}\right)^2-\left(x+\frac{3}{4}\right)^2\right]}dx$$

$$\Rightarrow I_2=-\frac{5\sqrt{2}}{4}\int\sqrt{\left(\frac{\sqrt{41}}{4}\right)^2-\left(x+\frac{3}{4}\right)^2}dx$$

Recall $\int\sqrt{a^2-x^2}dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c$

$$\Rightarrow I_2=-\frac{5\sqrt{2}}{4}\left[\frac{\left(x+\frac{3}{4}\right)}{2}\sqrt{\left(\frac{\sqrt{41}}{4}\right)^2-\left(x+\frac{3}{4}\right)^2}+\frac{\left(\frac{\sqrt{41}}{4}\right)^2}{2}\sin^{-1}\left(\frac{x+\frac{3}{4}}{\frac{\sqrt{41}}{4}}\right)\right]+c$$

$$\Rightarrow I_2=-\frac{5\sqrt{2}}{4}\left[\frac{(4x+3)}{8}\sqrt{2-\frac{3}{2}x-x^2}+\frac{41}{32}\sin^{-1}\left(\frac{4x+3}{\sqrt{41}}\right)\right]+c$$

$$\Rightarrow I_2=-\frac{5\sqrt{2}}{32}(4x+3)\sqrt{2-\frac{3}{2}x-x^2}-\frac{205\sqrt{2}}{128}\sin^{-1}\left(\frac{4x+3}{\sqrt{41}}\right)+c$$

$$\therefore I_2=-\frac{5}{32}(4x+3)\sqrt{4-3x-2x^2}-\frac{205\sqrt{2}}{128}\sin^{-1}\left(\frac{4x+3}{\sqrt{41}}\right)+c$$

Substituting I_1 and I_2 in I, we get

$$I=-\frac{1}{2}(4-3x-2x^2)^{\frac{3}{2}}-\frac{5}{32}(4x+3)\sqrt{4-3x-2x^2}-\frac{205\sqrt{2}}{128}\sin^{-1}\left(\frac{4x+3}{\sqrt{41}}\right)+c$$

Thus, $\int(3x+1)\sqrt{4-3x-2x^2}dx=-\frac{1}{2}(4-3x-2x^2)^{\frac{3}{2}}-\frac{5}{32}(4x+3)\sqrt{4-3x-2x^2}-\frac{205\sqrt{2}}{128}\sin^{-1}\left(\frac{4x+3}{\sqrt{41}}\right)+c$

14. Question

Evaluate the following integrals –

$$\int(2x+5)\sqrt{10-4x-3x^2}\,dx$$

Answer

Let $I=\int(2x+5)\sqrt{10-4x-3x^2}dx$

Let us assume, $2x + 5 = \lambda \frac{d}{dx}(10 - 4x - 3x^2) + \mu$

$\Rightarrow 2x + 5 = \lambda \left[\frac{d}{dx}(10) - \frac{d}{dx}(4x) - \frac{d}{dx}(3x^2)\right] + \mu$

$\Rightarrow 2x + 5 = \lambda \left[\frac{d}{dx}(10) - 4\frac{d}{dx}(x) - 3\frac{d}{dx}(x^2)\right] + \mu$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$\Rightarrow 2x + 5 = \lambda(0 - 4 - 3 \times 2x^{2-1}) + \mu$

$\Rightarrow 2x + 5 = \lambda(-4 - 6x) + \mu$

$\Rightarrow 2x + 5 = -6\lambda x + \mu - 4\lambda$

Comparing the coefficient of x on both sides, we get

$-6\lambda = 2 \Rightarrow \lambda = -\frac{2}{6} = -\frac{1}{3}$

Comparing the constant on both sides, we get

$\mu - 4\lambda = 5$

$\Rightarrow \mu - 4\left(-\frac{1}{3}\right) = 5$

$\Rightarrow \mu + \frac{4}{3} = 5$

$\therefore \mu = \frac{11}{3}$

Hence, we have $2x + 5 = -\frac{1}{3}(-4 - 6x) + \frac{11}{3}$

Substituting this value in I, we can write the integral as

$I = \int \left[-\frac{1}{3}(-4 - 6x) + \frac{11}{3}\right]\sqrt{10 - 4x - 3x^2}dx$

$\Rightarrow I = \int \left[-\frac{1}{3}(-4 - 6x)\sqrt{10 - 4x - 3x^2} + \frac{11}{3}\sqrt{10 - 4x - 3x^2}\right]dx$

$\Rightarrow I = -\int \frac{1}{3}(-4 - 6x)\sqrt{10 - 4x - 3x^2}dx + \int \frac{11}{3}\sqrt{10 - 4x - 3x^2}dx$

$\Rightarrow I = -\frac{1}{3}\int (-4 - 6x)\sqrt{10 - 4x - 3x^2}dx + \frac{11}{3}\int \sqrt{10 - 4x - 3x^2}dx$

Let $I_1 = -\frac{1}{3}\int(-4 - 6x)\sqrt{10 - 4x - 3x^2}dx$

Now, put $10 - 4x - 3x^2 = t$

$\Rightarrow (-4 - 6x)dx = dt$ (Differentiating both sides)

Substituting this value in I_1, we can write

$I_1 = -\frac{1}{3}\int \sqrt{t}dt$

$\Rightarrow I_1 = -\frac{1}{3}\int t^{\frac{1}{2}}dt$

Recall $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow I_1 = -\frac{1}{3}\left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) + c$$

$$\Rightarrow I_1 = -\frac{1}{3}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + c$$

$$\Rightarrow I_1 = -\frac{1}{3} \times \frac{2}{3} t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = -\frac{2}{9} t^{\frac{3}{2}} + c$$

$$\therefore I_1 = -\frac{2}{9}(10 - 4x - 3x^2)^{\frac{3}{2}} + c$$

Let $I_2 = \frac{11}{3}\int \sqrt{10 - 4x - 3x^2}dx$

We can write $10 - 4x - 3x^2 = -(3x^2 + 4x - 10)$

$$\Rightarrow 10 - 4x - 3x^2 = -3\left[x^2 + \frac{4}{3}x - \frac{10}{3}\right]$$

$$\Rightarrow 10 - 4x - 3x^2 = -3\left[x^2 + 2(x)\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^2 - \frac{10}{3}\right]$$

$$\Rightarrow 10 - 4x - 3x^2 = -3\left[\left(x + \frac{2}{3}\right)^2 - \frac{4}{9} - \frac{10}{3}\right]$$

$$\Rightarrow 10 - 4x - 3x^2 = -3\left[\left(x + \frac{2}{3}\right)^2 - \frac{34}{9}\right]$$

$$\Rightarrow 10 - 4x - 3x^2 = 3\left[\frac{34}{9} - \left(x + \frac{2}{3}\right)^2\right]$$

$$\Rightarrow 10 - 4x - 3x^2 = 3\left[\left(\frac{\sqrt{34}}{3}\right)^2 - \left(x + \frac{2}{3}\right)^2\right]$$

Hence, we can write I_2 as

$$I_2 = \frac{11}{3}\int \sqrt{3\left[\left(\frac{\sqrt{34}}{3}\right)^2 - \left(x + \frac{2}{3}\right)^2\right]}dx$$

$$\Rightarrow I_2 = \frac{11\sqrt{3}}{3}\int \sqrt{\left(\frac{\sqrt{34}}{3}\right)^2 - \left(x + \frac{2}{3}\right)^2}\,dx$$

Recall $\int \sqrt{a^2 - x^2}dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

$$\Rightarrow I_2 = \frac{11\sqrt{3}}{3}\left[\frac{\left(x + \frac{2}{3}\right)}{2}\sqrt{\left(\frac{\sqrt{34}}{3}\right)^2 - \left(x + \frac{2}{3}\right)^2} + \frac{\left(\frac{\sqrt{34}}{3}\right)^2}{2}\sin^{-1}\left(\frac{x + \frac{2}{3}}{\frac{\sqrt{34}}{3}}\right)\right] + c$$

$$\Rightarrow I_2 = \frac{11\sqrt{3}}{3}\left[\frac{(3x + 2)}{6}\sqrt{\frac{10}{3} - \frac{4}{3}x - x^2} + \frac{34}{18}\sin^{-1}\left(\frac{3x + 2}{\sqrt{34}}\right)\right] + c$$

$$\Rightarrow I_2 = -\frac{11\sqrt{3}}{18}(3x+2)\sqrt{\frac{10}{3}-\frac{4}{3}x-x^2} - \frac{374\sqrt{3}}{54}\sin^{-1}\left(\frac{3x+2}{\sqrt{34}}\right) + c$$

$$\therefore I_2 = -\frac{11}{18}(3x+2)\sqrt{10-4x-3x^2} - \frac{187\sqrt{3}}{27}\sin^{-1}\left(\frac{3x+2}{\sqrt{34}}\right) + c$$

Substituting I_1 and I_2 in I, we get

$$I = -\frac{2}{9}(10-4x-3x^2)^{\frac{3}{2}} - \frac{11}{18}(3x+2)\sqrt{10-4x-3x^2} - \frac{187\sqrt{3}}{27}\sin^{-1}\left(\frac{3x+2}{\sqrt{34}}\right) + c$$

Thus, $\int (2x+5)\sqrt{10-4x-3x^2}dx = -\frac{2}{9}(10-4x-3x^2)^{\frac{3}{2}} - \frac{11}{18}(3x+2)\sqrt{10-4x-3x^2} - \frac{187\sqrt{3}}{27}\sin^{-1}\left(\frac{3x+2}{\sqrt{34}}\right) + c$

Exercise 19.30

1. Question

Evaluate the following integral:

$$\int \frac{2x+1}{(x+1)(x-2)}\, dx$$

Answer

Here the denominator is already factored.

So let

$$\frac{2x+1}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2} \ldots\ldots \text{(i)}$$

$$\Rightarrow \frac{2x+1}{(x+1)(x-2)} = \frac{A(x-2)+B(x+1)}{(x+1)(x-2)}$$

⇒ 2x + 1 = A(x – 2) + B(x + 1)……(ii)

We need to solve for A and B. One way to do this is to pick values for x which will cancel each variable.

Put x = 2 in the above equation, we get

⇒ 2(2) + 1 = A(2 – 2) + B(2 + 1)

⇒ 3B = 5

$$\Rightarrow B = \frac{5}{3}$$

Now put x = – 1 in equation (ii), we get

⇒ 2(– 1) + 1 = A((– 1) – 2) + B((– 1) + 1)

⇒ – 3A = – 1

$$\Rightarrow A = \frac{1}{3}$$

We put the values of A and B values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[\frac{A}{x+1} + \frac{B}{x-2}\right] dx$$

$$\Rightarrow \int \left[\frac{\frac{1}{3}}{x+1} + \frac{\frac{5}{3}}{x-2}\right] dx$$

Split up the integral,

$$\Rightarrow \frac{1}{3}\int \left[\frac{1}{x+1}\right] dx + \frac{5}{3}\int \left[\frac{1}{x-2}\right] dx$$

Let substitute u = x + 1 ⇒ du = dx and z = x – 2 ⇒ dz = dx, so the above equation becomes,

$$\Rightarrow \frac{1}{3}\int \left[\frac{1}{u}\right] du + \frac{5}{3}\int \left[\frac{1}{z}\right] dz$$

On integrating we get

$$\Rightarrow \frac{1}{3}\log|u| + \frac{5}{3}\log|z| + C$$

Substituting back, we get

$$\Rightarrow \frac{1}{3}\log|x+1| + \frac{5}{3}\log|x-2| + C$$

Note: the absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int \frac{2x+1}{(x+1)(x-2)} dx = \frac{1}{3}\log|x+1| + \frac{5}{3}\log|x-2| + C$$

2. Question

Evaluate the following integral:

$$\int \frac{1}{x(x-2)(x-4)} dx$$

Answer

Here the denominator is already factored.

So let

$$\frac{1}{x(x-2)(x-4)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-4} \ldots\ldots(i)$$

$$\Rightarrow \frac{1}{x(x-2)(x-4)} = \frac{A(x-2)(x-4) + Bx(x-4) + Cx(x-2)}{x(x-2)(x-4)}$$

⇒ 1 = A(x – 2)(x – 4) + Bx(x – 4) + Cx(x – 2)……(ii)

We need to solve for A, B and C. One way to do this is to pick values for x which will cancel each variable.

Put x = 0 in the above equation, we get

⇒ 1 = A(0 – 2)(0 – 4) + B(0)(0 – 4) + C(0)(0 – 2)

⇒ 1 = 8A + 0 + 0

$$\Rightarrow A = \frac{1}{8}$$

Now put x = 2 in equation (ii), we get

⇒ 1 = A(2 – 2)(2 – 4) + B(2)(2 – 4) + C(2)(2 – 2)

⇒ 1 = 0 – 4B + 0

$\Rightarrow B = -\frac{1}{4}$

Now put x = 4 in equation (ii), we get

⇒ 1 = A(4 – 2)(4 – 4) + B(4)(4 – 4) + C(4)(4 – 2)

⇒ 1 = 0 + 0 + 8C

$\Rightarrow C = \frac{1}{8}$

We put the values of A, B, and C values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[\frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-4}\right] dx$$

$$\Rightarrow \int \left[\frac{\frac{1}{8}}{x} + \frac{-\frac{1}{4}}{x-2} + \frac{\frac{1}{8}}{x-4}\right] dx$$

Split up the integral,

$$\Rightarrow \frac{1}{8}\int \left[\frac{1}{x}\right] dx - \frac{1}{4}\int \left[\frac{1}{x-2}\right] dx + \frac{1}{8}\int \left[\frac{1}{x-4}\right] dx$$

Let substitute u = x – 4 ⇒ du = dx and z = x – 2 ⇒ dz = dx, so the above equation becomes,

$$\Rightarrow \frac{1}{8}\int \left[\frac{1}{x}\right] dx - \frac{1}{4}\int \left[\frac{1}{z}\right] dz + \frac{1}{8}\int \left[\frac{1}{u}\right] du$$

On integrating we get

$$\Rightarrow \frac{1}{8}\log|x| - \frac{1}{4}\log|z| + \frac{1}{8}\log|u| + C$$

Substituting back, we get

$$\Rightarrow \frac{1}{8}\log|x| - \frac{1}{4}\log|x-2| + \frac{1}{8}\log|x-4| + C$$

We will take $\frac{1}{8}$ common, we get

$$\Rightarrow \frac{1}{8}[\log|x| - 2\log|x-2| + \log|x-4| + C]$$

Applying the logarithm rule we can rewrite the above equation as

$$\Rightarrow \frac{1}{8}\left[\log\left|\frac{x}{(x-2)^2}\right| + \log|x-4| + C\right]$$

$$\Rightarrow \frac{1}{8}\left[\log\left|\frac{x(x-4)}{(x-2)^2}\right|\right] + C$$

Note: the absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int \frac{1}{x(x-2)(x-4)} dx = \frac{1}{8}\left[\log\left|\frac{x(x-4)}{(x-2)^2}\right|\right] + C$$

3. Question

Evaluate the following integral:

$$\int \frac{x^2+x-1}{x^2+x-6}\,dx$$

Answer

First we simplify numerator, we get

$$\frac{x^2 + x - 1}{x^2 + x - 6}$$

$$= \frac{x^2 + x - 6 + 5}{x^2 + x - 6}$$

$$= \frac{x^2 + x - 6}{x^2 + x - 6} + \frac{5}{x^2 + x - 6}$$

$$= 1 + \frac{5}{x^2 + x - 6}$$

Now we will factorize denominator by splitting the middle term, we get

$$1 + \frac{5}{x^2 + x - 6}$$

$$= 1 + \frac{5}{x^2 + 3x - 2x - 6}$$

$$= 1 + \frac{5}{x(x + 3) - 2(x + 3)}$$

$$= 1 + \frac{5}{(x + 3)(x - 2)}$$

Now the denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{5}{(x + 3)(x - 2)} = \frac{A}{x + 3} + \frac{B}{x - 2} \ldots\ldots \text{(i)}$$

$$\Rightarrow \frac{5}{(x + 3)(x - 2)} = \frac{A(x - 2) + B(x + 3)}{(x + 3)(x - 2)}$$

⇒ 5 = A(x – 2) + B(x + 3)……(ii)

We need to solve for A and B. One way to do this is to pick values for x which will cancel each variable.

Put x = 2 in the above equation, we get

⇒ 5 = A(2 – 2) + B(2 + 3)

⇒ 5 = 0 + 5B

⇒ B = 1

Now put x = – 3 in equation (ii), we get

⇒ 5 = A((– 3) – 2) + B((– 3) + 3)

⇒ 5 = – 5A

⇒ A = – 1

We put the values of A and B values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[1 + \frac{A}{x + 3} + \frac{B}{x - 2}\right] dx$$

$$\Rightarrow \int \left[1 + \frac{-1}{x + 3} + \frac{1}{x - 2}\right] dx$$

Split up the integral,

$$\Rightarrow \int 1dx - \int \left[\frac{1}{x + 3}\right] dx + \int \left[\frac{1}{x - 2}\right] dx$$

Let substitute u = x + 3 ⇒ du = dx and z = x – 2 ⇒ dz = dx, so the above equation becomes,

$$\Rightarrow \int 1dx - \int \left[\frac{1}{u}\right] du + \int \left[\frac{1}{z}\right] dz$$

On integrating we get

⇒ x – log|u| + log|z| + C

Substituting back, we get

⇒ x – log|x + 3| + log|x – 2| + C

Applying the logarithm rule, we can rewrite the above equation as

$$\Rightarrow x + \log\left|\frac{x - 2}{x + 3}\right| + C$$

Note: the absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int \frac{x^2 + x - 1}{x^2 + x - 6} dx = x + \log\left|\frac{x - 2}{x + 3}\right| + C$$

4. Question

Evaluate the following integral:

$$\int \frac{3 + 4x - x^2}{(x + 2)(x - 1)} dx$$

Answer

First we simplify numerator, we get

$$\frac{3 + 4x - x^2}{(x + 2)(x - 1)}$$

$$= \frac{-(x^2 - 4x - 3)}{x^2 + x - 2}$$

$$= \frac{-(x^2 + x - 5x - 2 - 1)}{x^2 + x - 2}$$

$$= \frac{-(x^2 + x - 2)}{x^2 + x - 2} + \frac{5x + 1}{x^2 + x - 2}$$

$$= -1 + \frac{5x + 1}{(x + 2)(x - 1)}$$

Now the denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{5x + 1}{(x + 2)(x - 1)} = \frac{A}{x + 2} + \frac{B}{x - 1} \ldots\ldots (i)$$

$$\Rightarrow \frac{5x + 1}{(x + 2)(x - 1)} = \frac{A(x - 1) + B(x + 2)}{(x + 2)(x - 1)}$$

$\Rightarrow 5x + 1 = A(x - 1) + B(x + 2)$......(ii)

We need to solve for A and B. One way to do this is to pick values for x which will cancel each variable.

Put x = 1 in the above equation, we get

$\Rightarrow 5(1) + 1 = A(1 - 1) + B(1 + 2)$

$\Rightarrow 6 = 0 + 3B$

$\Rightarrow B = 2$

Now put x = – 2 in equation (ii), we get

$\Rightarrow 5(-2) + 1 = A((-2) - 1) + B((-2) + 2)$

$\Rightarrow -9 = -3A + 0$

$\Rightarrow A = 3$

We put the values of A and B values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[-1 + \frac{5x + 1}{(x + 2)(x - 1)}\right] dx$$

$$\Rightarrow \int \left[-1 + \frac{A}{x + 2} + \frac{B}{x - 1}\right] dx$$

$$\Rightarrow \int \left[-1 + \frac{3}{x + 2} + \frac{2}{x - 1}\right] dx$$

Split up the integral,

$$\Rightarrow -\int 1dx + 3\int \left[\frac{1}{x + 2}\right] dx + 2\int \left[\frac{1}{x - 1}\right] dx$$

Let substitute $u = x + 2 \Rightarrow du = dx$ and $z = x - 1 \Rightarrow dz = dx$, so the above equation becomes,

$$\Rightarrow -\int 1dx + 3\int \left[\frac{1}{u}\right] du + 2\int \left[\frac{1}{z}\right] dz$$

On integrating we get

$$\Rightarrow -x + 3\log|u| + 2\log|z| + C$$

Substituting back, we get

$$\Rightarrow -x + 3\log|x + 2| + 2\log|x - 1| + C$$

Note: the absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int \frac{3 + 4x - x^2}{(x + 2)(x - 1)} dx = -x + 3\log|x + 2| + 2\log|x - 1| + C$$

5. Question

Evaluate the following integral:

$$\int \frac{x^2 + 1}{x^2 - 1} dx$$

Answer

First we simplify numerator, we get

$$\frac{x^2 + 1}{x^2 - 1}$$

$$= \frac{x^2 - 1 + 2}{x^2 - 1}$$

$$= \frac{x^2 - 1}{x^2 - 1} + \frac{2}{x^2 - 1}$$

$$= 1 + \frac{2}{(x - 1)(x + 1)}$$

Now the denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{2}{(x + 1)(x - 1)} = \frac{A}{x + 1} + \frac{B}{x - 1} \ldots\ldots (i)$$

$$\Rightarrow \frac{2}{(x + 2)(x - 1)} = \frac{A(x - 1) + B(x + 1)}{(x + 2)(x - 1)}$$

$$\Rightarrow 2 = A(x - 1) + B(x + 1) \ldots\ldots (ii)$$

We need to solve for A and B. One way to do this is to pick values for x which will cancel each variable.

Put x = 1 in the above equation, we get

⇒ 2 = A(1 – 1) + B(1 + 1)

⇒ 2 = 0 + 2B

⇒ B = 1

Now put x = – 1 in equation (ii), we get

⇒ 2 = A((– 1) – 1) + B((– 1) + 1)

⇒ 2 = – 2A + 0

⇒ A = – 1

We put the values of A and B values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[1 + \frac{2}{(x - 1)(x + 1)}\right] dx$$

$$\Rightarrow \int \left[1 + \frac{A}{x + 1} + \frac{B}{x - 1}\right] dx$$

$$\Rightarrow \int \left[1 + \frac{-1}{x + 1} + \frac{1}{x - 1}\right] dx$$

Split up the integral,

$$\Rightarrow \int 1dx - \int \left[\frac{1}{x + 1}\right] dx + \int \left[\frac{1}{x - 1}\right] dx$$

Let substitute u = x + 1 ⇒ du = dx and z = x – 1 ⇒ dz = dx, so the above equation becomes,

$$\Rightarrow \int 1dx - \int \left[\frac{1}{u}\right] du + \int \left[\frac{1}{z}\right] dz$$

On integrating we get

$$\Rightarrow x - \log|u| + \log|z| + C$$

Substituting back, we get

$\Rightarrow$ $x - \log|x + 1| + \log|x - 1| + C$

Applying the logarithm rule we get

$\Rightarrow$ $x + \log\left|\frac{x - 1}{x + 1}\right| + C$

Note: the absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int \frac{x^2 + 1}{x^2 - 1} dx = x + \log\left|\frac{x - 1}{x + 1}\right| + C$$

6. Question

Evaluate the following integral:

$$\int \frac{x^2}{(x-1)(x-2)(x-3)} dx$$

Answer

Denominator is already factorized, so let

$$\frac{x^2}{(x - 1)(x - 2)(x - 3)} = \frac{A}{x - 1} + \frac{B}{x - 2} + \frac{C}{x - 3} \ldots\ldots (i)$$

$$\Rightarrow \frac{x^2}{(x - 1)(x - 2)(x - 3)} = \frac{A(x - 2)(x - 3) + B(x - 1)(x - 3) + C(x - 1)(x - 2)}{(x - 1)(x - 2)(x - 3)}$$

$\Rightarrow$ $x^2 = A(x - 2)(x - 3) + B(x - 1)(x - 3) + C(x - 1)(x - 2) \ldots\ldots (ii)$

We need to solve for A, B and C. One way to do this is to pick values for x which will cancel each variable.

Put x = 1 in the above equation, we get

$\Rightarrow$ $1^2 = A(1 - 2)(1 - 3) + B(1 - 1)(1 - 3) + C(1 - 1)(1 - 2)$

$\Rightarrow$ 1 = 2A + 0 + 0

$\Rightarrow$ $A = \frac{1}{2}$

Now put x = 2 in equation (ii), we get

$\Rightarrow$ $2^2 = A(2 - 2)(2 - 3) + B(2 - 1)(2 - 3) + C(2 - 1)(2 - 2)$

$\Rightarrow$ 4 = 0 – B + 0

$\Rightarrow$ B = – 4

Now put x = 3 in equation (ii), we get

$\Rightarrow$ $3^2 = A(3 - 2)(3 - 3) + B(3 - 1)(3 - 3) + C(3 - 1)(3 - 2)$

$\Rightarrow$ 9 = 0 + 0 + 2C

$\Rightarrow$ $C = \frac{9}{2}$

We put the values of A, B, and C values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[\frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}\right] dx$$

$$\Rightarrow \int \left[\frac{\frac{1}{2}}{x-1} + \frac{-4}{x-2} + \frac{\frac{9}{2}}{x-3}\right] dx$$

Split up the integral,

$$\Rightarrow \frac{1}{2}\int \left[\frac{1}{x-1}\right] dx - 4\int \left[\frac{1}{x-2}\right] dx + \frac{9}{2}\int \left[\frac{1}{x-3}\right] dx$$

Let substitute u = x – 1 ⇒ du = dx, y = x – 2 ⇒ dy = dx and z = x – 3 ⇒ dz = dx, so the above equation becomes,

$$\Rightarrow \frac{1}{2}\int \left[\frac{1}{u}\right] du - 4\int \left[\frac{1}{y}\right] dy + \frac{9}{2}\int \left[\frac{1}{z}\right] dz$$

On integrating we get

$$\Rightarrow \frac{1}{2}\log|u| - 4\log|y| + \frac{9}{2}\log|z| + C$$

Substituting back, we get

$$\Rightarrow \frac{1}{2}\log|x-1| - 4\log|x-2| + \frac{9}{2}\log|x-3| + C$$

Note: the absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int \frac{x^2}{(x-1)(x-2)(x-3)} dx = \frac{1}{2}\log|x-1| - 4\log|x-2| + \frac{9}{2}\log|x-3| + C$$

7. Question

Evaluate the following integral:

$$\int \frac{5x}{(x+1)(x^2-4)} dx$$

Answer

$$\frac{5x}{(x+1)(x^2-4)} = \frac{5x}{(x+1)(x-2)(x+2)}$$

The denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{5x}{(x+1)(x-2)(x+2)} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x+2} \ldots\ldots (i)$$

$$\Rightarrow \frac{5x}{(x+1)(x-2)(x+2)} = \frac{A(x-2)(x+2) + B(x+1)(x+2) + C(x+1)(x-2)}{(x+1)(x-2)(x+2)}$$

⇒ 5x = A(x – 2)(x + 2) + B(x + 1)(x + 2) + C(x + 1)(x – 2)……(ii)

We need to solve for A, B and C. One way to do this is to pick values for x which will cancel each variable.

Put x = – 1 in the above equation, we get

⇒ 5(– 1) = A((– 1) – 2)((– 1) + 2) + B((– 1) + 1)((– 1) + 2) + C((– 1) + 1)((– 1) – 2)

⇒ – 5 = – 3A + 0 + 0

$\Rightarrow A = \frac{5}{3}$

Now put x = – 2 in equation (ii), we get

$\Rightarrow 5(-2) = A((-2)-2)((-2)+2) + B((-2)+1)((-2)+2) + C((-2)+1)((-2)-2)$

$\Rightarrow -10 = 0 + 0 + 4C$

$\Rightarrow C = -\frac{10}{4} = -\frac{5}{2}$

Now put x = 2 in equation (ii), we get

$\Rightarrow 5(2) = A((2)-2)((2)+2) + B((2)+1)((2)+2) + C((2)+1)((2)-2)$

$\Rightarrow 10 = 0 + 12B + 0$

$\Rightarrow B = \frac{10}{12} = \frac{5}{6}$

We put the values of A, B, and C values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int\left[\frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x+2}\right]dx$$

$$\Rightarrow \int\left[\frac{\frac{5}{3}}{x+1} + \frac{-\frac{5}{2}}{x-2} + \frac{\frac{5}{6}}{x+2}\right]dx$$

Split up the integral,

$$\Rightarrow \frac{5}{3}\int\left[\frac{1}{x+1}\right]dx - \frac{5}{2}\int\left[\frac{1}{x-2}\right]dx + \frac{5}{6}\int\left[\frac{1}{x+2}\right]dx$$

Let substitute u = x + 1 ⇒ du = dx, y = x – 2 ⇒ dy = dx and z = x + 2 ⇒ dz = dx, so the above equation becomes,

$$\Rightarrow \frac{5}{3}\int\left[\frac{1}{u}\right]du - \frac{5}{2}\int\left[\frac{1}{y}\right]dy + \frac{5}{6}\int\left[\frac{1}{z}\right]dz$$

On integrating we get

$$\Rightarrow \frac{5}{3}\log|u| - \frac{5}{2}\log|y| + \frac{5}{6}\log|z| + C$$

Substituting back, we get

$$\Rightarrow \frac{5}{3}\log|x+1| - \frac{5}{2}\log|x-2| + \frac{5}{6}\log|x+2| + C$$

Note: the absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int\frac{5x}{(x+1)(x^2-4)}dx = \frac{5}{3}\log|x+1| - \frac{5}{2}\log|x-2| + \frac{5}{6}\log|x+2| + C$$

8. Question

Evaluate the following integral:

$$\int\frac{x^2+1}{x(x^2-1)}dx$$

Answer

$$\frac{x^2+1}{x(x^2-1)} = \frac{x^2+1}{x(x-1)(x+1)}$$

The denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{x^2+1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} \ldots \ldots (i)$$

$$\Rightarrow \frac{x^2+1}{x(x-1)(x+1)} = \frac{A(x-1)(x+1) + Bx(x+1) + Cx(x-1)}{x(x-1)(x+1)}$$

$$\Rightarrow x^2 + 1 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1) \ldots \ldots (ii)$$

We need to solve for A, B and C. One way to do this is to pick values for x which will cancel each variable.

Put x = 0 in the above equation, we get

$\Rightarrow 0^2 + 1 = A(0 - 1)(0 + 1) + B(0)(0 + 1) + C(0)(0 - 1)$

$\Rightarrow 1 = - A + 0 + 0$

$\Rightarrow A = - 1$

Now put x = – 1 in equation (ii), we get

$\Rightarrow (-1)^2 + 1 = A((-1) - 1)((-1) + 1) + B(-1)((-1) + 1) + C(-1)((-1) - 1)$

$\Rightarrow 2 = 0 + 0 + C$

$\Rightarrow C = 1$

Now put x = 1 in equation (ii), we get

$\Rightarrow 1^2 + 1 = A(1 - 1)(1 + 1) + B(1)(1 + 1) + C(1)(1 - 1)$

$\Rightarrow 2 = 0 + 2B + 0$

$\Rightarrow B = 1$

We put the values of A, B, and C values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[\frac{x^2+1}{x(x-1)(x+1)}\right] dx$$

$$\Rightarrow \int \left[\frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}\right] dx$$

$$\Rightarrow \int \left[\frac{-1}{x} + \frac{1}{x-1} + \frac{1}{x+1}\right] dx$$

Split up the integral,

$$\Rightarrow -\int \left[\frac{1}{x}\right] dx + \int \left[\frac{1}{x-1}\right] dx + \int \left[\frac{1}{x+1}\right] dx$$

Let substitute u = x + 1 ⇒ du = dx, y = x – 1 ⇒ dy = dx, so the above equation becomes,

$$\Rightarrow -\int \left[\frac{1}{x}\right] dx + \int \left[\frac{1}{y}\right] dy + \int \left[\frac{1}{u}\right] du$$

On integrating we get

$$\Rightarrow -\log|x| + \log|y| + \log|u| + C$$

Substituting back, we get

$$\Rightarrow -\log|x| + \log|x-1| + \log|x+1| + C$$

Applying the rules of logarithm we get

$$\Rightarrow -\log|x| + \log|(x-1)(x+1)| + C$$

$$\Rightarrow \log\left|\frac{x^2-1}{x}\right| + C$$

Note: the absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int \frac{x^2+1}{x(x^2-1)}dx = \log\left|\frac{x^2-1}{x}\right| + C$$

9. Question

Evaluate the following integral:

$$\int \frac{2x-3}{(x^2-1)(2x+3)}\,dx$$

Answer

$$\frac{2x-3}{(x^2-1)(2x+3)} = \frac{2x-3}{(x-1)(x+1)(2x+3)}$$

The denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{2x-3}{(x-1)(x+1)(2x+3)} = \frac{A}{(x-1)} + \frac{B}{x+1} + \frac{C}{2x+3} \ldots\ldots(i)$$

$$\Rightarrow \frac{2x-3}{(x-1)(x+1)(2x+3)} = \frac{A(x+1)(2x+3) + B(x-1)(2x+3) + C(x-1)(x+1)}{(x-1)(x+1)(2x+3)}$$

⇒ 2x – 3 = A(x + 1)(2x + 3) + B(x – 1)(2x + 3) + C(x – 1)(x + 1)......(ii)

We need to solve for A, B and C. One way to do this is to pick values for x which will cancel each variable.

Put x = – 1 in the above equation, we get

⇒ 2(– 1) – 3 = A((– 1) + 1)(2(– 1) + 3) + B((– 1) – 1)(2(– 1) + 3) + C((– 1) – 1)((– 1) + 1)

⇒ – 5 = 0 – 2B + 0

$$\Rightarrow B = \frac{5}{2}$$

Now put x = 1 in equation (ii), we get

⇒ 2(1) – 3 = A((1) + 1)(2(1) + 3) + B((1) – 1)(2(1) + 3) + C((1) – 1)((1) + 1)

⇒ – 1 = 10A + 0 + 0

$$\Rightarrow A = -\frac{1}{10}$$

Now put $x = -\frac{3}{2}$ in equation (ii), we get

$$\Rightarrow 2\left(-\frac{3}{2}\right)-3$$
$$= A\left(\left(-\frac{3}{2}\right)+1\right)\left(2\left(-\frac{3}{2}\right)+3\right) + B\left(\left(-\frac{3}{2}\right)-1\right)\left(2\left(-\frac{3}{2}\right)+3\right) + C\left(\left(-\frac{3}{2}\right)-1\right)\left(\left(-\frac{3}{2}\right)+1\right)$$

$$\Rightarrow -6 = 0 + 0 + \frac{5}{4}C$$

$$\Rightarrow C = -\frac{24}{5}$$

We put the values of A, B, and C values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[\frac{2x-3}{(x-1)(x+1)(2x+3)}\right] dx$$

$$\Rightarrow \int \left[\frac{A}{(x-1)} + \frac{B}{x+1} + \frac{C}{2x+3}\right] dx$$

$$\Rightarrow \int \left[\frac{-\frac{1}{10}}{(x-1)} + \frac{\frac{5}{2}}{x+1} + \frac{-\frac{24}{5}}{2x+3}\right] dx$$

Split up the integral,

$$\Rightarrow -\frac{1}{10}\int \left[\frac{1}{x-1}\right] dx + \frac{5}{2}\int \left[\frac{1}{x+1}\right] dx - \frac{24}{5}\int \left[\frac{1}{2x+3}\right] dx$$

Let substitute

$u = x + 1 \Rightarrow du = dx$,

$y = x - 1 \Rightarrow dy = dx$ and

$z = 2x + 3 \Rightarrow dz = 2dx \Rightarrow dx = \frac{dz}{2}$ so the above equation becomes,

$$\Rightarrow -\frac{1}{10}\int \left[\frac{1}{y}\right] dy + \frac{5}{2}\int \left[\frac{1}{u}\right] du - \frac{24}{5}\int \frac{\left[\frac{1}{z}\right] dz}{2}$$

On integrating we get

$$\Rightarrow -\frac{1}{10}\log|y| + \frac{5}{2}\log|u| - \frac{12}{5}\log|z| + C$$

Substituting back, we get

$$\Rightarrow -\frac{1}{10}\log|x-1| + \frac{5}{2}\log|x+1| - \frac{12}{5}\log|2x+3| + C$$

Note: the absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int \frac{2x-3}{(x^2-1)(2x+3)} dx = -\frac{1}{10}\log|x-1| + \frac{5}{2}\log|x+1| - \frac{12}{5}\log|2x+3| + C$$

10. Question

Evaluate the following integral:

$$\int \frac{x^3}{(x-1)(x-2)(x-3)} dx$$

Answer

First we simplify numerator, we will rewrite denominator as shown below

$$\frac{x^3}{(x-1)(x-2)(x-3)} = \frac{x^3}{x^3 - 6x^2 + 11x - 6}$$

Add and subtract numerator with (– 6x^2 + 11x – 6), we get

$$\frac{x^3 - 6x^2 + 11x - 6 + (6x^2 - 11x + 6)}{x^3 - 6x^2 + 11x - 6}$$

$$\Rightarrow = 1 + \frac{6x^2 - 11x + 6}{x^3 - 6x^2 + 11x - 6}$$

$$\Rightarrow = 1 + \frac{6x^2 - 11x + 6}{(x-1)(x-2)(x-3)}$$

The denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{6x^2 - 11x + 6}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{x-2} + \frac{C}{x-3} \ldots\ldots(i)$$

$$\Rightarrow \frac{6x^2 - 11x + 6}{(x-1)(x-2)(x-3)} = \frac{A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

⇒ 6x^2 – 11x + 6 = A(x – 2)(x – 3) + B(x – 1)(x – 3) + C(x – 1)(x – 2)……(ii)

We need to solve for A, B and C. One way to do this is to pick values for x which will cancel each variable.

Put x = 1 in the above equation, we get

⇒ 6(1)^2 – 11(1) + 6 = A(1 – 2)(1 – 3) + B(1 – 1)(1 – 3) + C(1 – 1)(1 – 2)

⇒ 1 = 2A + 0 + 0

$$\Rightarrow A = \frac{1}{2}$$

Now put x = 2 in equation (ii), we get

6(2)^2 – 11(2) + 6 = A(2 – 2)(2 – 3) + B(2 – 1)(2 – 3) + C(2 – 1)(2 – 2)

⇒ 8 = 0 – B + 0

⇒ B = – 8

Now put x = 3 in equation (ii), we get

⇒ 6(3)^2 – 11(3) + 6 = A(3 – 2)(3 – 3) + B(3 – 1)(3 – 3) + C(3 – 1)(3 – 2)

⇒ 27 = 0 + 0 + 2C

$$\Rightarrow C = \frac{27}{2}$$

We put the values of A, B, and C values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[1 + \frac{6x^2 - 11x + 6}{(x-1)(x-2)(x-3)}\right] dx$$

$$\Rightarrow \int \left[1 + \frac{A}{(x-1)} + \frac{B}{x-2} + \frac{C}{x-3}\right] dx$$

$$\Rightarrow \int \left[1 + \frac{\frac{1}{2}}{(x-1)} + \frac{-8}{x-2} + \frac{\frac{27}{2}}{x-3}\right] dx$$

Split up the integral,

$$\Rightarrow \int 1dx + \frac{1}{2}\int \left[\frac{1}{x-1}\right] dx - 8\int \left[\frac{1}{x-2}\right] dx + \frac{27}{2}\int \left[\frac{1}{x-3}\right] dx$$

Let substitute

u = x - 1 ⇒ du = dx,

y = x - 2 ⇒ dy = dx and

z = x - 3 ⇒ dz = dx, so the above equation becomes,

$$\Rightarrow \int 1dx + \frac{1}{2}\int \left[\frac{1}{u}\right] du - 8\int \left[\frac{1}{y}\right] dy + \frac{27}{2}\int \left[\frac{1}{z}\right] dz$$

On integrating we get

$$\Rightarrow x + \frac{1}{2}\log|u| - 8\log|y| + \frac{27}{2}\log|z| + C$$

Substituting back, we get

$$\Rightarrow x + \frac{1}{2}\log|x-1| - 8\log|x-2| + \frac{27}{2}\log|x-3| + C$$

Note: the absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int \frac{x^3}{(x-1)(x-2)(x-3)} dx$$
$$= x + \frac{1}{2}\log|x-1| - 8\log|x-2| + \frac{27}{2}\log|x-3| + C$$

11. Question

Evaluate the following integral:

$$\int \frac{\sin 2x}{(1+\sin x)(2+\sin x)} dx$$

Answer

The denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{\sin 2x}{(1 + \sin x)(2 + \sin x)} = \frac{A}{(1 + \sin x)} + \frac{B}{2 + \sin x} \ldots\ldots (i)$$

$$\Rightarrow \frac{\sin 2x}{(1 + \sin x)(2 + \sin x)} = \frac{A(2 + \sin x) + B(1 + \sin x)}{(1 + \sin x)(2 + \sin x)}$$

$$\Rightarrow \sin 2x = A(2 + \sin x) + B(1 + \sin x) = 2A + A\sin x + B + B\sin x$$

$$\Rightarrow 2\sin x \cos x = \sin x (A + B) + (2A + B) \ldots\ldots (ii)$$

We need to solve for A and B.

We will equate similar terms, we get.

2A + B = 0 ⇒ B = – 2A

And A + B = 2 cos x

Substituting the value of B, we get

A – 2A = 2 cos x ⇒ A = – 2 cos x

Hence B = – 2A = – 2(– 2 cos x)

⇒ B = 4cos x

We put the values of A and B values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[\frac{\sin 2x}{(1 + \sin x)(2 + \sin x)}\right] dx$$

$$\Rightarrow \int \left[\frac{A}{(1 + \sin x)} + \frac{B}{2 + \sin x}\right] dx$$

$$\Rightarrow \int \left[\frac{-2\cos x}{(1 + \sin x)} + \frac{4\cos x}{2 + \sin x}\right] dx$$

Split up the integral,

$$\Rightarrow -\int \frac{2\cos x}{(1 + \sin x)} dx + \int \frac{4\cos x}{2 + \sin x} dx$$

Let substitute

$u = \sin x \Rightarrow du = \cos x\, dx,$

so the above equation becomes,

$$\Rightarrow -2\int \frac{1}{(1 + u)} du + 4\int \frac{1}{2 + u} du$$

Now substitute

v = 1 + u ⇒ dv = du

z = 2 + u ⇒ dz = du

So above equation becomes,

$$\Rightarrow -2\int \frac{1}{(v)} dv + 4\int \frac{1}{z} dz$$

On integrating we get

$$\Rightarrow -2\log|v| + 4\log|z| + C$$

Substituting back, we get

$$\Rightarrow 4\log|2 + u| - 2\log|1 + u| + C$$

$$\Rightarrow 4\log|2 + \sin x| - 2\log|1 + \sin x| + C$$

Applying logarithm rule, we get

$$\Rightarrow \log|(2 + \sin x)^4| - \log|(1 + \sin x)^2| + C$$

$$\Rightarrow \log\left|\frac{(2 + \sin x)^4}{(1 + \sin x)^2}\right| + C$$

Note: the absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int \frac{\sin 2x}{(1+\sin x)(2+\sin x)}dx = \log\left|\frac{(2+\sin x)^4}{(1+\sin x)^2}\right| + C$$

12. Question

Evaluate the following integral:

$$\int \frac{2x}{(x^2+1)(x^2+3)}\,dx$$

Answer

Denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{2x}{(x^2+1)(x^2+3)} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{x^2+3} \ldots\ldots (i)$$

$$\Rightarrow \frac{2x}{(x^2+1)(x^2+3)} = \frac{(Ax+B)(x^2+3)+(Cx+D)(x^2+1)}{(x^2+1)(x^2+3)}$$

$$\Rightarrow 2x = (Ax+B)(x^2+3)+(Cx+D)(x^2+1)$$

$$\Rightarrow 2x = Ax^3 + 3Ax + Bx^2 + 3B + Cx^3 + Cx + Dx^2 + D$$

$$\Rightarrow 2x = (A+C)x^3 + (B+D)x^2 + (3A+C)x + (3B+D) \ldots\ldots (ii)$$

By equating similar terms, we get

$A + C = 0 \Rightarrow A = -C$ ....(iii)

$B + D = 0 \Rightarrow B = -D$ ....(iv)

$3A + C = 2$

$\Rightarrow 3(-C) + C = 2$ (from equation(iii))

$\Rightarrow C = -1$

So equation(iii) becomes $A = 1$

And also $3B + D = 0$ (from equation (ii))

$\Rightarrow 3(-D) + D = 0$ (from equation (iv))

$\Rightarrow D = 0$

So equation (iv) becomes, $B = 0$

We put the values of A, B, C and D values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[\frac{2x}{(x^2+1)(x^2+3)}\right]dx$$

$$\Rightarrow \int \left[\frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{x^2+3}\right]dx$$

$$\Rightarrow \int \left[\frac{(1)x+0}{(x^2+1)} + \frac{(-1)x+0}{x^2+3}\right]dx$$

Split up the integral,

$$\Rightarrow \int \frac{x}{(x^2+1)}dx - \int \left[\frac{x}{x^2+3}\right]dx$$

Let substitute

$u = x^2 + 1 \Rightarrow du = 2xdx \Rightarrow dx = \frac{1}{2x}du$

$v = x^2 + 3 \Rightarrow dv = 2xdx \Rightarrow dx = \frac{1}{2x}dv$

so the above equation becomes,

$\Rightarrow \frac{1}{2}\int \frac{1}{(u)}du - \frac{1}{2}\int \left[\frac{1}{v}\right]dv$

On integrating we get

$\Rightarrow \frac{1}{2}\log|u| - \frac{1}{2}\log|v| + C$

Substituting back, we get

$\Rightarrow \frac{1}{2}\log|x^2 + 1| - \frac{1}{2}\log|x^2 + 3| + C$

$\Rightarrow \frac{1}{2}[\log|x^2 + 1| - \log|x^2 + 3|] + C$

Applying the logarithm rule we get

$\Rightarrow \frac{1}{2}\left[\log\left|\frac{(x^2 + 1)}{x^2 + 3}\right|\right] + C$

Note: the absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int \frac{2x}{(x^2 + 1)(x^2 + 3)}dx = \frac{1}{2}\left[\log\left|\frac{(x^2 + 1)}{x^2 + 3}\right|\right] + C$$

13. Question

Evaluate the following integral:

$$\int \frac{1}{x \log x(2 + \log x)}dx$$

Answer

Let substitute $u = \log x \Rightarrow du = \frac{1}{x}dx$, so the given equation becomes

$$\int \frac{1}{x\log x(2 + \log x)}dx = \int \frac{1}{u(2 + u)}du \ldots..(i)$$

Denominator is factorised, so let separate the fraction through partial fraction, hence let

$$\frac{1}{u(2 + u)} = \frac{A}{u} + \frac{B}{(2 + u)} \ldots\ldots(ii)$$

$$\Rightarrow \frac{1}{u(2 + u)} = \frac{A(2 + u) + Bu}{u(2 + u)}$$

$\Rightarrow 1 = A(2 + u) + Bu$……(ii)

We need to solve for A and B. One way to do this is to pick values for x which will cancel each variable.

Put u = – 2 in above equation, we get

$\Rightarrow 1 = A(2 + (-2)) + B(-2)$

$\Rightarrow 1 = -2B$

$\Rightarrow B = -\frac{1}{2}$

Now put u = 0 in equation (ii), we get

$\Rightarrow 1 = A(2 + 0) + B(0)$

$\Rightarrow 1 = 2A + 0$

$\Rightarrow A = \frac{1}{2}$

We put the values of A and B values back into our partial fractions in equation (ii) and replace this as the integrand. We get

$$\int \left[\frac{1}{u(2 + u)}\right] du$$

$$\Rightarrow \int \left[\frac{A}{u} + \frac{B}{(2 + u)}\right] du$$

$$\Rightarrow \int \left[\frac{\frac{1}{2}}{u} + \frac{-\frac{1}{2}}{(2 + u)}\right] du$$

Split up the integral,

$$\Rightarrow \frac{1}{2}\int \frac{1}{u} du - \frac{1}{2}\int \left[\frac{1}{2 + u}\right] du$$

Let substitute

$z = 2 + u \Rightarrow dz = du$, so the above equation becomes,

$$\Rightarrow \frac{1}{2}\int \frac{1}{u} du - \frac{1}{2}\int \left[\frac{1}{z}\right] dz$$

On integrating we get

$$\Rightarrow \frac{1}{2}\log|u| - \frac{1}{2}\log|z| + C$$

Substituting back the value of z, we get

$$\Rightarrow \frac{1}{2}\log|u| - \frac{1}{2}\log|2 + u| + C$$

Now substitute back the value of u, we get

$$\Rightarrow \frac{1}{2}[\log|\log x| - \log|2 + \log x|] + C$$

Applying the rules of logarithm we get

$$\Rightarrow \frac{1}{2}\log\left|\frac{\log x}{2 + \log x}\right| + C$$

Note: the absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int \frac{1}{x\log x(2 + \log x)} dx = \frac{1}{2}\log\left|\frac{\log x}{2 + \log x}\right| + C$$

14. Question

Evaluate the following integral:

$$\int \frac{x^2+x+1}{(x^2+1)(x+2)}dx$$

Answer

Denominator is factorised, so let separate the fraction through partial fraction, hence let

$$\frac{x^2+x+1}{(x^2+1)(x+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x+2} \ldots\ldots(i)$$

$$\Rightarrow \frac{x^2+x+1}{(x^2+1)(x+2)} = \frac{(Ax+B)(x+2)+(Cx+D)(x^2+1)}{(x^2+1)(x+2)}$$

$$\Rightarrow x^2+x+1 = (Ax+B)(x+2)+(Cx+D)(x^2+1)$$

$$\Rightarrow x^2+x+1 = Ax^2+2Ax+Bx+2B+Cx^3+Cx+Dx^2+D$$

$$\Rightarrow x^2+x+1 = Cx^3+(A+D)x^2+(2A+B+C)x+(2B+D) \ldots\ldots(ii)$$

We need to solve for A, B, C and D. We will equate the like terms we get,

C = 0............(iii)

A + D = 1⇒ A = 1 – D.........(iv)

2A + B + C = 1

⇒ 2(1 – D) + B + 0 = 1 (from equation (iii) and (iv))

⇒ B = 2D – 1..........(v)

2B + D = 1

⇒ 2(2D – 1) + D = 1 (from equation (v), we get

⇒ 4D – 2 + D = 1

⇒ 5D = 3

$\Rightarrow D = \frac{3}{5}$..............(vi)

Equation (vi) in (v) and (iv), we get

$$B = 2\left(\frac{3}{5}\right) - 1 = \frac{1}{5}$$

$$A = 1 - \frac{3}{5} = \frac{2}{5}$$

We put the values of A, B, C, and D values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[\frac{x^2+x+1}{(x^2+1)(x+2)}\right]dx$$

$$\Rightarrow \int \left[\frac{Ax+B}{x^2+1} + \frac{Cx+D}{x+2}\right]dx$$

$$\Rightarrow \int \left[\frac{\left(\frac{2}{5}\right)x+\frac{1}{5}}{x^2+1} + \frac{(0)x+\frac{3}{5}}{x+2}\right]dx$$

Split up the integral,

$$\Rightarrow \frac{1}{5}\int \frac{2x}{x^2+1}dx + \frac{1}{5}\int \frac{1}{x^2+1}dx + \frac{3}{5}\int \left[\frac{1}{x+2}\right]dx$$

Let substitute

$u = x^2 + 1 \Rightarrow du = 2xdx$,

$y = x + 2 \Rightarrow dy = dx$, so the above equation becomes,

$$\Rightarrow \frac{1}{5}\int \frac{1}{u}du + \frac{1}{5}\int \frac{1}{x^2+1}dx + \frac{3}{5}\int \left[\frac{1}{y}\right]dy$$

On integrating we get

$$\Rightarrow \frac{1}{5}\log|u| + \frac{1}{5}\tan^{-1}x + \frac{3}{5}\log|y| + C$$

(the standard integral of $\frac{1}{x^2+1} = \tan^{-1}x$)

Substituting back, we get

$$\Rightarrow \frac{1}{5}\log|x^2+1| + \frac{1}{5}\tan^{-1}x + \frac{3}{5}\log|x+2| + C$$

Note: the absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int \frac{x^2+x+1}{(x^2+1)(x+2)}dx = \frac{1}{5}\log|x^2+1| + \frac{1}{5}\tan^{-1}x + \frac{3}{5}\log|x+2| + C$$

15. Question

Evaluate the following integral:

$\int \frac{ax^2+bx+c}{(x-a)(x-b)(x-c)}dx$, where a, b, c are distinct.

Answer

Denominator is factorised, so let separate the fraction through partial fraction, hence let

$$\frac{ax^2+bx+c}{(x-a)(x-b)(x-c)} = \frac{A}{(x-a)} + \frac{B}{x-b} + \frac{C}{x-c} \ldots\ldots (i)$$

$$\Rightarrow \frac{ax^2+bx+c}{(x-a)(x-b)(x-c)} = \frac{A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b)}{(x-a)(x-b)(x-c)}$$

$\Rightarrow ax^2 + bx + c = A(x - b)(x - c) + B(x - a)(x - c) + C(x - a)(x - b)$......(ii)

We need to solve for A, B and C. One way to do this is to pick values for x which will cancel each variable.

Put x = a in the above equation, we get

$\Rightarrow a(a)^2 + b(a) + c = A(a - b)(a - c) + B(a - a)(a - c) + C(a - a)(a - b)$

$\Rightarrow a^3 + ab + c = (a - b)(a - c)A + 0 + 0$

$$\Rightarrow A = \frac{a^3+ab+c}{(a-b)(a-c)}$$

Now put x = b in equation (ii), we get

$\Rightarrow a(b)^2 + b(b) + c = A(b - b)(b - c) + B(b - a)(b - c) + C(b - a)(b - b)$

$\Rightarrow ab^2 + b^2 + c = 0 + (b - a)(b - c)B + 0$

$$\Rightarrow B = \frac{a^3 + ab + c}{(a-b)(a-c)}$$

Now put x = c in equation (ii), we get

$$\Rightarrow a(c)^2 + b(c) + c = A(c-b)(c-c) + B(c-a)(c-c) + C(c-a)(c-b)$$

$$\Rightarrow ac^2 + bc + c = 0 + 0 + (c-a)(c-b)C$$

$$\Rightarrow C = \frac{ac^2 + bc + c}{(c-a)(c-b)}$$

We put the values of A, B, and C values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[\frac{ax^2 + bx + c}{(x-a)(x-b)(x-c)}\right] dx$$

$$\Rightarrow \int \left[\frac{A}{(x-a)} + \frac{B}{x-b} + \frac{C}{x-c}\right] dx$$

$$\Rightarrow \int \left[\frac{\frac{a^3 + ab + c}{(a-b)(a-c)}}{(x-a)} + \frac{\frac{a^3 + ab + c}{(a-b)(a-c)}}{x-b} + \frac{\frac{ac^2 + bc + c}{(c-a)(c-b)}}{x-c}\right] dx$$

Split up the integral,

$$\Rightarrow \frac{a^3 + ab + c}{(a-b)(a-c)} \int \frac{1}{x-a} dx + \frac{a^3 + ab + c}{(a-b)(a-c)} \int \left[\frac{1}{x-b}\right] dx + \frac{ac^2 + bc + c}{(c-a)(c-b)} \int \left[\frac{1}{x-c}\right] dx$$

Let substitute

u = x - a ⇒ du = dx,

y = x - b ⇒ dy = dx and

z = x - c ⇒ dz = dx, so the above equation becomes,

$$\Rightarrow \frac{a^3 + ab + c}{(a-b)(a-c)} \int \frac{1}{u} du + \frac{a^3 + ab + c}{(a-b)(a-c)} \int \left[\frac{1}{y}\right] dy + \frac{ac^2 + bc + c}{(c-a)(c-b)} \int \left[\frac{1}{z}\right] dz$$

On integrating we get

$$\Rightarrow \frac{a^3 + ab + c}{(a-b)(a-c)} \log|u| + \frac{a^3 + ab + c}{(a-b)(a-c)} \log|y| + \frac{ac^2 + bc + c}{(c-a)(c-b)} \log|z| + C$$

Substituting back, we get

$$\Rightarrow \frac{a^3 + ab + c}{(a-b)(a-c)} \log|x-a| + \frac{a^3 + ab + c}{(a-b)(a-c)} \log|x-b| + \frac{ac^2 + bc + c}{(c-a)(c-b)} \log|x-c| + C$$

Note: the absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int \frac{ax^2 + bx + c}{(x-a)(x-b)(x-c)} dx$$
$$= \frac{a^3 + ab + c}{(a-b)(a-c)} \log|x-a| + \frac{a^3 + ab + c}{(a-b)(a-c)} \log|x-b|$$
$$+ \frac{ac^2 + bc + c}{(c-a)(c-b)} \log|x-c| + C$$

16. Question

Evaluate the following integral:

$$\int \frac{x}{(x^2+1)(x-1)} dx$$

Answer

Denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{x}{(x^2 + 1)(x-1)} = \frac{Ax + B}{(x^2 + 1)} + \frac{Cx + D}{x-1} \ldots\ldots (i)$$

$$\Rightarrow \frac{x}{(x^2 + 1)(x-1)} = \frac{(Ax + B)(x-1) + (Cx + D)(x^2 + 1)}{(x^2 + 1)(x-1)}$$

$\Rightarrow x = (Ax + B)(x - 1) + (Cx + D)(x^2 + 1)$

$\Rightarrow x = Ax^2 - Ax + Bx - B + Cx^2 + Cx + Dx^2 + D$

$\Rightarrow x = (C)\, x^2 + (A + D)\, x^2 + (B - A + C)x + (D - B)$......(ii)

By equating similar terms, we get

$C = 0$ ....(iii)

$A + D = 0 \Rightarrow A = - D$ ....(iv)

$B - A + C = 1$

$\Rightarrow B - (- D) + 0 = 2$ (from equation(iii) and (iv))

$\Rightarrow B = 2 - D$...........(v)

$D - B = 0 \Rightarrow D - (2 - D) = 0 \Rightarrow 2D = 2 \Rightarrow D = 1$

So equation(iv) becomes $A = - 1$

So equation (v) becomes, $B = 2 - 1 = 1$

We put the values of A, B, C, and D values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[\frac{x}{(x^2 + 1)(x-1)}\right] dx$$

$$\Rightarrow \int \frac{Ax + B}{(x^2 + 1)} + \frac{Cx + D}{x-1} dx$$

$$\Rightarrow \int \left[\frac{(-1)x + 1}{(x^2 + 1)} + \frac{(0)x + 1}{x-1}\right] dx$$

Split up the integral,

$$\Rightarrow \int \frac{1}{(x^2 + 1)} dx - \int \frac{x}{(x^2 + 1)} dx + \int \left[\frac{1}{x-1}\right] dx$$

Let substitute

$u = x^2 + 1 \Rightarrow du = 2xdx \Rightarrow xdx = \frac{1}{2}du$

$v = x - 1 \Rightarrow dv = dx$

so the above equation becomes,

$$\Rightarrow \int \frac{1}{(x^2 + 1)}dx - \frac{1}{2}\int \frac{1}{(u)}du + \int \left[\frac{1}{v}\right]dv$$

On integrating we get

$$\Rightarrow \tan^{-1}x - \frac{1}{2}\log|u| + \log|v| + C$$

(the standard integral of $\frac{1}{x^2+1} = \tan^{-1}x$)

Substituting back, we get

$$\Rightarrow \tan^{-1}x - \frac{1}{2}\log|x^2 + 1| + \log|x - 1| + C$$

Note: the absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int \frac{x}{(x^2 + 1)(x - 1)}dx = \tan^{-1}x - \frac{1}{2}\log|x^2 + 1| + \log|x - 1| + C$$

17. Question

Evaluate the following integral:

$$\int \frac{1}{(x-1)(x+1)(x+2)}dx$$

Answer

Denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{1}{(x - 1)(x + 1)(x + 2)} = \frac{A}{(x - 1)} + \frac{B}{x + 1} + \frac{C}{x + 2} \ldots\ldots (i)$$

$$\Rightarrow \frac{1}{(x - 1)(x + 1)(x + 2)} = \frac{A(x + 1)(x + 2) + B(x - 1)(x + 2) + C(x - 1)(x + 1)}{(x - 1)(x + 1)(x + 2)}$$

⇒ 1 = A(x + 1)(x + 2) + B(x – 1)(x + 2) + C(x – 1)(x + 1)......(ii)

We need to solve for A, B and C. One way to do this is to pick values for x which will cancel each variable.

Put x = 1 in the above equation, we get

⇒ 1 = A(1 + 1)(1 + 2) + B(1 – 1)(1 + 2) + C(1 – 1)(1 + 1)

⇒ 1 = 6A + 0 + 0

$$\Rightarrow A = \frac{1}{6}$$

Now put x = – 1 in equation (ii), we get

⇒ 1 = A(– 1 + 1)(– 1 + 2) + B(– 1 – 1)(– 1 + 2) + C(– 1 – 1)(– 1 + 1)

⇒ 1 = 0 – 2B + 0

$\Rightarrow B = -\frac{1}{2}$

Now put x = – 2 in equation (ii), we get

$\Rightarrow 1 = A(-2+1)(-2+2) + B(-2-1)(-2+2) + C(-2-1)(-2+1)$

$\Rightarrow 1 = 0 + 0 + 3C$

$\Rightarrow C = \frac{1}{3}$

We put the values of A, B, and C values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[\frac{1}{(x-1)(x+1)(x+2)}\right] dx$$

$$\Rightarrow \int \left[\frac{A}{(x-1)} + \frac{B}{x+1} + \frac{C}{x+2}\right] dx$$

$$\Rightarrow \int \left[\frac{\frac{1}{6}}{(x-1)} + \frac{-\frac{1}{2}}{x+1} + \frac{\frac{1}{3}}{x+2}\right] dx$$

Split up the integral,

$$\Rightarrow \frac{1}{6}\int \left[\frac{1}{(x-1)}\right] dx - \frac{1}{2}\int \left[\frac{1}{x+1}\right] dx + \frac{1}{3}\int \left[\frac{1}{x+2}\right] dx$$

Let substitute

$u = x - 1 \Rightarrow du = dx$,

$y = x + 1 \Rightarrow dy = dx$ and

$z = x + 2 \Rightarrow dz = dx$, so the above equation becomes,

$$\Rightarrow \frac{1}{6}\int \left[\frac{1}{u}\right] du - \frac{1}{2}\int \left[\frac{1}{y}\right] dy + \frac{1}{3}\int \left[\frac{1}{z}\right] dz$$

On integrating we get

$$\Rightarrow \frac{1}{6}\log|u| - \frac{1}{2}\log|y| + \frac{1}{3}\log|z| + C$$

Substituting back, we get

$$\Rightarrow \frac{1}{6}\log|x-1| - \frac{1}{2}\log|x+1| + \frac{1}{3}\log|x+2| + C$$

Note: the absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int \frac{1}{(x-1)(x+1)(x+2)} dx = \frac{1}{6}\log|x-1| - \frac{1}{2}\log|x+1| + \frac{1}{3}\log|x+2| + C$$

18. Question

Evaluate the following integral:

$$\int \frac{x^2}{(x^2+4)(x^2+9)} dx$$

Answer

Denominator is factorised, so let separate the fraction through partial fraction, hence let

$$\frac{x^2}{(x^2+4)(x^2+9)} = \frac{Ax+B}{(x^2+4)} + \frac{Cx+D}{x^2+9} \ldots\ldots (i)$$

$$\Rightarrow \frac{x^2}{(x^2+4)(x^2+9)} = \frac{(Ax+B)(x^2+9)+(Cx+D)(x^2+4)}{(x^2+4)(x^2+9)}$$

$$\Rightarrow x^2 = (Ax+B)(x^2+9)+(Cx+D)(x^2+4)$$

$$\Rightarrow x^2 = Ax^3+9Ax+Bx^2+9B+Cx^3+4Cx+Dx^2+4D$$

$$\Rightarrow x^2 = (A+C)x^3+(B+D)x^2+(9A+4C)x+(9B+4D) \ldots\ldots (ii)$$

By equating similar terms, we get

A + C = 0 ⇒ A = – C ....(iii)

B + D = 1⇒ B = 1 – D............(iv)

9A + 4C = 0

⇒ 9(– C) + 4C = 0 (from equation(iii))

⇒ C = 0...........(v)

$$9B+4D = 0 \Rightarrow 9(1-D)+4D = 0 \Rightarrow 5D = 9 \Rightarrow D = \frac{9}{5}$$

So equation(iv) becomes $B = 1-\frac{9}{5} = -\frac{4}{5}$

So equation (iii) becomes, A = 0

We put the values of A, B, C, and D values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \frac{x^2}{(x^2+4)(x^2+9)}dx$$

$$\Rightarrow \int \left[\frac{Ax+B}{(x^2+4)} + \frac{Cx+D}{x^2+9}\right]dx$$

$$\Rightarrow \int \left[\frac{(0)x-\frac{4}{5}}{(x^2+4)} + \frac{(0)x+\frac{9}{5}}{x^2+9}\right]dx$$

Split up the integral,

$$\Rightarrow -\frac{4}{5}\int \frac{1}{(x^2+4)}dx + \frac{9}{5}\int \frac{1}{(x^2+9)}dx$$

Let substitute

$u = \frac{x}{2} \Rightarrow du = \frac{1}{2}dx \Rightarrow dx = 2du$ in first partthe

$v = \frac{x}{3} \Rightarrow dv = \frac{1}{3}dx \Rightarrow dx = 3dv$ in second parthe t

so the above equation becomes,

$$\Rightarrow \frac{9}{5}\int \frac{3}{((3v)^2+9)}dv - \frac{4}{5}\int \frac{2}{((2u)^2+4)}du$$

$$\Rightarrow \frac{9}{5}\int \frac{3}{(9v^2+9)}dv - \frac{4}{5}\int \frac{2}{(4u^2+4)}du$$

$$\Rightarrow \frac{3}{5}\int \frac{1}{v^2 + 1}dv - \frac{2}{5}\int \frac{1}{u^2 + 1}du$$

On integrating we get

$$\Rightarrow \frac{3}{5}\tan^{-1} v - \frac{2}{5}\tan^{-1} u + C$$

(the standard integral of $\frac{1}{x^2+1} = \tan^{-1} x$)

Substituting back, we get

$$\Rightarrow \frac{3}{5}\tan^{-1}\left(\frac{x}{3}\right) - \frac{2}{5}\tan^{-1}\left(\frac{x}{2}\right) + C$$

Note: the absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int \frac{x^2}{(x^2 + 4)(x^2 + 9)}dx = \frac{3}{5}\tan^{-1}\left(\frac{x}{3}\right) - \frac{2}{5}\tan^{-1}\left(\frac{x}{2}\right) + C$$

19. Question

Evaluate the following integral:

$$\int \frac{5x^2 - 1}{x(x-1)(x+1)}dx$$

Answer

Denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{5x^2 - 1}{x(x-1)(x + 1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x + 1} \ldots\ldots(i)$$

$$\Rightarrow \frac{5x^2 - 1}{x(x-1)(x + 1)} = \frac{A(x-1)(x + 1) + Bx(x + 1) + Cx(x-1)}{x(x-1)(x + 1)}$$

$\Rightarrow 5x^2 - 1 = A(x - 1)(x + 1) + Bx(x + 1) + Cx(x - 1)$……(ii)

We need to solve for A, B and C. One way to do this is to pick values for x which will cancel each variable.

Put x = 0 in the above equation, we get

$\Rightarrow 5(0)^2 - 1 = A(0 - 1)(0 + 1) + B(0)(0 + 1) + C(0)(0 - 1)$

$\Rightarrow A = 1$

Now put x = 1 in equation (ii), we get

$\Rightarrow 5(1)^2 - 1 = A(1 - 1)(1 + 1) + B(1)(1 + 1) + C(1)(1 - 1)$

$\Rightarrow 4 = 0 + 2B + 0$

$\Rightarrow B = 2$

Now put x = – 1 in equation (ii), we get

$\Rightarrow 5(-1)^2 - 1 = A(-1 - 1)(-1 + 1) + B(-1)(-1 + 1) + C(-1)(-1 - 1)$

$\Rightarrow 4 = 0 + 0 + 2C$

$\Rightarrow C = 2$

We put the values of A, B, and C values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[\frac{5x^2 - 1}{x(x-1)(x+1)}\right] dx$$

$$\Rightarrow \int \left[\frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}\right] dx$$

$$\Rightarrow \int \left[\frac{1}{x} + \frac{2}{x-1} + \frac{2}{x+1}\right] dx$$

Split up the integral,

$$\Rightarrow \int \left[\frac{1}{x}\right] dx + 2\int \left[\frac{1}{x-1}\right] dx + 2\int \left[\frac{1}{x+1}\right] dx$$

Let substitute

u = x – 1 ⇒ du = dx,

y = x + 1 ⇒ dy = dx, so the above equation becomes,

$$\Rightarrow \int \left[\frac{1}{x}\right] dx + 2\int \left[\frac{1}{u}\right] du + 2\int \left[\frac{1}{y}\right] dy$$

On integrating we get

$$\Rightarrow \log|x| + 2\log|u| + 2\log|y| + C$$

Substituting back, we get

$$\Rightarrow \log|x| + 2\log|x-1| + 2\log|x+1| + C$$

Applying logarithm rule, we get

$$\Rightarrow \log|x| + \log|(x-1)^2| + \log|(x+1)^2| + C$$

$$\Rightarrow \log|x(x^2-1)^2| + C$$

Note: the absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int \frac{5x^2 - 1}{x(x-1)(x+1)} dx = \log|x(x^2-1)^2| + C$$

20. Question

Evaluate the following integral:

$$\int \frac{x^2 + 6x - 8}{x^3 - 4x} dx$$

Answer

Denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{x^2 + 6x - 8}{x^3 - 4x}$$

$$= \frac{x^2 + 6x - 8}{x(x^2 - 4)}$$

$$\frac{x^2 + 6x - 8}{x(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2} \ldots\ldots (i)$$

$$\Rightarrow \frac{x^2 + 6x - 8}{x(x-2)(x+2)} = \frac{A(x-2)(x+2) + Bx(x+2) + Cx(x-2)}{x(x-2)(x+2)}$$

$\Rightarrow x^2 + 6x - 8 = A(x - 2)(x + 2) + Bx(x + 2) + Cx(x - 2)$......(ii)

We need to solve for A, B and C. One way to do this is to pick values for x which will cancel each variable.

Put x = 0 in the above equation, we get

$\Rightarrow 0^2 + 6(0) - 8 = A(0 - 2)(0 + 2) + B(0)(0 + 2) + C(0)(0 - 2)$

$\Rightarrow -8 = -4A + 0 + 0$

$\Rightarrow A = 2$

Now put x = 2 in equation (ii), we get

$\Rightarrow 2^2 + 6(2) - 8 = A(2 - 2)(2 + 2) + B(2)(2 + 2) + C(2)(2 - 2)$

$\Rightarrow 8 = 0 + 8B + 0$

$\Rightarrow B = 1$

Now put x = – 2 in equation (ii), we get

$\Rightarrow (-2)^2 + 6(-2) - 8 = A((-2) - 2)((-2) + 2) + B(-2)((-2) + 2) + C(-2)((-2) - 2)$

$\Rightarrow -16 = 0 + 0 + 8C$

$\Rightarrow C = -2$

We put the values of A, B, and C values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[\frac{x^2 + 6x - 8}{x(x-2)(x+2)}\right] dx$$

$$\Rightarrow \int \left[\frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}\right] dx$$

$$\Rightarrow \int \left[\frac{2}{x} + \frac{1}{x-2} + \frac{-2}{x+2}\right] dx$$

Split up the integral,

$$\Rightarrow 2\int \left[\frac{1}{x}\right] dx + \int \left[\frac{1}{x-2}\right] dx - 2\int \left[\frac{1}{x+2}\right] dx$$

Let substitute

$u = x - 2 \Rightarrow du = dx$,

$y = x + 2 \Rightarrow dy = dx$, so the above equation becomes,

$$\Rightarrow 2\int \left[\frac{1}{x}\right] dx + \int \left[\frac{1}{u}\right] du - 2\int \left[\frac{1}{y}\right] dy$$

On integrating we get

$$\Rightarrow 2\log|x| + \log|u| - 2\log|y| + C$$

Substituting back, we get

$$\Rightarrow \log|x| + \log|x-2| - 2\log|x+2| + C$$

Applying logarithm rule, we get

$$\Rightarrow \log|x(x-2)| - \log|(x+2)^2| + C$$

$$\Rightarrow \log\left|\frac{x(x-2)}{(x+2)^2}\right| + C$$

Note: the absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int \frac{x^2 + 6x - 8}{x(x-2)(x+2)} dx = \log\left|\frac{x(x-2)}{(x+2)^2}\right| + C$$

21. Question

Evaluate the following integral:

$$\int \frac{x^2 + 1}{(2x+1)(x^2-1)} dx$$

Answer

Denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{x^2 + 1}{(2x+1)(x^2-1)}$$

$$= \frac{x^2 + 1}{(2x+1)(x-1)(x+1)}$$

$$\frac{x^2 + 1}{(2x+1)(x-1)(x+1)} = \frac{A}{2x+1} + \frac{B}{x-1} + \frac{C}{x+1} \ldots\ldots(i)$$

$$\Rightarrow \frac{x^2 + 1}{(2x+1)(x-1)(x+1)} = \frac{A(x-1)(x+1) + B(2x+1)(x+1) + C(2x+1)(x-1)}{(2x+1)(x-1)(x+1)}$$

$\Rightarrow x^2 + 1 = A(x - 1)(x + 1) + B(2x + 1)(x + 1) + C(2x + 1)(x - 1)$......(ii)

We need to solve for A, B and C. One way to do this is to pick values for x which will cancel each variable.

Put x = 1 in the above equation, we get

$\Rightarrow 1^2 + 1 = A(1 - 1)(1 + 1) + B(2(1) + 1)(1 + 1) + C(2(1) + 1)(1 - 1)$

$\Rightarrow 2 = 0 + 6B + 0$

$$\Rightarrow B = \frac{1}{3}$$

Now put $x = -\frac{1}{2}$ in equation (ii), we get

$$\Rightarrow \left(-\frac{1}{2}\right)^2 + 1 = A\left(\left(-\frac{1}{2}\right) - 1\right)\left(-\frac{1}{2} + 1\right) + B\left(2\left(-\frac{1}{2}\right) + 1\right)\left(-\frac{1}{2} + 1\right) + C\left(2\left(-\frac{1}{2}\right) + 1\right)\left(-\frac{1}{2} - 1\right)$$

$$\Rightarrow \frac{5}{4} = -\frac{3}{4}A + 0 + 0$$

$$\Rightarrow A = -\frac{5}{3}$$

Now put x = – 1 in equation (ii), we get

$\Rightarrow (-1)^2 + 1 = A(-1 - 1)(-1 + 1) + B(2(-1) + 1)(-1 + 1) + C(2(-1) + 1)(-1 - 1)$

$\Rightarrow 2 = 0 + 0 + 2C$

$\Rightarrow C = 1$

We put the values of A, B, and C values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[\frac{x^2 + 1}{(2x + 1)(x - 1)(x + 1)}\right] dx$$

$$\Rightarrow \int \left[\frac{A}{2x + 1} + \frac{B}{x - 1} + \frac{C}{x + 1}\right] dx$$

$$\Rightarrow \int \left[\frac{-\frac{5}{3}}{2x + 1} + \frac{\frac{1}{3}}{x - 1} + \frac{1}{x + 1}\right] dx$$

Split up the integral,

$$\Rightarrow -\frac{5}{3}\int \left[\frac{1}{2x + 1}\right] dx + \frac{1}{3}\int \left[\frac{1}{x - 1}\right] dx + \int \left[\frac{1}{x + 1}\right] dx$$

Let substitute

$u = x - 1 \Rightarrow du = dx,$

$y = x + 1 \Rightarrow dy = dx$ and

$z = 2x + 1 \Rightarrow dz = 2dx$ so the above equation becomes,

$$\Rightarrow -\frac{5}{3}\int \frac{\left[\frac{1}{z}\right] dz}{2} + \frac{1}{3}\int \left[\frac{1}{u}\right] du + \int \left[\frac{1}{y}\right] dy$$

On integrating we get

$$\Rightarrow -\frac{5}{6}\log|z| + \frac{1}{3}\log|u| + \log|y| + C$$

Substituting back, we get

$$\Rightarrow -\frac{5}{6}\log|2x + 1| + \frac{1}{3}\log|x - 1| + \log|x + 1| + C$$

Note: the absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int \frac{x^2 + 1}{(2x + 1)(x^2 - 1)} dx = -\frac{5}{6}\log|2x + 1| + \frac{1}{3}\log|x - 1| + \log|x + 1| + C$$

22. Question

Evaluate the following integral:

$$\int \frac{1}{x\left\{6(\log x)^2 + 7\log x + 2\right\}} dx$$

Answer

Let substitute $u = \log x \Rightarrow du = \frac{1}{x}dx$, so the given equation becomes

$$\int \frac{1}{x\{6(\log x)^2 + 7\log x + 2\}} dx = \int \frac{1}{\{6u^2 + 7u + 2\}} du \ldots.. (i)$$

Factorizing the denominator, we get

$$\int \frac{1}{(2u+1)(3u+2)} du$$

The denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{1}{(2u+1)(3u+2)} = \frac{A}{2u+1} + \frac{B}{(3u+2)} \ldots\ldots(ii)$$

$$\Rightarrow \frac{1}{(2u+1)(3u+2)} = \frac{A(3u+2) + B(2u+1)}{(2u+1)(3u+2)}$$

$\Rightarrow 1 = A(3u + 2) + B(2u + 1)$......(ii)

We need to solve for A and B. One way to do this is to pick values for x which will cancel each variable.

Put $u = -\frac{2}{3}$ in the above equation, we get

$$\Rightarrow 1 = A\left(3\left(-\frac{2}{3}\right) + 2\right) + B\left(2\left(-\frac{2}{3}\right) + 1\right)$$

$$\Rightarrow 1 = -\frac{1}{3}B$$

$$\Rightarrow B = -3$$

Now put $u = -\frac{1}{2}$ in equation (ii), we get

$$\Rightarrow 1 = A\left(3\left(-\frac{1}{2}\right) + 2\right) + B\left(2\left(-\frac{1}{2}\right) + 1\right)$$

$$\Rightarrow 1 = \frac{1}{2}A$$

$$\Rightarrow A = 2$$

We put the values of A and B values back into our partial fractions in equation (ii) and replace this as the integrand. We get

$$\int \left[\frac{1}{(2u+1)(3u+2)}\right] du$$

$$\Rightarrow \int \left[\frac{A}{2u+1} + \frac{B}{(3u+2)}\right] du$$

$$\Rightarrow \int \left[\frac{2}{2u+1} + \frac{-3}{(3u+2)}\right] du$$

Split up the integral,

$$\Rightarrow 2\int \frac{1}{2u+1} du - 3\int \left[\frac{1}{3u+2}\right] du$$

Let substitute

$z = 2u + 1 \Rightarrow dz = 2du$ and $y = 3u + 2 \Rightarrow dy = 3du$ so the above equation becomes,

$$\Rightarrow \int \frac{1}{z} dz - \int \left[\frac{1}{y}\right] dy$$

On integrating we get

$$\Rightarrow \log|z| - \log|y| + C$$

Substituting back the value of z, we get

$\Rightarrow \log|2u + 1| - \log|3u + 2| + C$

Now substitute back the value of u, we get

$\Rightarrow \log|2(\log x) + 1| - \log|3(\log x) + 2| + C$

Applying the rules of logarithm we get

$\Rightarrow \log\left|\frac{2(\log x) + 1}{3(\log x) + 2}\right| + C$

Note: the absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int \frac{1}{x\{6(\log x)^2 + 7\log x + 2\}}dx = \log\left|\frac{2(\log x) + 1}{3(\log x) + 2}\right| + C + C$$

23. Question

Evaluate the following integral:

$$\int \frac{1}{x\left(x^n + 1\right)}dx$$

Answer

$\frac{1}{x(x^n + 1)}$

Multiply numerator and denominator by x^{n-1}, we get

$$\int \frac{1}{x(x^n + 1)}dx \Rightarrow \int \frac{x^{n-1}}{x(x^n + 1)x^{n-1}}dx \Rightarrow \int \frac{x^{n-1}}{x^n(x^n + 1)}dx$$

Let $x^n = t \Rightarrow nx^{n-1}dx = dt$

So the above equation becomes,

$$\int \frac{x^{n-1}}{x^n(x^n + 1)}dx \Rightarrow \frac{1}{n}\int \frac{1}{t(t + 1)}dt$$

The denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{1}{t(t + 1)} = \frac{A}{t} + \frac{B}{t + 1} \ldots\ldots (i)$$

$$\Rightarrow \frac{1}{t(t + 1)} = \frac{A(t + 1) + Bt}{t(t + 1)}$$

$\Rightarrow 1 = A(t + 1) + Bt$……(ii)

Put $t = 0$ in above equations we get

$1 = A(0 + 1) + B(0)$

$\Rightarrow A = 1$

Now put $t = -1$ in equation (ii) we get

$1 = A(-1 + 1) + B(-1)$

$\Rightarrow B = -1$

We put the values of A and B values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \frac{x^{n-1}}{x^n(x^n+1)}dx \Rightarrow \frac{1}{n}\int \frac{1}{t(t+1)}dt$$

$$\Rightarrow \frac{1}{n}\int \left[\frac{A}{t}+\frac{B}{t+1}\right]dt$$

$$\Rightarrow \frac{1}{n}\int \left[\frac{1}{t}+\frac{-1}{t+1}\right]dt$$

Split up the integral,

$$\Rightarrow \frac{1}{n}\left[\int \frac{1}{t}dt - \int \frac{1}{t+1}dt\right]$$

Let substitute

$u = t + 1 \Rightarrow du = dt$, so the above equation becomes,

$$\Rightarrow \frac{1}{n}\left[\int \frac{1}{t}dt - \int \frac{1}{u}du\right]$$

On integrating we get

$$\Rightarrow \frac{1}{n}[\log t - \log u] + C$$

Substituting back the values of u, we get

$$\Rightarrow \frac{1}{n}[\log|t| - \log(|t+1|)] + C$$

Substituting back the values of t, we get

$$\Rightarrow \frac{1}{n}[\log|x^n| - \log|x^n+1|] + C$$

Applying the logarithm rules, we get

$$\Rightarrow \frac{1}{n}\left[\log\left|\frac{x^n}{x^n+1}\right|\right] + C$$

Note: the absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int \frac{1}{x(x^n+1)}dx = \frac{1}{n}\left[\log\left|\frac{x^n}{x^n+1}\right|\right] + C$$

24. Question

Evaluate the following integral:

$$\int \frac{x}{\left(x^2-a^2\right)\left(x^2-b^2\right)}dx$$

Answer

Denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{x}{(x^2-a^2)(x^2-b^2)} = \frac{Ax+B}{(x^2-a^2)} + \frac{Cx+D}{(x^2-b^2)} \ldots \ldots (i)$$

$$\Rightarrow \frac{x}{(x^2-a^2)(x^2-b^2)} = \frac{(Ax+B)(x^2-b^2)+(Cx+D)(x^2-a^2)}{(x^2-a^2)(x^2-b^2)}$$

$$\Rightarrow x = (Ax+B)(x^2-b^2) + (Cx+D)(x^2-a^2)$$

$\Rightarrow\ x = Ax^3 - Ab^2x + Bx^2 - b^2B + Cx^3 - a^2Cx + Dx^2 - a^2D$

$\Rightarrow\ x = (A + C)x^3 + (B + D)x^2 + (-Ab^2 - Ca^2)x + (-b^2B - a^2D) \ldots\ldots (ii)$

By equating similar terms, we get

A + C = 0 ⇒ A = – C ....(iii)

B + D = 0⇒ B = – D............(iv)

$-Ab^2 - Ca^2 = 1$

$\Rightarrow -(-C)b^2 - Ca^2 = 1$ (from equation(iii))

$\Rightarrow\ C = \frac{1}{b^2-a^2}$..............(v)

$-b^2B - a^2D = 0$

$\Rightarrow -b^2(-D) - a^2D = 0$

⇒ D = 0

So equation(iv) becomes B = 0

So equation (iii) becomes, $A = -\frac{1}{b^2-a^2}$

We put the values of A, B, C, and D values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \frac{x}{(x^2-a^2)(x^2-b^2)}dx$$

$$\Rightarrow \int \left[\frac{Ax + B}{(x^2-a^2)} + \frac{Cx + D}{(x^2-b^2)}\right]dx$$

$$\Rightarrow \int \left[\frac{\left(-\frac{1}{b^2-a^2}\right)x + 0}{(x^2-a^2)} + \frac{\left(\frac{1}{b^2-a^2}\right)x + 0}{(x^2-b^2)}\right]dx$$

Split up the integral,

$$\Rightarrow -\frac{1}{b^2-a^2}\int \frac{1}{(x^2-a^2)}dx + \frac{1}{b^2-a^2}\int \frac{1}{(x^2-b^2)}dx$$

Let substitute

$u = x^2 - a^2 \Rightarrow du = 2dx$

$v = x^2 - b^2 \Rightarrow dv = 2dx$, so the above equation becomes,

$$\Rightarrow -\frac{1}{b^2-a^2}\int \frac{\frac{1}{u}du}{2} + \frac{1}{b^2-a^2}\int \frac{\frac{1}{v}dv}{2}$$

$$\Rightarrow -\frac{1}{2(b^2-a^2)}\int \frac{1}{u}du + \frac{1}{2(b^2-a^2)}\int \frac{1}{v}dv$$

On integrating we get

$$\Rightarrow -\frac{1}{2(b^2-a^2)}\log|u| + \frac{1}{2(b^2-a^2)}\log|v| + C$$

Substituting back, we get

$$\Rightarrow \frac{1}{2(b^2-a^2)}[\log|x^2-b^2|-\log|x^2-a^2|] + C$$

Applying the logarithm rule we get

$$\Rightarrow \frac{1}{2(b^2-a^2)}\left[\log\left|\frac{x^2-b^2}{x^2-a^2}\right|\right] + C$$

Note: the absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int \frac{x}{(x^2-a^2)(x^2-b^2)}dx = \frac{1}{2(b^2-a^2)}\left[\log\left|\frac{x^2-b^2}{x^2-a^2}\right|\right] + C$$

25. Question

Evaluate the following integral:

$$\int \frac{x^2+1}{(x^2+4)(x^2+25)}dx$$

Answer

Denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{x^2+1}{(x^2+4)(x^2+25)} = \frac{Ax+B}{(x^2+4)} + \frac{Cx+D}{x^2+25} \ldots\ldots (i)$$

$$\Rightarrow \frac{x^2+1}{(x^2+4)(x^2+25)} = \frac{(Ax+B)(x^2+25)+(Cx+D)(x^2+4)}{(x^2+4)(x^2+25)}$$

$$\Rightarrow x^2+1 = (Ax+B)(x^2+25)+(Cx+D)(x^2+4)$$

$$\Rightarrow x^2+1 = Ax^3+25Ax+Bx^2+25B+Cx^3+4Cx+Dx^2+4D$$

$$\Rightarrow x^2+1 = (A+C)x^3+(B+D)x^2+(25A+4C)x+(25B+4D) \ldots\ldots (ii)$$

By equating similar terms, we get

A + C = 0 ⇒ A = – C ....(iii)

B + D = 1⇒ B = 1 – D............(iv)

25A + 4C = 0

⇒ 25(– C) + 4C = 0 (from equation(iii))

⇒ C = 0...........(v)

$$25B+4D = 1 \Rightarrow 25(1-D)+4D = 1 \Rightarrow 21D = 24 \Rightarrow D = \frac{24}{21} = \frac{8}{7}$$

So equation(iv) becomes $B = 1-\frac{8}{7} = -\frac{1}{7}$

So equation (iii) becomes, A = 0

We put the values of A, B, C, and D values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \frac{x^2+1}{(x^2+4)(x^2+25)}dx$$

$$\Rightarrow \int\left[\frac{Ax + B}{(x^2 + 4)} + \frac{Cx + D}{x^2 + 25}\right]dx$$

$$\Rightarrow \int\left[\frac{(0)x - \frac{1}{7}}{(x^2 + 4)} + \frac{(0)x + \frac{8}{7}}{x^2 + 25}\right]dx$$

Split up the integral,

$$\Rightarrow -\frac{1}{7}\int\frac{1}{(x^2 + 4)}dx + \frac{8}{7}\int\frac{1}{(x^2 + 25)}dx$$

Let substitute

$u = \frac{x}{2} \Rightarrow du = \frac{1}{2}dx \Rightarrow dx = 2du$ in first partthe

$v = \frac{x}{5} \Rightarrow dv = \frac{1}{5}dx \Rightarrow dx = 5dv$ in second parthe t

so the above equation becomes,

$$\Rightarrow \frac{8}{7}\int\frac{5}{((5v)^2 + 25)}dv - \frac{1}{7}\int\frac{2}{((2u)^2 + 4)}du$$

$$\Rightarrow \frac{8}{7}\int\frac{5}{(25v^2 + 25)}dv - \frac{1}{7}\int\frac{2}{(4u^2 + 4)}du$$

$$\Rightarrow \frac{8}{35}\int\frac{1}{v^2 + 1}dv - \frac{1}{14}\int\frac{1}{u^2 + 1}du$$

On integrating we get

$$\Rightarrow \frac{8}{35}\tan^{-1}v - \frac{1}{14}\tan^{-1}u + C$$

(the standard integral of $\frac{1}{x^2+1} = \tan^{-1}x$)

Substituting back, we get

$$\Rightarrow \frac{8}{35}\tan^{-1}\left(\frac{x}{5}\right) - \frac{1}{14}\tan^{-1}\left(\frac{x}{2}\right) + C$$

Note: the absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int\frac{x^2 + 1}{(x^2 + 4)(x^2 + 25)}dx = \frac{8}{35}\tan^{-1}\left(\frac{x}{5}\right) - \frac{1}{14}\tan^{-1}\left(\frac{x}{2}\right) + C$$

26. Question

Evaluate the following integral:

$$\int\frac{x^3 + x + 1}{x^2 - 1}$$

Answer

Let

$$I = \int\frac{x^3 + x + 1}{x^2 - 1}dx = \int\left(x + \frac{2x + 1}{x^2 - 1}\right)dx$$

Now,

Let $\frac{2x+1}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1}$

2x + 1 = A(x – 1) + B(x + 1)

Put x = 1

2 + 1 = A × 0 + B × 2

3 = 2B

$B = \frac{3}{2}$

Put x = – 1

– 2 + 1 = – 2A + B × 0

– 1 = – 2A

$A = \frac{1}{2}$

$I = \int xdx + \frac{1}{2}\int \frac{dx}{x + 1} + \frac{3}{2}\int \frac{dx}{x-1}$

$\int \frac{dx}{x} = \log|x| \text{ and } \int xdx = \frac{x^2}{2}$

Therefore,

$I = \frac{x^2}{2} + \frac{1}{2}\log|x + 1| + \frac{3}{2}\log|x-1| + c$

27. Question

Evaluate the following integral:

$\int \frac{3x-2}{(x+1)^2(x+3)}$

Answer

$I = \int \frac{3x-2}{(x + 1)^2(x + 3)} dx$

$\frac{3x-2}{(x + 1)^2(x + 3)} = \frac{A}{x + 1} + \frac{B}{(x + 1)^2} + \frac{c}{x + 3}$

3x – 2 = A(x + 1)(x + 3) + B(x + 3) + C(x + 1)^2

Put x = – 1

– 3 – 2 = A × 0 + B × (– 1 + 3) + C × 0

– 5 = 2B

$B = -\frac{5}{2}$

Put x = – 3

– 9 – 2 = C × (– 2)(– 2)

– 11 = 4C

$C = -\frac{11}{4}$

Equating coefficients of constants

– 2 = 3A + 3B + C

$$-2 = 3A + 3 \times \frac{-5}{2} - \frac{11}{4}$$

$$A = \frac{11}{4}$$

Thus,

$$I = \frac{11}{4}\int \frac{dx}{x + 1} - \frac{5}{2}\int \frac{dx}{(x + 1)^2} - \frac{11}{4}\int \frac{dx}{x + 3}$$

$$I = \frac{11}{4}\log|x + 1| - \frac{5}{2(x + 1)} - \frac{11}{4}\log|x + 3| + C$$

28. Question

Evaluate the following integral:

$$\int \frac{2x+1}{(x+2)(x-3)^2}$$

Answer

$$I = \int \frac{2x + 1}{(x + 2)(x-3)^2}\,dx$$

$$\frac{2x + 1}{(x + 2)(x-3)^2} = \frac{A}{x + 2} + \frac{B}{x-3} + \frac{c}{(x-3)^2}$$

$2x + 1 = A(x - 3)^2 + B(x + 2)(x - 3) + C(x + 2)$

$2x + 1 = Ax^2 - 3Ax + 9A + Bx^2 - 5Bx - 6B + Cx + 2C$

Put x = 3

7 = 5C

$$C = \frac{7}{5}$$

Put x = – 2

– 3 = 0A

– 11 = 4C

$$C = -\frac{11}{4}$$

Equating coefficients of constants

– 2 = 3A + 3B + C

$$-2 = 3A + 3 \times \frac{-5}{2} - \frac{11}{4}$$

$$A = \frac{11}{4}$$

Thus,

$$I = \frac{11}{4}\int \frac{dx}{x + 1} - \frac{5}{2}\int \frac{dx}{(x + 1)^2} - \frac{11}{4}\int \frac{dx}{x + 3}$$

$$I = \frac{11}{4}\log|x + 1| - \frac{5}{2(x + 1)} - \frac{11}{4}\log|x + 3| + C$$

29. Question

Evaluate the following integral:

$$\int \frac{x^2+1}{(x-2)^2(x+3)} dx$$

Answer

$$I = \int \frac{x^2 + 2}{(x-2)^2(x + 3)} dx$$

$$\frac{x^2 + 2}{(x-2)^2(x + 3)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{c}{x + 3}$$

$X^2 + 1 = A(x - 2)(x + 3) + B(x + 3) + C(x - 2)^2$

Put x = 2

4 + 1 = B × 5

5 = 5B

$$B = \frac{5}{5} = 1$$

Put x = - 3

10 = C × 25

$$C = \frac{10}{25} = \frac{2}{5}$$

Equating coefficients of constants

1 = - 6A + 3B + 4C

$$1 = -6A + 3 + \frac{8}{5}$$

$$A = \frac{3}{5}$$

Thus,

$$I = \frac{3}{5}\int \frac{dx}{x-2} - \int \frac{dx}{(x-2)^2} - \frac{2}{5}\int \frac{dx}{x + 3}$$

$$I = \frac{3}{5}\log|x-2| - \frac{1}{(x-2)} + \frac{2}{5}\log|x + 3| + C$$

30. Question

Evaluate the following integral:

$$\int \frac{x}{(x-1)^2(x+2)} dx$$

Answer

$$I = \int \frac{x}{(x-1)^2(x + 2)} dx$$

$$\frac{x}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{c}{x+2}$$

x = A(x – 1)(x + 2) + B(x + 2) + C(x – 1)^2

Put x = – 2

– 2 = 9C

$$C = -\frac{2}{9}$$

Put x = 1

1 = 3B

$$B = \frac{1}{3}$$

Equating coefficients of constants

0 = – 2A + 2B + C

$$0 = -2A + 2 * \frac{1}{3} - \frac{2}{9}$$

$$A = \frac{2}{9}$$

Thus,

$$I = \frac{2}{9}\int \frac{dx}{x-1} + \frac{1}{3}\int \frac{dx}{(x-1)^2} - \frac{2}{9}\int \frac{dx}{x+2}$$

$$I = \frac{2}{9}\log|x-1| + \frac{1}{3}\left(\frac{-1}{(x-1)}\right) - \frac{2}{9}\log|x+2| + C$$

$$= \frac{2}{9}\log\left|\frac{x-1}{x+2}\right| - \frac{1}{3(x-1)} + C$$

31. Question

Evaluate the following integral:

$$\int \frac{x^2}{(x-1)(x+1)^2} dx$$

Answer

$$I = \int \frac{x^2}{(x-1)(x+1)^2} dx$$

$$\frac{x^2}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

Put x = 1

1 = 4A

$$A = \frac{1}{4}$$

Put x = – 1

1 = – 2C

$C = -\frac{1}{2}$

Equating coefficients of x^2

1 = A + B

$1 = \frac{1}{4} + B$

$B = \frac{3}{4}$

Thus,

$I = \frac{1}{4}\int \frac{dx}{x-1} + \frac{3}{4}\int \frac{dx}{x+1} - \frac{1}{2}\int \frac{dx}{(x+1)^2}$

$I = \frac{1}{4}\log|x-1| + \frac{3}{4}\log|x+1| + \frac{1}{2(x+1)} + C$

32. Question

Evaluate the following integral:

$\int \frac{x^2+x-1}{(x+1)^2(x+2)} dx$

Answer

$I = \int \frac{x^2+x-1}{(x+1)^2(x+2)} dx$

$\frac{x^2+x-1}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{c}{x+2}$

$X^2 + x - 1 = A(x+1)(x+2) + B(x+2) + C(x+1)^2$

Put x = - 2

1 = C

$C = 1$

Put x = - 1

- 1 = B

$B = -1$

Equating coefficients of constants

- 1 = 2A + 2B + C

$-1 = 2A - 2 + 1$

$A = 0$

Thus,

$I = 0 \times \int \frac{dx}{x+1} + (-1)\int \frac{dx}{(x+1)^2} + \int \frac{dx}{x+2}$

$I = -\left(\frac{-1}{(x+1)}\right) + \log|x+2| + C$

$= \left(\frac{1}{(x + 1)}\right) + \log|x + 2| + C$

33. Question

Evaluate the following integral:

$$\int \frac{2x^2+7x-3}{x^2(2x+1)}dx$$

Answer

$$I = \int \frac{2x^2 + 7x - 3}{x^2(2x + 1)}dx$$

$$\frac{2x^2 + 7x - 3}{x^2(2x + 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{2x + 1}$$

$$2x^2 + 7x - 3 = Ax(2x + 1) + B(2x + 1) + Cx^2$$

Equating constants

- 3 = B

Equating coefficients of x

7 = A + 2B

7 = A - 6

A = 13

Equating coefficients of x^2

2 = 2A + C

2 = 26 + C

C = - 24

Thus,

$$I = \int \frac{13dx}{x} - \int \frac{3dx}{x^2} - 24\int \frac{dx}{2x + 1}$$

$$I = 13\log|x| + \frac{3}{x} - 12\log|2x + 1| + C$$

34. Question

Evaluate the following integral:

$$\int \frac{5x^2+20x+6}{x^3+2x^2+x}dx$$

Answer

$$I = \int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} = \int \frac{5x^2 + 20x + 6}{x(x + 1)^2}$$

$$\frac{5x^2 + 20x + 6}{x(x + 1)^2} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2}$$

$$5x^2 + 20x + 6 = A(x + 1)^2 + Bx(x + 1) + Cx$$

Equating constants

6 = A

Equating coefficients of x^2

5 = A + B

B = – 1

Equating coefficients of x

20 = 2A + B + C

20 = 12 – 1 + C

C = 9

$$I = \int \frac{6dx}{x} - \int \frac{dx}{x+1} + 9\int \frac{dx}{(x+1)^2}$$

$$I = 6\log|x| - \log|x+1| - \frac{9}{x+1} + C$$

35. Question

Evaluate the following integral:

$$\int \frac{18}{(x+2)(x^2+4)} dx$$

Answer

$$I = \int \frac{18}{(x+2)(x^2+4)}$$

$$\frac{18}{(x+2)(x^2+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4}$$

$18 = A(x^2 + 4) + (Bx + C)(x + 2)$

Equating constants

18 = 4A + 2C

Equating coefficients of x

0 = 2B + C

Equating coefficients of x^2

0 = A + B

Solving, we get

$$A = \frac{9}{4}, \quad B = -\frac{9}{4}, \quad C = \frac{9}{2}$$

Thus,

$$I = \frac{9}{4}\int \frac{dx}{x+2} + \left(-\frac{9}{4}\right)\int \frac{xdx}{x^2+4} + \frac{9}{2}\int \frac{dx}{x^2+4}$$

$$I = \frac{9}{4}\log|x+2| - \frac{9}{8}\log|x^2+4| + \frac{9}{4}\tan^{-1}\left(\frac{x}{2}\right) + C$$

36. Question

Evaluate the following integral:

$$\int \frac{5}{(x^2+1)(x+2)} dx$$

Answer

$$I = \int \frac{5}{(x^2 + 1)(x + 2)}$$

$$\frac{5}{(x^2 + 1)(x + 2)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x + 2}$$

$5 = (Ax + B)(x + 2) + C(x^2 + 1)$

Equating constants

$5 = 2B + C$

Equating coefficients of x

$0 = 2A + B$

Equating coefficients of x^2

$0 = A + C$

Solving, we get

$A = -1, B = 2, C = 1$

Thus

$$I = \int \frac{-x + 2}{x^2 + 1} dx + \int \frac{dx}{x + 2}$$

$$= \int \frac{-xdx}{x^2 + 1} + 2\int \frac{dx}{x^2 + 1} + \int \frac{dx}{x + 2}$$

$$I = -\frac{1}{2}\log|x^2 + 1| + 2\tan^{-1}x + \log|x + 2| + C$$

37. Question

Evaluate the following integral:

$$\int \frac{x}{(x+1)(x^2+1)} dx$$

Answer

$$I = \int \frac{x}{(x + 1)(x^2 + 1)}$$

$$\frac{x}{(x + 1)(x^2 + 1)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1}$$

$x = A(x^2 + 1) + (Bx + C)(x + 1)$

Equating constants

$0 = A + C$

Equating coefficients of x

$1 = B + C$

Equating coefficients of x^2

0 = A + B

Solving, we get

$A = -\frac{1}{2}\ B = \frac{1}{2}\ C = \frac{1}{2}$

Thus

$$I = -\frac{1}{2}\int \frac{dx}{x + 1} + \frac{1}{2}\int \frac{xdx}{x^2 + 1} + \frac{1}{2}\int \frac{dx}{x^2 + 1}$$

$$I = -\frac{1}{2}\log|x + 1| + \frac{1}{4}\log|x^2 + 1| + \frac{1}{2}\tan^{-1}x + C$$

38. Question

Evaluate the following integral:

$$\int \frac{1}{1+x+x^2+x^3}dx$$

Answer

$$I = \int \frac{1}{1 + x + x^2 + x^3} = \int \frac{dx}{(x^2 + 1)(x + 1)}$$

$$\frac{1}{(x^2 + 1)(x + 1)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x + 1}$$

$1 = (Ax + B)(x + 1) + C(x^2 + 1)$

Equating constants

1 = B + C

Equating coefficients of x

0 = A + B

Equating coefficients of x^2

0 = A + C

Solving, we get

$A = -\frac{1}{2}\ B = \frac{1}{2}\ C = \frac{1}{2}$

Thus

$$I = -\frac{1}{2}\int \frac{xdx}{x^2 + 1} + \frac{1}{2}\int \frac{dx}{x^2 + 1} + \frac{1}{2}\int \frac{dx}{x + 1}$$

$$I = -\frac{1}{4}\log|x^2 + 1| + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\log|x + 1| + C$$

39. Question

Evaluate the following integral:

$$\int \frac{1}{(x+1)^2(x^2+1)}dx$$

Answer

$I = \frac{1}{(x+1)^2(x^2+1)}$

$\frac{1}{(x+1)^2(x^2+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1}$

$1 = A(x+1)(x^2+1) + B(x^2+1) + (Cx+D)(x+1)^2$

$= Ax^3 + Ax^2 + Ax + A + Bx^2 + B + Cx^3 + 2Cx^2 + Cx + Dx^2 + 2D + D$

$= (A+C)x^3 + (A+B+2C+D)x^2 + (A+C+2D)x + (A+B+D)$

Equating constants

$1 = A + B + D$

Equating coefficients of x^3

$0 = A + C$

Equating coefficients of x^2

$0 = A + B + 2C + D$

Equating coefficients of x

$0 = A + C + 2D$

Solving we get

$A = \frac{1}{2}\ B = \frac{1}{2}\ C = -\frac{1}{2}\ D = 0$

Thus,

$I = \frac{1}{2}\int \frac{dx}{x+1} + \frac{1}{2}\int \frac{dx}{(x+1)^2} - \frac{1}{2}\int \frac{dx}{x^2+1}$

$I = \frac{1}{2}\log|x+1| - \frac{1}{2(x+1)} - \frac{1}{4}\log|x^2+1| + C$

40. Question

Evaluate the following integral:

$\int \frac{2x}{x^3-1}dx$

Answer

$I = \int \frac{2x}{x^3-1}dx = \int \frac{2x}{(x-1)(x^2+x+1)}dx$

$\frac{2x}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$

$2x = A(x^2+x+1) + (Bx+C)(x-1)$

$= (A+B)x^2 + (A-B+C)x + (A-C)$

Equating constants,

$A - C = 0$

Equating coefficients of x

$2 = A - B + C$

Equating coefficients of x^2

0 = A + B

On solving,

We get

$A = \frac{2}{3}\ B = -\frac{2}{3}\ C = \frac{2}{3}$

$I = \frac{2}{3}\int \frac{dx}{x-1} - \frac{2}{3}\int \frac{(x-1)dx}{x^2 + x + 1}$

$= \frac{2}{3}\int \frac{dx}{x-1} - \frac{2}{3}.\frac{1}{2}\int \frac{(2x-2)dx}{x^2 + x + 1}$

$= \frac{2}{3}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{(2x + 1)dx}{x^2 + x + 1} + \int \frac{dx}{x^2 + x + 1}$

$= \frac{2}{3}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{(2x + 1)dx}{x^2 + x + 1} + \int \frac{dx}{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$= \frac{2}{3}\log|x-1| - \frac{1}{3}\log|x^2 + x + 1| + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x + 1}{\sqrt{3}}\right) + C$

41. Question

Evaluate the following integral:

$\int \frac{1}{(x^2+1)(x^2+4)}dx$

Answer

$I = \int \frac{1}{(x^2 + 1)(x^2 + 4)}dx$

$\frac{1}{(x^2 + 1)(x^2 + 4)} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{x^2 + 4}$

$1 = (Ax + B)(x^2 + 4) + (Cx + D)(x^2 + 1)$

$= (A + C)\, x^3 + (B + D)x^2 + (4A + C)x + 4B + D$

Equating similar terms

A + C = 0

B + D = 0

4A + C = 0

4B + D = 1

We get, $A = 0\ B = \frac{1}{3}\ C = 0\ D = -\frac{1}{3}$

Thus,

$I = \int \frac{\frac{1}{3}dx}{x^2 + 1} - \int \frac{\frac{1}{3}dx}{x^2 + 4}$

$= \frac{1}{3}\tan^{-1}x - \frac{1}{6}\tan^{-1}\frac{x}{2} + C$

42. Question

Evaluate the following integral:

$$\int \frac{x^2}{\left(x^2+1\right)\left(3x^2+4\right)}dx$$

Answer

$$I = \int \frac{x^2}{(x^2 + 1)(3x^2 + 4)}dx$$

$$\frac{x^2}{(x^2 + 1)(3x^2 + 4)} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{3x^2 + 4}$$

$x^2 = (Ax + B)(3x^2 + 4) + (Cx + D)(x^2 + 1)$

$= (3A + C)\, x^3 + (3B + D)x^2 + (4A + C)x + 4B + D$

Equating similar terms

$3A + C = 0$

$3B + D = 1$

$4A + C = 0$

$4B + D = 0$

Solving we get,

$A = 0,\ B = -1,\ C = 0, D = 4$

Thus,

$$I = \int \frac{-dx}{x^2 + 1} - \int \frac{4dx}{3x^2 + 4}$$

$$I = -\tan^{-1}x + \frac{4}{3}\int \frac{dx}{x^2 + \left(\frac{2}{\sqrt{3}}\right)^2}$$

$$I = -\tan^{-1}x + \frac{4}{3}.\frac{\sqrt{3}}{2}\tan^{-1}\frac{\sqrt{3}x}{2} + C$$

$$I = \frac{2}{\sqrt{3}}\tan^{-1}\frac{\sqrt{3}x}{2} - \tan^{-1}x + C$$

43. Question

Evaluate the following integral:

$$\int \frac{3x+5}{x^3-x^2-x+1}dx$$

Answer

$$I = \int \frac{3x + 5}{x^3 - x^2 - x + 1}dx = \int \frac{3x + 5}{(x-1)^2(x + 1)}$$

$$\frac{3x + 5}{(x-1)^2(x + 1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x + 1}$$

$3x + 5 = A(x - 1)(x + 1) + B(x + 1) + C(x - 1)^2$

Put x = 1

8 = 2B

B = 4

Put x = - 1

- 3 + 5 = 4C

2 = 4C

$C = \frac{1}{2}$

Put x = 0

5 = - A + B + C

$A = \frac{1}{2}$

$$\int \frac{3x + 5}{(x-1)^2(x + 1)} dx = \frac{1}{2}\int \frac{dx}{x-1} + 4\int \frac{dx}{(x-1)^2} + \frac{1}{2}\int \frac{dx}{x + 1}$$

$$= -\frac{1}{2}\ln|x-1| - \frac{4}{(x-1)} + \frac{1}{2}\ln|x + 1| + C$$

$$= \frac{1}{2}\ln\left|\frac{x + 1}{x-1}\right| - \frac{4}{(x-1)} + C$$

44. Question

Evaluate the following integral:

$$\int \frac{x^3 - 1}{x^3 + x} dx$$

Answer

$$I = \int \frac{x^3 - 1}{x^3 + x} dx = \int 1 - \frac{x + 1}{x^3 + x} dx$$

$$= \int 1dx - \int \frac{x + 1}{x^3 + x} dx$$

$$\frac{x + 1}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$$

X + 1 = A(x^2 + 1) + (Bx + C)(x)

Equating constants

A = 1

Equating coefficients of x

1 = C

Equating coefficients of x^2

0 = A + B

B = - 1

$$I = -\int \frac{dx}{x} - \int \frac{-x + 1dx}{x^2 + 1} + \int dx$$

$$I = -\int \frac{dx}{x} + \int \frac{xdx}{x^2 + 1} - \int \frac{dx}{x^2 + 1} + \int dx$$

$$= -\log|x| + \frac{1}{2}\log|x^2 + 1| - \tan^{-1}x + x + c$$

$$I = x - \log|x| + \frac{1}{2}\log|x^2 + 1| - \tan^{-1}x + c$$

45. Question

Evaluate the following integral:

$$\int \frac{x^2 + x + 1}{(x+1)^2 (x+2)} dx$$

Answer

$$I = \int \frac{x^2 + x + 1}{(x + 1)^2(x + 2)} dx$$

$$\frac{x^2 + x + 1}{(x + 1)^2(x + 2)} = \frac{A}{x + 1} + \frac{B}{(x + 1)^2} + \frac{c}{x + 2}$$

$X^2 + x + 1 = A(x + 1)(x + 2) + B(x + 2) + C(x + 1)^2$

Put x = – 2

3 = C

$C = 3$

Put x = – 1

1 = B

$B = 1$

Equating coefficients of constants

1 = 2A + 2B + C

$1 = 2A + 2 + 3$

$A = -2$

Thus,

$$I = 2 * \int \frac{dx}{x + 1} + (1)\int \frac{dx}{(x + 1)^2} + 3\int \frac{dx}{x + 2}$$

$$I = -2 \ln|x + 1| - \left(\frac{1}{(x + 1)}\right) + 3\ln|x + 2| + C$$

46. Question

Evaluate the following integral:

$$\int \frac{1}{x(x^4 + 1)} dx$$

Answer

Let

$$I = \int \frac{1}{x(x^4 + 1)} dx$$

$$\frac{1}{x(x^4 + 1)} = \frac{A}{x} + \frac{Bx^3 + Cx^2 + Dx + E}{x^4 + 1}$$

$1 = A(x^4 + 1) + (Bx^3 + Cx^2 + Dx + E)(x)$

Equating constants

$A = 1$

Equating coefficients of x^4

$0 = A + B$

$0 = 1 + B$

$B = -1$

Equating coefficients of x^2

$D = 0$

Equating coefficients of x

$E = 0$

Thus,

$$I = \int \frac{dx}{x} + \int -\frac{x^2 dx}{x^4 + 1}$$

$$= \log|x| - \frac{1}{4}\log|x^4 + 1| + C$$

$$= \frac{4}{4}\log|x| - \frac{1}{4}\log|x^4 + 1| + C$$

$$= \frac{1}{4}\log|x^4| - \frac{1}{4}\log|x^4 + 1| + C$$

$$\frac{1}{4}\log\left|\frac{x^4}{x^4 + 1}\right| + C$$

47. Question

Evaluate the following integral:

$$\int \frac{1}{x\left(x^3 + 8\right)} dx$$

Answer

Consider the integral,

$$I = \int \frac{1}{x(x^3 + 8)} dx$$

Rewriting the above integral, we have

$$I = \int \frac{x^2}{x^3(x^3 + 8)} dx$$

$$I = \frac{1}{3}\int \frac{3x^2}{x^3(x^3 + 8)} dx$$

Substitute $x^3 = t$

$3x^2dx = dt$

$$I = \frac{1}{3}\int \frac{dt}{t(t + 8)}$$

$$\frac{1}{t(t + 8)} = \frac{A}{t} + \frac{B}{t + 8}$$

$1 = A(t + 8) + Bt$

Equating constants

$1 = 8A$

$$A = \frac{1}{8}$$

Equating coefficients of t

$0 = A + B$

$$B = -\frac{1}{8}$$

$$I = \frac{1}{3}\int \frac{dt}{t(t + 8)}$$

$$= \frac{1}{3}\int \left(\frac{\frac{1}{8}}{t} - \frac{\frac{1}{8}}{t + 8}\right)dt$$

$$= \frac{1}{3} \times \frac{1}{8}\int \frac{dt}{t} - \frac{1}{3} \times \frac{1}{8}\int \frac{dt}{t + 8}$$

$$= \frac{1}{24}\log t - \frac{1}{24}\log|t + 8| + C$$

$$= \frac{1}{24}\log x^3 - \frac{1}{24}\log|x^3 + 8| + C$$

$$= \frac{3}{24}\log x - \frac{1}{24}\log|x^3 + 8| + C$$

$$= \frac{1}{8}\log x - \frac{1}{24}\log|x^3 + 8| + C$$

48. Question

Evaluate the following integral:

$$\int \frac{3}{(1-x)(1+x^2)}dx$$

Answer

$$I = \int \frac{3}{(1 - x)(1 + x^2)}dx$$

$$\frac{3}{(1 - x)(1 + x^2)} = \frac{A}{1 - x} + \frac{Bx + C}{1 + x^2}$$

$3 = A(1 + x^2) + (Bx + C)(1 - x)$

Equating similar terms

A - B = 0

B - C = 0

A + C = 3

Solving

$A = \frac{3}{2}, B = \frac{3}{2}, C = \frac{3}{2}$

Thus,

$I = \frac{3}{2}\int \frac{dx}{1-x} + \frac{3}{2}\int \frac{xdx}{1+x^2} + \frac{3}{2}\int \frac{dx}{1+x^2}$

$= -\frac{3}{2}\log|1-x| + \frac{3}{2}\log|1+x^2| + \frac{3}{2}\tan^{-1}x + C$

$I = \frac{3}{4}\left[\log\left|\frac{1+x^2}{(1-x)^2}\right| + 2\tan^{-1}x\right] + C$

49. Question

Evaluate the following integral:

$\int \frac{\cos x}{(1-\sin x)^3(2+\sin x)}dx$

Answer

Let

Sin x = t

Cos x dx = dt

$I = \int \frac{\cos x}{(1-\sin x)^3(2+\sin x)}dx$

$= \int \frac{dt}{(1-t)^3(2+t)}$

$\frac{1}{(1-t)^3(2+t)} = \frac{A}{1-t} + \frac{B}{(1-t)^2} + \frac{C}{(1-t)^3} + \frac{D}{2+t}$

$1 = A(1-t)^2(2+t) + B(1-t)(2+t) + C(2+t) + D(1-t)^3$

Put t = 1

1 = 3C

$C = \frac{1}{3}$

Put t = - 2

1 = 27D

$D = \frac{1}{27}$

$A = -\frac{1}{27}\ B = \frac{1}{9}$

$$\int \frac{dt}{(1-t)^3(2+t)}$$
$$= -\frac{1}{27}\int \frac{1}{1-t}dt + \frac{1}{9}\int \frac{dt}{(1-t)^2} + \frac{1}{3}\int \frac{dt}{(1-t)^3} + \frac{1}{27}\int \frac{dt}{2+t}$$

$$= -\frac{1}{27}\log|1-t| + \frac{1}{9(1-t)} + \frac{1}{6(1-t)^2} + \frac{1}{27}\log|2+t| + C$$

Put t = sin x

$$= -\frac{1}{27}\log|1-\sin x| + \frac{1}{9(1-\sin x)} + \frac{1}{6(1-\sin x)^2} + \frac{1}{27}\log|2+\sin x| + C$$

50. Question

Evaluate the following integral:

$$\int \frac{2x^2+1}{x^2(x^2+4)}dx$$

Answer

$$I = \int \frac{2x^2+1}{x^2(x^2+4)}dx$$

Put $x^2 = t$

2xdx = dt

$$\frac{2t+1}{t(t+4)} = \frac{A}{t} + \frac{B}{t+4}$$

2t + 1 = A(t + 4) + Bt

Equating constants

1 = 4A

$$A = \frac{1}{4}$$

Equating coefficients of t

2 = A + B

$$B = 2 - \frac{1}{4} = \frac{7}{4}$$

$$\frac{2x^2+1}{x^2(x^2+4)} = \frac{1}{4x^2} + \frac{7}{4(x^2+4)}$$

Thus we have

$$\int \frac{2x^2+1}{x^2(x^2+4)}dx = \frac{1}{4}\int \frac{dx}{x^2} + \frac{7}{4}\int \frac{dx}{x^2+4}$$

$$= -\frac{1}{4x} + \frac{7}{8}\tan^{-1}\left(\frac{x}{2}\right) + C$$

51. Question

Evaluate the following integral:

$$\int \frac{\cos x}{(1-\sin x)(2-\sin x)} dx$$

Answer

We have,

$$I = \int \frac{\cos x}{(1-\sin x)(2-\sin x)} dx$$

Let 1 – sin x = t

⇒ – cos x dx = dt

$$\therefore I = -\int \frac{dt}{t(1+t)}$$

$$\Rightarrow I = -\int \frac{(1+t)-t}{t(1+t)} dt$$

$$\Rightarrow I = -\int \left(\frac{1}{t} - \frac{1}{1+t}\right) dt$$

⇒ I= – (ln t – ln(1 + t)) + c

⇒ I= ln (1 + t) – ln t + c

$$\Rightarrow I = \frac{\ln(1+t)}{\ln t} + c$$

$$\Rightarrow I = \frac{\ln(2-\sin x)}{\ln(1-\sin x)} + c$$

$$\text{Therefore, } \int \frac{\cos x}{(1-\sin x)(2-\sin x)} dx = \frac{\ln(2-\sin x)}{\ln(1-\sin x)} + c$$

52. Question

Evaluate the following integral:

$$\int \frac{2x+1}{(x-2)(x-3)} dx$$

Answer

$$\text{Let, } I = \int \frac{2x+1}{(x-2)(x-3)} dx$$

$$\text{Now, let } \frac{2x+1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$

⇒ 2x + 1=A(x – 3) + B(x – 2)

⇒ 2x + 1=(A + B)x – 3A – 2B

Equating similar terms, we get,

A + B=2 and 3A + 2B= – 1

So, A= – 5, B=7

$$\therefore I = -5\int \frac{dx}{x-2} + 7\int \frac{dx}{x-3}$$

⇒ I = – 5 log |x – 2| + 7 log |x – 3| + c

$\Rightarrow I = \log|x - 2|^{-5} + \log|x - 3|^7 + c$

$$\Rightarrow I = \log\left|\frac{(x-3)^7}{(x-2)^5}\right| + c$$

Hence, $\int \frac{2x + 1}{(x-2)(x-3)} dx = \log\left|\frac{(x-3)^7}{(x-2)^5}\right| + c$

53. Question

Evaluate the following integral:

$$\int \frac{1}{(x^2+1)(x^2+2)} dx$$

Answer

$$\text{Let, } I = \int \frac{1}{(x^2 + 1)(x^2 + 2)} dx$$

Let, $x^2=y$

$$\text{Then, } \frac{1}{(y + 1)(y + 2)} = \frac{A}{y + 1} + \frac{B}{y + 2}$$

$\Rightarrow 1=A(y + 2) + B(y + 1)$

$\Rightarrow 1=(A + B)y + 2A + B$

On equating similar terms, we get,

$A + B=0$, and $2A + B=1$

We get, $A=1$, $B= - 1$

$$\therefore I = \int \frac{dx}{x^2 + 1} - \int \frac{dx}{x^2 + 2}$$

$$\Rightarrow I = \tan^{-1} x - \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + c$$

$$\text{So, } \int \frac{1}{(x^2 + 1)(x^2 + 2)} dx = \tan^{-1} x - \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + c$$

54. Question

Evaluate the following integral:

$$\int \frac{1}{x(x^4-1)} dx$$

Answer

$$\text{Let, } I = \int \frac{1}{x(x^4 - 1)} dx$$

$$\text{Let, } \frac{1}{x(x^4 - 1)} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{x - 1} + \frac{D}{x^2 + 1}$$

$\Rightarrow 1= A(x + 1)(x - 1)(x^2 + 1) + Bx(x - 1)(x^2 + 1) + cx(x + 1)(x^2 + 1) + Dx(x + 1)(x - 1)$

For, $x=0$, $A= - 1$

$$\text{For, } x = 1, C = \frac{1}{4}$$

$\text{For}, x = -1, B = \frac{1}{4}$

$\text{For}, x = 2, D = \frac{1}{4}$

$\therefore I = -\int \frac{dx}{x} + \frac{1}{4}\int \frac{dx}{x+1} + \frac{1}{4}\int \frac{dx}{x-1} + \frac{1}{4}\int \frac{dx}{x^2+1}$

$\Rightarrow I = -\ln|x| + \frac{1}{4}\ln|(x+1)| + \frac{1}{4}\ln|x-1| + \frac{1}{4}\tan^{-1}x + c$

$\Rightarrow I = -\ln|x| + \frac{1}{4}(\ln|x^2-1|) + \frac{1}{4}\tan^{-1}x + c$

$\Rightarrow I = -\frac{1}{4}\ln|x^4| + \frac{1}{4}\ln(x^2-1) + \frac{1}{4}\tan^{-1}x + c$

$\Rightarrow I = \frac{1}{4}\ln\left|\frac{x^2-1}{x^4}\right| + + \frac{1}{4}\tan^{-1}x + c$

$\text{Thus}, \int \frac{1}{x(x^4-1)}dx = \frac{1}{4}\ln\left|\frac{x^4-1}{x^4}\right| + c$

55. Question

Evaluate the following integral:

$$\int \frac{1}{x^4-1}dx$$

Answer

$\text{Let}, I = \int \frac{1}{(x^4-1)}dx$

$\text{Let}, \frac{1}{(x^4-1)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x^2+1}$

$\Rightarrow 1 = A(x-1)(x^2+1) + B(x+1)(x^2+1) + c(x+1)(x-1)$

$\text{For}, x = 1, B = \frac{1}{4}$

$\text{For}, x = -1, A = \frac{1}{4}$

$\text{For}, x = 0, A = -\frac{1}{2}$

$\therefore I = -\frac{1}{4}\int \frac{dx}{x+1} + \frac{1}{4}\int \frac{dx}{x-1} - \frac{1}{2}\int \frac{dx}{x^2+1}$

$\Rightarrow I = -\frac{1}{4}\ln|(x+1)| + \frac{1}{4}\ln|x-1| - \frac{1}{2}\tan^{-1}x + c$

$\Rightarrow I = \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\tan^{-1}x + c$

$\text{So}, \int \frac{1}{(x^4-1)}dx = \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\tan^{-1}x + c$

56. Question

Evaluate the following integral:

$$\int \frac{2x}{(x^2+1)(x^2+2)^2} dx$$

Answer

$$\text{Let, } I = \int \frac{2x}{(x^2 + 1)(x^2 + 2)^2} dx$$

Let $x^2 + 2 = t \Rightarrow 2x\, dx = dt$

$$\therefore I = \int \frac{dt}{(t-1)t^2}$$

$$\text{Now, let, } \frac{1}{(t-1)t^2} = \frac{A}{t-1} + \frac{B}{t} + \frac{C}{t^2}$$

$\Rightarrow 1 = At^2 + B\, t\,(t - 1) + C(t - 1)$

For t=1, A=1

For t=0, C= - 1

For t= - 1, B= - 1

$$\therefore I = \int \frac{dt}{t-1} - \int \frac{dt}{t} - \int \frac{dt}{t^2}$$

$$\Rightarrow\ I = \log|t-1| - \log|t| + \frac{1}{t} + c$$

$$\text{So, } \int \frac{2x}{(x^2 + 1)(x^2 + 2)^2} dx = \log|t-1| - \log|t| + \frac{1}{t} + c$$

57. Question

Evaluate the following integral:

$$\int \frac{x^2}{(x-1)(x^2+1)} dx$$

Answer

$$\text{Let, } I = \int \frac{x^2}{(x-1)(x^2 + 1)} dx$$

$$\text{Let } \frac{x^2}{(x-1)(x^2 + 1)} = \frac{A}{x-1} + \frac{B}{x^2 + 1}$$

$\Rightarrow x^2 = A(x^2 + 1) + B(x - 1)$

$$\text{For, } x = 1, A = \frac{1}{2}$$

$$\text{For, } x = 0, B = \frac{1}{2}$$

$$\therefore I = \frac{1}{2}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{x^2 + 1}$$

$$\Rightarrow\ I = \frac{1}{2}\log|x-1| + \frac{1}{2}\tan^{-1}x + c$$

Hence, $\int \frac{x^2}{(x-1)(x^2+1)} dx = \frac{1}{2}\log|x-1| + \frac{1}{2}\tan^{-1}x + c$

58. Question

Evaluate the following integral:

$$\int \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx$$

Answer

Let, $I = \int \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx$

Let $x^2 = y$

Thus, $\frac{x^2}{(x^2+a^2)(x^2+b^2)} = \frac{y}{(y+a^2)(y+b^2)}$

Now, let $\frac{y}{(y+a^2)(y+b^2)} = \frac{A}{y+a^2} + \frac{B}{y+b^2}$

$\Rightarrow y = A(y + b^2) + B(y + a^2)$

$\Rightarrow y = y(A + B) + (Ab^2 + Ba^2)$

Equating the coefficients, we get,

$A + B=1$, and $Ab^2 + Ba^2 = 0$

On solving we get, $A = -\frac{a^2}{b^2-a^2}$, $\quad B = \frac{b^2}{b^2-a^2}$

$\therefore I = -\frac{a^2}{b^2-a^2}\int \frac{dx}{x^2+a^2} + \frac{b^2}{b^2-a^2}\int \frac{dx}{x^2+b^2}$

$\Rightarrow I = \frac{b}{b^2-a^2}\tan^{-1}\left(\frac{x}{b}\right) - \frac{a}{b^2-a^2}\tan^{-1}\left(\frac{x}{a}\right) + c$

Thus, $\int \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx = \frac{b}{b^2-a^2}\tan^{-1}\left(\frac{x}{b}\right) - \frac{a}{b^2-a^2}\tan^{-1}\left(\frac{x}{a}\right) + c$

59. Question

Evaluate the following integral:

$$\int \frac{1}{\cos x(5-4\sin x)} dx$$

Answer

Let, $I = \int \frac{dx}{\cos x\,(5-4\sin x)}$

Multiplying and dividing by cos x

Let, $I = \int \frac{\cos x\, dx}{\cos^2 x\,(5-4\sin x)}$

$\Rightarrow I = \int \frac{\cos x\, dx}{(1-\sin^2 x)(5-4\sin x)}$

Let, sin x = t, cos x dx = dt

$\therefore I = \int \frac{dt}{(1-t^2)(5-4t)}$

Now, let $\frac{1}{(1-t^2)(5-4t)} = \frac{A}{1-t} + \frac{B}{1+t} + \frac{C}{5-4t}$

$\Rightarrow 1 = A(1+t)(5-4t) + B(1-t)(5-4t) + C(1-t^2)$

For $t = 1, A = \frac{1}{2}$

For $t = -1, B = \frac{1}{18}$

For $t = \frac{5}{4}, C = -\frac{16}{9}$

$\therefore I = \frac{1}{2}\int \frac{dt}{1-t} + \frac{1}{18}\int \frac{dt}{1+t} - \frac{16}{9}\int \frac{dt}{5-4t}$

$\Rightarrow I = -\frac{1}{2}\log|1-t| + \frac{1}{18}\log|1+t| + \frac{4}{9}\log|5-4t| + c$

So, $I = -\frac{1}{2}\log|1-\sin x| + \frac{1}{18}\log|1+\sin x| + \frac{4}{9}\log|5-4\sin x| + c$

60. Question

Evaluate the following integral:

$$\int \frac{1}{\sin x(3+2\cos x)} dx$$

Answer

Let, $I = \int \frac{1}{\sin x(3+2\cos x)} dx$

Multiplying and dividing by sin x

$\therefore I = \int \frac{\sin x}{\sin^2 x(3+2\cos x)} dx$

$\therefore I = \int \frac{\sin x}{(1-\cos^2 x)(3+2\cos x)} dx$

Let cos x = t, – sin x dx = dt

So, $I = \int \frac{dt}{(t^2-1)(3+2t)}$

Now, let $\frac{1}{(t^2-1)(3+2t)} = \frac{A}{t-1} + \frac{B}{t+1} + \frac{C}{3+2t}$

$\Rightarrow 1 = A(t+1)(3+2t) + B(t-1)(3+2t) + C(t^2-1)$

For, $t = 1, A = \frac{1}{10}$

For, $t = -1, B = -\frac{1}{2}$

For, $t = -\frac{3}{2}, C = \frac{4}{5}$

$$\therefore I = \frac{1}{10}\int \frac{dt}{t-1} - \frac{1}{2}\int \frac{dt}{t+1} + \frac{4}{5}\int \frac{dt}{3+2t}$$

$$\Rightarrow I = \frac{1}{10}\log|t-1| - \frac{1}{2}\log|t+1| + \frac{2}{5}\log|3+2t| + c$$

61. Question

Evaluate the following integral:

$$\int \frac{1}{\sin x + \sin 2x}\,dx$$

Answer

$$\text{Let, } I = \int \frac{1}{\sin x + \sin 2x}\,dx$$

$$\Rightarrow I = \int \frac{1}{\sin x + 2\sin x \cos x}\,dx$$

Multiplying and dividing by sin x

$$\Rightarrow I = \int \frac{\sin x}{\sin^2 x + 2\sin^2 x . \cos x}\,dx$$

$$\Rightarrow I = \int \frac{\sin x}{1-\cos^2 x + 2(1-\cos^2 x)\cos x}\,dx$$

Let cos x = t, – sin x dx = dt

$$\therefore I = \int \frac{dt}{(t^2-1) + 2(t^2-1)t}$$

$$\Rightarrow I = \int \frac{dt}{(t^2-1)(1+2t)}$$

$$\text{Let, } \frac{1}{(t^2-1)(1+2t)} = \frac{A}{t-1} + \frac{B}{1+t} + \frac{C}{1+2t}$$

$$\Rightarrow 1 = A(1+t)(1+2t) + B(t-1)(1+2t) + C(t^2-1)$$

$$\text{For } t = 1, A = \frac{1}{6}$$

$$\text{For } t = -1, B = \frac{1}{2}$$

$$\text{For } t = -\frac{1}{2}, C = -\frac{4}{3}$$

$$\text{So, } I = \frac{1}{6}\int \frac{dt}{t-1} + \frac{1}{2}\int \frac{dt}{t+1} - \frac{4}{3}\int \frac{dt}{1+2t}$$

$$\Rightarrow I = \frac{1}{6}\log|t-1| + \frac{1}{2}\log|1+t| - \frac{2}{3}\log|1+2t| + c$$

$$\text{So, } I = \frac{1}{6}\log|\cos x - 1| + \frac{1}{2}\log|1+\cos x| - \frac{2}{3}\log|1+2\cos x| + c$$

62. Question

Evaluate the following integral:

$$\int \frac{x+1}{x\left(1+xe^{x}\right)} dx$$

Answer

Let, $I = \int \frac{x + 1}{x(1 + xe^{x})} dx$

$\Rightarrow, \quad I = \int \frac{(x + 1)(1 + xe^{x} - xe^{x})}{x(1 + xe^{x})} dx$

$\Rightarrow, \quad I = \int \frac{(x + 1)(1 + xe^{x})}{x(1 + xe^{x})} dx - \int \frac{(x + 1)(xe^{x})}{x(1 + xe^{x})} dx$

$\Rightarrow, \quad I = \int \frac{(x + 1)}{x} dx - \int \frac{(x + 1)(e^{x})}{(1 + xe^{x})} dx$

$\Rightarrow, \quad I = \log|xe^{x}| - \log|1 + xe^{x}| + c$

$\Rightarrow, \quad I = \log\left|\frac{xe^{x}}{1 + xe^{x}}\right| + c$

Hence, $\int \frac{x + 1}{x(1 + xe^{x})} dx = \log\left|\frac{xe^{x}}{1 + xe^{x}}\right| + c$

63. Question

Evaluate the following integral:

$$\int \frac{\left(x^{2}+1\right)\left(x^{2}+2\right)}{\left(x^{2}+3\right)\left(x^{2}+4\right)} dx$$

Answer

$\frac{(x^2 + 1)(x^2 + 2)}{(x^2 + 3)(x^2 + 4)} = \frac{x^4 + 3x^2 + 2}{x^4 + 7x^2 + 12}$

$= \frac{(x^4 + 7x^2 + 12) - 4x^2 - 10}{x^4 + 7x^2 + 12}$

$= 1 - \frac{4x^2 + 10}{x^4 + 7x^2 + 12}$

Now, $\frac{4x^2 + 10}{x^4 + 7x^2 + 12} = \frac{4x^2 + 10}{(x^2 + 3)(x^2 + 4)}$

Let, $\frac{4x^2 + 10}{(x^2 + 3)(x^2 + 4)} = \frac{Ax + B}{x^2 + 3} + \frac{CX + D}{x^2 + 4}$

$\Rightarrow 4x^2 + 10 = (Ax + B)(x^2 + 4) + (Cx + D)(x^2 + 3)$

For, x=0, 10 = 4B + 3D .... (i)

For, x=1, 14 = 5A + 5B + 4C + 4D .... (ii)

For, x= – 1,14 = – 5A + 5B – 4C + 4D .... (iii)

Also, by comparing coefficient of x^3 we get, 0=A + C (iv)

On solving, (i), (ii), (iii), (iv) we get,

A=0, B= – 2, C=0, D=6

So, $\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = 1 + \frac{2}{x^2+3} - \frac{6}{x^2+4}$

$\therefore \int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} dx = \int \left(1 + \frac{2}{x^2+3} - \frac{6}{x^2+4}\right) dx$

$= x + \frac{2}{\sqrt{3}} \tan^{-1} x - 3\tan^{-1}\frac{x}{2} + c$

Therefore, $\int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} dx = x + \frac{2}{\sqrt{3}} \tan^{-1} x - 3\tan^{-1}\frac{x}{2} + c$

64. Question

Evaluate the following integral:

$$\int \frac{4x^4+3}{(x^2+2)(x^2+3)(x^2+4)} dx$$

Answer

Let $I = \int \frac{4x^4+3}{(x^2+2)(x^2+3)(x^2+4)} dx$

Let $x^2=y$

$\therefore \frac{4x^4+3}{(x^2+2)(x^2+3)(x^2+4)} = \frac{4y^2+3}{(y+2)(y+3)(y+4)}$

Let, $\frac{4y^2+3}{(y+2)(y+3)(y+4)} = \frac{A}{y+2} + \frac{B}{y+3} + \frac{C}{y+4}$

$\Rightarrow 4y^2 + 3 = A(y+3)(y+4) + B(y+2)(y+4) + C(y+2)(y+3)$

For $y = -2, A = \frac{19}{2}$

For $y = -3, B = -39$

For $y = -4, C = \frac{67}{2}$

Thus, $I = \frac{19}{2}\int \frac{dx}{x^2+2} - 39\int \frac{dx}{x^2+3} + \frac{67}{2}\int \frac{dx}{x^2+4}$

$\Rightarrow I = \frac{19}{2\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) - \frac{39}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + \frac{67}{4} \tan^{-1}\left(\frac{x}{2}\right) + c$

65. Question

Evaluate the following integral:

$$\int \frac{x^4}{(x-1)(x^2+1)} dx$$

Answer

$\frac{x^4}{(x-1)(x^2+1)} = \frac{x^4}{x^3 - x^2 + x - 1}$

$= \frac{x(x^3 - x^2 + x - 1) + 1(x^3 - x^2 + x - 1) + 1}{x^3 - x^2 + x - 1}$

$$= x + 1 + \frac{1}{(x-1)(x^2+1)}$$

Now, let $\frac{1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$

$\Rightarrow 1 = A(x^2+1) + (Bx+C)(x-1)$

For, $x = 1, A = \frac{1}{2}$

For, $x = 0, C = A - 1 = -\frac{1}{2}$

For, $x = -1, B = -\frac{1}{2}$

$$\therefore \int \frac{x^4}{(x-1)(x^2+1)} dx = \int x dx + \int dx + \frac{1}{2}\int \frac{1}{x-1} dx - \frac{1}{2}\int \frac{x+1}{x^2+1} dx$$

$$= \frac{x^2}{2} + x + \frac{1}{2}\log|x-1| - \frac{1}{4}\log(x^2+1) - \frac{1}{2}\tan^{-1}x + c$$

66. Question

Evaluate the following integral:

$$\int \frac{x^2}{x^4 - x^2 - 12} dx$$

Answer

$$\frac{x^2}{x^4 - x^2 - 12} = \frac{x^2}{(x^2-4)(x^2+3)}$$

Let, $\frac{x^2}{(x^2-4)(x^2+3)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x^2+3}$

$\Rightarrow x^2 = A(x+2)(x^2+3) + B(x-2)(x^2+3) + C(x-2)(x+2)$

For, $x = 2, A = \frac{1}{7}$

For, $x = -2, B = -\frac{1}{7}$

For, $x = 0, C = \frac{3}{7}$

$$\therefore \int \frac{x^2}{x^4 - x^2 - 12} dx = \frac{1}{7}\int \frac{dx}{x-2} - \frac{1}{7}\int \frac{dx}{x+2} + \frac{3}{7}\int \frac{dx}{x^2+3}$$

$$= \frac{1}{7}\log|x-2| - \frac{1}{7}\log|x+2| + \frac{3}{7\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + c$$

67. Question

Evaluate the following integral:

$$\int \frac{x^2}{1-x^4} dx$$

Answer

Let, $I = \int \frac{x^2}{1-x^4} dx$

Let, $\frac{x^2}{1-x^4} = \frac{A}{1-x} + \frac{B}{1+x} + \frac{C}{1+x^2}$

$\Rightarrow x^2 = A(1+x)(x^2+1) + B(1-x)(x^2+1) + c(x+1)(1-x)$

For, $x = 1, A = \frac{1}{4}$

For, $x = -1, B = \frac{1}{4}$

For, $x = 0, C = -\frac{1}{2}$

$\therefore I = \frac{1}{4}\int \frac{dx}{1-x} + \frac{1}{4}\int \frac{dx}{1+x} - \frac{1}{2}\int \frac{dx}{1+x^2}$

$\Rightarrow I = -\frac{1}{4}\log|1-x| + \frac{1}{4}\log|1+x| - \frac{1}{2}\tan^{-1}x + c$

$\Rightarrow I = \frac{1}{4}\log\left|\frac{1+x}{1-x}\right| - \frac{1}{2}\tan^{-1}x + c$

Hence, $\int \frac{x^2}{1-x^4} dx = \frac{1}{4}\log\left|\frac{1+x}{1-x}\right| - \frac{1}{2}\tan^{-1}x + c$

68. Question

Evaluate the following integral:

$$\int \frac{x^2}{x^4 + x^2 - 2} dx$$

Answer

Let, $I = \int \frac{x^2}{x^4 + x^2 - 2} dx$

Let, $\frac{x^2}{x^4 + x^2 - 2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x^2+2}$

$\Rightarrow x^2 = A(x-1)(x^2+2) + B(x+1)(x^2+2) + C(x^2-1)$

For, $x = 1, A = \frac{1}{6}$

For, $x = -1, B = -\frac{1}{6}$

For, $x = 0, C = -\frac{2}{3}$

$\therefore I = \frac{1}{6}\int \frac{dx}{x+1} - \frac{1}{6}\int \frac{dx}{x-1} - \frac{2}{3}\int \frac{dx}{x^2+2}$

$\Rightarrow I = \frac{1}{6}\log|x+1| - \frac{1}{6}\log|x-1| - \frac{2}{3\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + c$

69. Question

Evaluate the following integral:

$$\int \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)}dx$$

Answer

$$\frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)} = \frac{x^4+5x^2+4}{x^4-2x^2-15}$$

$$= \frac{(x^4-2x^2-15)+7x^2+19}{x^4-2x^2-15}$$

$$= 1 + \frac{7x^2+19}{x^4-2x^2-15}$$

Now, $\frac{7x^2+19}{x^4-2x^2-15} = \frac{7x^2+19}{(x^2+3)(x^2-5)}$

Let, $\frac{7x^2+19}{x^4-2x^2-15} = \frac{Ax+B}{x^2+3} + \frac{CX+D}{x^2-5}$

$\Rightarrow 7x^2+19 = (Ax+B)(x^2-5)+(Cx+D)(x^2+3)$

For, x=0, 19 = – 5B + 3D .... (i)

For, x=1, 26 = – 4A – 4B + 4C + 4D .... (ii)

For, x= – 1,14 = 4A – 4B – 4C + 4D .... (iii)

Also, by comparing coefficient of x^3 we get, 0=A + C (iv)

On solving, (i), (ii), (iii), (iv) we get,

$A = 0, B = -\frac{11}{8}, C = 0, D = \frac{69}{8}$

So, $\frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)} = 1 - \frac{11}{8}\frac{1}{x^2+3} + \frac{69}{8}\frac{1}{x^2-5}$

$$\therefore \int \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)}dx = \int \left(1 - \frac{11}{8}\frac{1}{x^2+3} + \frac{69}{8}\frac{1}{x^2-5}\right)dx$$

$$= x - \frac{11}{8\sqrt{3}}\tan^{-1}x + \frac{69}{16\sqrt{5}}\log\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right| + c$$

Thus, $I = x - \frac{11}{8\sqrt{3}}\tan^{-1}x + \frac{69}{16\sqrt{5}}\log\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right| + c$

Exercise 19.31

1. Question

Evaluate the following integral:

$$\int \frac{x^2+1}{x^4+x^2+1}dx$$

Answer

re-writing the given equation as

$$\int \frac{1+\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}}dx$$

$$\int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+3}dx$$

Let $x-\frac{1}{x}$ as t

$$\left(1+\frac{1}{x^2}\right)=dt$$

$$\int \frac{1}{t^2+3}dt$$

Using identity $\int \frac{1}{x^2+1}dx = \arctan(x)$

$$\frac{1}{\sqrt{3}}\arctan\left(\frac{t}{\sqrt{3}}\right)+c$$

Substituting t as $x-\frac{1}{x}$

$$\frac{1}{\sqrt{3}}\arctan\left(\frac{\left(x-\frac{1}{x}\right)}{\sqrt{3}}\right)+c$$

2. Question

Evaluate the following integral:

$$\int \sqrt{\cot\theta}\ d\theta$$

Answer

let $\cot\theta$ as x^2

$$-\mathrm{cosec}^2\theta d\theta = 2xdx$$

$$d\theta = -\frac{2x}{1+\cot^2\theta}dx$$

$$d\theta = -\frac{2x}{1+x^4}dx$$

$$\int -\frac{2x^2}{1+x^4}dx$$

re-writing the given equation as

$$\int \frac{1+\frac{1}{x^2}+1-\frac{1}{x^2}}{\frac{1}{x^2}+x^2}dx$$

$$-\int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}dx - \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-2}dx$$

Let $x-\frac{1}{x}=t$ and $x+\frac{1}{x}=z$

So $\left(1+\frac{1}{x^2}\right)dx = dt$ and $\left(1-\frac{1}{x^2}\right)dx = dz$

$$-\int \frac{dt}{(t)^2+2} - \int \frac{dz}{(z)^2-2}$$

Using identity $\int \frac{1}{x^2+1}dx = \arctan(x)$ and $\int \frac{dz}{(z)^2-1} = \frac{1}{2}\log\left|\frac{z-1}{z+1}\right| + c$

$$-\frac{1}{2}\arctan\left(\frac{t}{\sqrt{2}}\right) - \frac{1}{2\sqrt{2}}\log\left|\frac{z-\sqrt{2}}{z+\sqrt{2}}\right| + c$$

Substituting t as $x-\frac{1}{x}$ and z as $x+\frac{1}{x}$

$$-\frac{1}{2}\arctan\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) - \frac{1}{2\sqrt{2}}\log\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right| + c$$

3. Question

Evaluate the following integral:

$$\int \frac{x^2+9}{x^4+81}dx$$

Answer

re-writing the given equation as

$$\int \frac{1+\frac{9}{x^2}}{x^2+\frac{81}{x^2}}dx$$

$$\int \frac{1+\frac{9}{x^2}}{\left(x-\frac{9}{x}\right)^2+18}dx$$

Let $x-\frac{9}{x} = t$

$$\left(1+\frac{9}{x^2}\right)dx = dt$$

$$\int \frac{dt}{t^2+18}$$

Using identity $\int \frac{1}{x^2+1}dx = \arctan(x)$

$$\frac{1}{3\sqrt{2}}\arctan\left(\frac{t}{3\sqrt{2}}\right) + c$$

Substituting t as $x-\frac{1}{x}$

$$\frac{1}{3\sqrt{2}}\arctan\left(\frac{x-\frac{1}{x}}{3\sqrt{2}}\right) + c$$

4. Question

Evaluate the following integral:

$$\int \frac{1}{x^4+x^2+1} dx$$

Answer

re-writing the given equation as

$$\int \frac{\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}} dx$$

$$\frac{1}{2}\int \frac{1+\frac{1}{x^2}+\frac{1}{x^2}-1}{x^2+1+\frac{1}{x^2}} dx$$

$$\frac{1}{2}\left[\int \frac{1+\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}} dx+\int \frac{-1+\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}} dx\right]$$

$$\frac{1}{2}\left[\int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+3} dx+\int \frac{-1+\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-1} dx\right]$$

Let $x-\frac{1}{x}=t$ and $x+\frac{1}{x}=z$

$\left(1+\frac{1}{x^2}\right)dx=dt$ and $\left(1-\frac{1}{x^2}\right)dx=dz$

$$\frac{1}{2}\left[\int \frac{dt}{(t)^2+3}-\int \frac{dz}{(z)^2-1}\right]$$

Using identity $\int \frac{1}{x^2+1} dx=\arctan(x)$ and $\int \frac{dz}{(z)^2-1}=\frac{1}{2}\log\left|\frac{z-1}{z+1}\right|+c$

$$\frac{1}{2}\left[\frac{1}{\sqrt{3}}(\arctan\left(\frac{t}{\sqrt{3}}\right)-\frac{1}{2}\log\left|\frac{z-1}{z+1}\right|\right]$$

Substituting t as $x-\frac{1}{x}$ and z as $x+\frac{1}{x}$

$$\frac{1}{2}\left[\frac{1}{\sqrt{3}}(\arctan\left(\frac{x-\frac{1}{x}}{\sqrt{3}}\right)-\frac{1}{2}\log\left|\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right|\right]$$

5. Question

Evaluate the following integral:

$$\int \frac{x^2-3x+1}{x^4+x^2+1} dx$$

Answer

re-writing the given equation as

$$\int \frac{1-\frac{3}{x}+\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}} dx$$

$$\int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+3} dx-\int \frac{3x}{x^4+x^2+1} dx$$

Substituting t as $x - \frac{1}{x}$ and z as x^2

$\left(1 + \frac{1}{x^2}\right) dx = dt$ and $2xdx = dz$

$$\int \frac{dt}{(t)^2 + 3} - \frac{3}{2}\int \frac{dz}{z^2 + z + 1}$$

$$\int \frac{dt}{(t)^2 + 3} - \frac{3}{2}\int \frac{dz}{\left(z + \frac{1}{2}\right)^2 + \frac{3}{4}}$$

Using identity $\int \frac{1}{x^2+1} dx = \arctan(x)$

$$\frac{1}{\sqrt{3}} \arctan\left(\frac{t}{\sqrt{3}}\right) - \sqrt{3} \arctan\left(\frac{2z+1}{\sqrt{3}}\right) + c$$

Substituting t as $x - \frac{1}{x}$ and z as x^2

$$\frac{1}{\sqrt{3}} \arctan\left(\frac{x - \frac{1}{x}}{\sqrt{3}}\right) - \sqrt{3} \arctan\left(\frac{2x^2+1}{\sqrt{3}}\right) + c$$

6. Question

Evaluate the following integral:

$$\int \frac{x^2 + 1}{x^4 - x^2 + 1} dx$$

Answer

re-writing the given equation as

$$\int \frac{1 + \frac{1}{x^2}}{x^2 - 1 + \frac{1}{x^2}} dx$$

$$\int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 1} dx$$

Substituting t as $x - \frac{1}{x}$

$$\left(1 + \frac{1}{x^2}\right) dx = dt$$

$$\int \frac{dt}{t^2 + 1}$$

Using identity $\int \frac{1}{x^2+1} dx = \arctan(x)$

$$\arctan t + c$$

Substituting t as $x - \frac{1}{x}$

$$\arctan\left(x - \frac{1}{x}\right) + c$$

7. Question

Evaluate the following integral:

$$\int \frac{x^2 - 1}{x^4 + 1} dx$$

Answer

re-writing the given equation as

$$\int \frac{1 - \frac{1}{x^2}}{x^2 - \frac{1}{x^2}} dx$$

$$\int \frac{1 - \frac{1}{x^2}}{\left(x + \frac{1}{x}\right)^2 - 2} dx$$

Assume $t = x + \frac{1}{x}$

$$dt = \left(1 - \frac{1}{x^2}\right) dx$$

$$\int \frac{dt}{t^2 - 2}$$

Using identity $\int \frac{dz}{(z)^2 - 1} = \frac{1}{2} \log \left|\frac{z-1}{z+1}\right| + c$

$$\frac{1}{2\sqrt{2}} \log \frac{t - \sqrt{2}}{t + \sqrt{2}} + c$$

Substituting t as $x + \frac{1}{x}$

$$\frac{1}{2\sqrt{2}} \log \frac{x + \frac{1}{x} - \sqrt{2}}{x + \frac{1}{x} + \sqrt{2}} + c$$

8. Question

Evaluate the following integral:

$$\int \frac{x^2 + 1}{x^4 + 7x^2 + 1} dx$$

Answer

re-writing the given equation as

$$\int \frac{1 + \frac{1}{x^2}}{x^2 + 7 + \frac{1}{x^2}} dx$$

$$\int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 9} dx$$

Assume $t = x - \frac{1}{x}$

$$dt = \left(1 + \frac{1}{x^2}\right) dx$$

$$\int \frac{dt}{(t)^2 + 9}$$

Using identity $\int \frac{1}{x^2+1} dx = \arctan(x)$

$$\frac{1}{3}\arctan\left(\frac{t}{3}\right) + c$$

Substituting t as $x - \frac{1}{x}$

$$\frac{1}{3}\arctan\left(\frac{x - \frac{1}{x}}{3}\right) + c$$

9. Question

Evaluate the following integral:

$$\int \frac{(x-1)^2}{x^4 + x^2 + 1} dx$$

Answer

re-writing the given equation as

$$\int \frac{x^2 - 2x + 1}{x^4 + x^2 + 1} dx$$

$$\int \frac{1 - \frac{2}{x} + \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx$$

$$\int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 3} dx - \int \frac{2x}{x^4 + x^2 + 1} dx$$

Substituting t as $x - \frac{1}{x}$ and z as x^2

$\left(1 + \frac{1}{x^2}\right) dx = dt$ and $2xdx = dz$

$$\int \frac{dt}{(t)^2 + 3} - \frac{3}{2}\int \frac{dz}{z^2 + z + 1}$$

$$\int \frac{dt}{(t)^2 + 3} - \int \frac{dz}{\left(z + \frac{1}{2}\right)^2 + \frac{3}{4}}$$

Using identity $\int \frac{1}{x^2+1} dx = \arctan(x)$

$$\frac{1}{\sqrt{3}}\arctan\left(\frac{t}{\sqrt{3}}\right) - \frac{2}{\sqrt{3}}\arctan\left(\frac{2z + 1}{\sqrt{3}}\right) + c$$

Substituting t as $x - \frac{1}{x}$ and z as x^2

$$\frac{1}{\sqrt{3}}\arctan\left(\frac{x - \frac{1}{x}}{\sqrt{3}}\right) - \frac{2}{\sqrt{3}}\arctan\left(\frac{2x^2 + 1}{\sqrt{3}}\right) + c$$

10. Question

Evaluate the following integral:

$$\int \frac{1}{x^4 + 3x^2 + 1} dx$$

Answer

re-writing the given equation as

$$\int \frac{\frac{1}{x^2}}{x^2 + 3 + \frac{1}{x^2}} dx$$

$$\frac{1}{2}\int \frac{\left(1 + \frac{1}{x^2}\right) - \left(1 - \frac{1}{x^2}\right)}{x^2 + 3 + \frac{1}{x^2}} dx$$

$$\frac{1}{2}\left[\int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 5} dx - \int \frac{1 - \frac{1}{x^2}}{\left(x + \frac{1}{x}\right)^2 + 1} dx\right]$$

Assume $t = x - \frac{1}{x}$ and $z = x + \frac{1}{x}$

$dt = \left(1 + \frac{1}{x^2}\right)dx$ and $dz = \left(1 - \frac{1}{x^2}\right)dx$

$$\frac{1}{2}\left[\int \frac{dt}{(t)^2 + 5} - \int \frac{dz}{(z)^2 + 1}\right]$$

Using identity $\int \frac{1}{x^2+1} dx = \arctan(x)$

$$\frac{1}{2\sqrt{5}}\arctan\left(\frac{t}{\sqrt{5}}\right) - \frac{1}{2}\arctan(z) + c$$

Substituting t as $x - \frac{1}{x}$ and z as $x + \frac{1}{x}$

$$\frac{1}{2\sqrt{5}}\arctan\left(\frac{x - \frac{1}{x}}{\sqrt{5}}\right) - \frac{1}{2}\arctan\left(x + \frac{1}{x}\right) + c$$

11. Question

Evaluate the following integral:

$$\int \frac{1}{\sin^4 x + \sin^2 x \cos^2 x + \cos^4 x} dx$$

Answer

Re-writing the given equation as

Multiplying $\sec^4 x$ in both numerator and denominator

$$\int \frac{\sec^4 x}{\tan^4 x + \tan^2 x + 1} dx$$

$$= \int \frac{(\tan^2 x + 1)\sec^2 x}{\tan^4 x + \tan^2 x + 1} dx$$

Assume tanx = t

$\sec^2 x dx = dt$

$$= \int \frac{(t^2+1)dt}{t^4+t^2+1}$$

$$= \int \frac{1+\frac{1}{t^2}}{t^2+1+\frac{1}{t^2}} dt$$

$$= \int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+3} dt$$

Assume $z = t - \frac{1}{t}$

$\Rightarrow dz = 1 + \frac{1}{t^2}$

$$= \int \frac{dz}{z^2+3}$$

Using identity $\int \frac{1}{x^2+1} dx = \arctan(x)$

$$= \frac{1}{\sqrt{3}} \arctan\left(\frac{z}{\sqrt{3}}\right) + c$$

$$= \frac{1}{\sqrt{3}} \arctan\left(\frac{t-\frac{1}{t}}{\sqrt{3}}\right) + c$$

$$= \frac{1}{\sqrt{3}} \arctan\left(\frac{\tan x - \frac{1}{\tan x}}{\sqrt{3}}\right) + c$$

Exercise 19.32

1. Question

Evaluate the following integral:

$$\int \frac{1}{(x-1)\sqrt{x+2}} dx$$

Answer

assume $x+2=t^2$

$dx=2tdt$

$$\int \frac{2dt}{(t^2-3)}$$

Using identity $\int \frac{dz}{(z)^2-1} = \frac{1}{2}\log\left|\frac{z-1}{z+1}\right| + c$

$$\frac{1}{\sqrt{3}} \log\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right| + c$$

$$\frac{1}{\sqrt{3}} \log\left|\frac{\sqrt{(x+2)}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}}\right| + c$$

2. Question

Evaluate the following integral:

$$\int \frac{1}{(x-1)\sqrt{2x+3}}\,dx$$

Answer

assume $2x+3=t^2$

dx=tdt

$$\int \frac{dt}{\frac{t^2-3}{2}-1}$$

$$\int \frac{2dt}{(t^2-5)}$$

Using identity $\int \frac{dz}{(z)^2-1} = \frac{1}{2}\log\left|\frac{z-1}{z+1}\right| + c$

$$\frac{1}{\sqrt{5}}\log\left|\frac{t-\sqrt{5}}{t+\sqrt{5}}\right| + c$$

$$\frac{1}{\sqrt{5}}\log\left|\frac{\sqrt{(2x+3)}\ -\sqrt{5}}{\sqrt{2x+3}+\sqrt{5}}\right| + c$$

3. Question

Evaluate the following integral:

$$\int \frac{x+1}{(x-1)\sqrt{x+2}}\,dx$$

Answer

re-writing the given equation as

$$\int \frac{(x-1)+2}{(x-1)\sqrt{x+2}}dx$$

Now splitting the integral in two parts

$$\int \frac{(x-1)}{(x-1)\sqrt{x+2}}dx + \int \frac{2}{(x-1)\sqrt{x+2}}dx$$

For the first part using identity $\int x^n dx = \frac{x^{n+1}}{n+1}$

$$2\sqrt{x+2}$$

For the second part

assume $x+2=t^2$

dx=2tdt

$$\int \frac{4dt}{(t^2-3)}$$

Using identity $\int \frac{dz}{(z)^2-1} = \frac{1}{2}\log\left|\frac{z-1}{z+1}\right| + c$

$$\frac{2}{\sqrt{3}}\log\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right| + c$$

$$\frac{2}{\sqrt{3}}\log\left|\frac{\sqrt{(x+2)}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}}\right|+c$$

Hence integral is

$$2\sqrt{x+2}+\frac{2}{\sqrt{3}}\log\left|\frac{\sqrt{(x+2)}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}}\right|+c$$

4. Question

Evaluate the following integral:

$$\int\frac{x^2}{(x-1)\sqrt{x+2}}\,dx$$

Answer

re-writing the given equation as

$$\int\frac{(x^2-1)+1}{(x-1)\sqrt{x+2}}dx$$

$$\int\frac{(x^2-1)}{(x-1)\sqrt{x+2}}dx+\int\frac{1}{(x-1)\sqrt{x+2}}dx$$

$$\int\frac{(x+1)}{\sqrt{x+2}}dx+\int\frac{1}{(x-1)\sqrt{x+2}}dx$$

$$\int\frac{(1)}{\sqrt{x+2}}dx+\int\sqrt{x+2}\,dx+\int\frac{1}{(x-1)\sqrt{x+2}}dx$$

For the first- and second-part using identity $\int x^n dx=\frac{x^{n+1}}{n+1}$

$$\frac{2}{3}(x+2)^{\frac{3}{2}}+2\sqrt{x+2}$$

For the second part

assume $x+2=t^2$

dx=2tdt

$$\int\frac{4dt}{(t^2-3)}$$

Using identity $\int\frac{dz}{(z)^2-1}=\frac{1}{2}\log\left|\frac{z-1}{z+1}\right|+c$

$$\frac{2}{\sqrt{3}}\log\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right|+c$$

$$\frac{2}{\sqrt{3}}\log\left|\frac{\sqrt{(x+2)}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}}\right|+c$$

Hence integral is

$$\frac{2}{3}(x+2)^{\frac{3}{2}}+2\sqrt{x+2}+\frac{2}{\sqrt{3}}\log\left|\frac{\sqrt{(x+2)}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}}\right|+c$$

5. Question

Evaluate the following integral:

$$\int \frac{x}{(x-3)\sqrt{x+1}}\,dx$$

Answer

re-writing the given equation as

$$\int \frac{(x-3)+3}{(x-3)\sqrt{x+1}}dx$$

$$\int \frac{(x-3)}{(x-3)\sqrt{x+1}}dx + \int \frac{3}{(x-3)\sqrt{x+1}}dx$$

For the first part using identity $\int x^n dx = \frac{x^{n+1}}{n+1}$

$2\sqrt{x+1} + c$

For the second part

assume x+1=t^2

dx=2tdt

$$\int \frac{2dt}{(t^2-4)}$$

Using identity $\int \frac{dz}{(z)^2-1} = \frac{1}{2}\log\left|\frac{z-1}{z+1}\right| + c$

$$\frac{1}{2}\log\left|\frac{t-2}{t+2}\right| + c$$

$$\frac{1}{2}\log\left|\frac{\sqrt{(x+2)}\ -2}{\sqrt{x+2}+2}\right| + c$$

Hence integral is

$$\frac{1}{2}\log\left|\frac{\sqrt{(x+2)}\ -2}{\sqrt{x+2}+2}\right| + c + 2\sqrt{x+1}$$

6. Question

Evaluate the following integral:

$$\int \frac{1}{(x^2+1)\sqrt{x}}\,dx$$

Answer

let x=t^2

dx=2tdt

$$\int \frac{2dt}{t^4+1}$$

Dividing by t^2 in both numerator and denominator

$$\int \frac{\left[\left(1+\frac{1}{t^2}\right)-\left(1-\frac{1}{t^2}\right)\right]dt}{t^2+\frac{1}{t^2}}$$

$$\int \frac{\left[\left(1+\frac{1}{t^2}\right)\right]dt}{\left(t-\frac{1}{t}\right)^2+2} - \int \frac{\left(1-\frac{1}{t^2}\right)dt}{\left(t+\frac{1}{t}\right)^2-2}$$

Let $t-\frac{1}{t}=z$ and $t+\frac{1}{t}=y$

$\left(1+\frac{1}{t^2}\right)dt = dz$ and $\left(1-\frac{1}{t^2}\right)dt = dy$

$$\int \frac{dz}{z^2+2} - \int \frac{dy}{y^2-2}$$

Using identity $\int \frac{1}{x^2+1}dx = \arctan(x)$ and $\int \frac{dz}{(z)^2-1} = \frac{1}{2}\log\left|\frac{z-1}{z+1}\right| + c$

$$\frac{1}{\sqrt{2}}\arctan\left(\frac{z}{\sqrt{2}}\right) - \frac{1}{2\sqrt{2}}\log\left|\frac{y-\sqrt{2}}{y+\sqrt{2}}\right| + c$$

Substituting $t-\frac{1}{t}=z$ and $t+\frac{1}{t}=y$

$$\frac{1}{\sqrt{2}}\arctan\left(\frac{t-\frac{1}{t}}{\sqrt{2}}\right) - \frac{1}{2\sqrt{2}}\log\left|\frac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}}\right| + c$$

$$\frac{1}{\sqrt{2}}\arctan\left(\frac{\sqrt{x}-\frac{1}{\sqrt{x}}}{\sqrt{2}}\right) - \frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{x}+\frac{1}{\sqrt{x}}-\sqrt{2}}{\sqrt{x}+\frac{1}{\sqrt{x}}+\sqrt{2}}\right| + c$$

7. Question

Evaluate the following integral:

$$\int \frac{x}{\left(x^2+2x+2\right)\sqrt{x+1}}dx$$

Answer

assume $x+1=t^2$

$dx=2tdt$

$$\int \frac{2(t^2-1)dt}{t^4+1}$$

Dividing by t^2 in both numerator and denominator

$$\int \frac{2\left(1-\frac{1}{t^2}\right)dt}{t^2+\frac{1}{t^2}}$$

$$\int \frac{2\left(1-\frac{1}{t^2}\right)dt}{\left(t+\frac{1}{t}\right)^2-2}$$

Let $\left(t+\frac{1}{t}\right)=z$

$\left(1-\frac{1}{t^2}\right)dt = dz$

$$\int \frac{2dz}{z^2 - 2}$$

Using identity $\int \frac{dz}{(z)^2-1} = \frac{1}{2}\log\left|\frac{z-1}{z+1}\right| + c$

$$\frac{1}{\sqrt{2}}\log\left|\frac{z-\sqrt{2}}{z+\sqrt{2}}\right| + c$$

Substituting $\left(t + \frac{1}{t}\right) = z$

$$\frac{1}{\sqrt{2}}\log\left|\frac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}}\right| + c$$

Substituting $t = \sqrt{x+1}$

$$\frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{x+1}+\frac{1}{\sqrt{x+1}}-\sqrt{2}}{\sqrt{x+1}+\frac{1}{\sqrt{x+1}}+\sqrt{2}}\right| + c$$

8. Question

Evaluate the following integral:

$$\int \frac{1}{(x-1)\sqrt{x^2+1}}\,dx$$

Answer

assume $x - 1 = \frac{1}{t}$

$$dx = -\frac{1}{t^2}dt$$

$$-\int \frac{dt}{\sqrt{2t^2+2t+1}}$$

$$-\frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{\left(t+\frac{1}{2}\right)^2+\frac{1}{4}}}$$

Using identity $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left(x + \sqrt{x^2+a^2}\right) + c$

$$-\frac{1}{\sqrt{2}}\log\left(t+\frac{1}{2}+\sqrt{\left(t+\frac{1}{2}\right)^2+\frac{1}{4}}\right) + c$$

Substituting $t = \frac{1}{x-1}$

$$-\frac{1}{\sqrt{2}}\log\left(\frac{1}{x-1}+\frac{1}{2}+\sqrt{\left(\frac{1}{x-1}+\frac{1}{2}\right)^2+\frac{1}{4}}\right) + c$$

9. Question

Evaluate the following integral:

$$\int \frac{1}{(x+1)\sqrt{x^2+x+1}} dx$$

Answer

assume $x+1=\frac{1}{t}$

$dx=-\frac{1}{t^2}dt$

$-\int \frac{dt}{\sqrt{1+t-t^2}}$

$-\int \frac{dt}{\sqrt{\frac{5}{4}-\left(t-\frac{1}{2}\right)^2}}$

Using identity $\int \frac{dx}{\sqrt{a^2-x^2}}=\arcsin\left(\frac{x}{a}\right)+c$

$-\arcsin\left(\frac{\left(t-\frac{1}{2}\right)}{\frac{\sqrt{5}}{2}}\right)+c$

Substituting $t=\frac{1}{x+1}$

$-\arcsin\left(\frac{\left(\frac{1}{x+1}-\frac{1}{2}\right)}{\frac{\sqrt{5}}{2}}\right)+c$

10. Question

Evaluate the following integral:

$$\int \frac{1}{(x^2-1)\sqrt{x^2+1}} dx$$

Answer

assume $x=\frac{1}{t}$

$dx=-\frac{1}{t^2}dt$

$-\int \frac{tdt}{(1-t^2)(\sqrt{1+t^2}}$

Let $1+t^2=u^2$

$tdt=udu$

$\int \frac{udu}{(u^2-2)u}$

$\int \frac{du}{(u^2-2)}$

Using identity $\int \frac{dz}{(z)^2-1}=\frac{1}{2}\log\left|\frac{z-1}{z+1}\right|+c$

$$\frac{1}{2\sqrt{2}}\log\left|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right|+c$$

Substituting $u=\sqrt{1+t^2}$

$$\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{1+t^2}-\sqrt{2}}{\sqrt{1+t^2}+\sqrt{2}}\right|+c$$

Substituting $t=\frac{1}{x}$

$$\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{1+\frac{1}{x^2}}-\sqrt{2}}{\sqrt{1+\frac{1}{x^2}}+\sqrt{2}}\right|+c$$

11. Question

Evaluate the following integral:

$$\int\frac{x}{\left(x^2+4\right)\sqrt{x^2+1}}dx$$

Answer

assume $x^2+1=u^2$

$xdx=udu$

$$\int\frac{udu}{(u^2+3)u}$$

$$\int\frac{du}{(u^2+3)}$$

Using identity $\int\frac{1}{x^2+1}dx=\arctan(x)$

$$\frac{1}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right)+c$$

Substituting $u=\sqrt{1+x^2}$

$$\frac{1}{\sqrt{3}}\arctan\left(\frac{\sqrt{1+x^2}}{\sqrt{3}}\right)+c$$

12. Question

Evaluate the following integral:

$$\int\frac{1}{\left(1+x^2\right)\sqrt{1-x^2}}dx$$

Answer

assume $x=\frac{1}{t}$

$$dx=-\frac{1}{t^2}dt$$

$$-\int\frac{tdt}{(t^2+1)(\sqrt{t^2-1}}$$

Let $t^2 - 1 = u^2$

$tdt = udu$

$$-\int \frac{udu}{(u^2+2)u}$$

$$-\int \frac{du}{(u^2+2)}$$

Using identity $\int \frac{1}{x^2+1}dx = \arctan(x)$

$$-\frac{1}{\sqrt{2}}\arctan\left(\frac{u}{\sqrt{2}}\right) + c$$

Substituting $u = \sqrt{t^2-1}$

$$-\frac{1}{\sqrt{2}}\arctan\left(\frac{\sqrt{t^2-1}}{\sqrt{2}}\right) + c$$

Substituting $t = \frac{1}{x}$

$$-\frac{1}{\sqrt{2}}\arctan\left(\frac{\sqrt{\frac{1}{x^2}-1}}{\sqrt{2}}\right) + c$$

13. Question

Evaluate the following integral:

$$\int \frac{1}{\left(2x^2+3\right)\sqrt{x^2-4}}dx$$

Answer

assume $x = \frac{1}{t}$

$$dx = -\frac{1}{t^2}dt$$

$$-\int \frac{tdt}{(3t^2+2)(\sqrt{1-4t^2}}$$

Assume $1-4t^2=u^2$

$-4tdt=udu$

$$-\frac{1}{4}\int \frac{udu}{\left(\frac{11-3u^2}{4}\right)u}$$

$$-\frac{1}{3}\int \frac{du}{\left(\frac{11}{3}-u^2\right)}$$

Using identity $\int \frac{dz}{(z)^2-1} = \frac{1}{2}\log\left|\frac{z-1}{z+1}\right| + c$

$$\frac{1}{2\sqrt{33}}\log\left|\frac{u-\sqrt{\frac{11}{3}}}{u+\sqrt{\frac{11}{3}}}\right|+c$$

Substituting $u=\sqrt{1-4t^2}$

$$\frac{1}{2\sqrt{33}}\log\left|\frac{\sqrt{1-4t^2}-\sqrt{\frac{11}{3}}}{\sqrt{1-4t^2}+\sqrt{\frac{11}{3}}}\right|+c$$

Substituting $t=\frac{1}{x}$

$$\frac{1}{2\sqrt{33}}\log\left|\frac{\sqrt{1-\frac{4}{x^2}}-\sqrt{\frac{11}{3}}}{\sqrt{1-\frac{4}{x^2}}+\sqrt{\frac{11}{3}}}\right|+c$$

14. Question

Evaluate the following integral:

$$\int\frac{x}{\left(x^2+4\right)\sqrt{x^2+9}}dx$$

Answer

assume $x^2+9=u^2$

$xdx=udu$

$$\int\frac{udu}{(u^2-5)u}$$

$$\int\frac{du}{(u^2-5)}$$

Using identity $\int\frac{dz}{(z)^2-1}=\frac{1}{2}\log\left|\frac{z-1}{z+1}\right|+c$

$$\frac{1}{2\sqrt{5}}\log\left|\frac{u-\sqrt{5}}{u+\sqrt{5}}\right|+c$$

Substituting $u=\sqrt{9+x^2}$

$$\frac{1}{2\sqrt{5}}\log\left|\frac{\sqrt{9+x^2}-\sqrt{5}}{\sqrt{9+x^2}+\sqrt{5}}\right|+c$$

Very short answer

16. Question

Write a value of $\int\frac{1}{1+2e^x}dx$

Answer

Take e^x out from the denominator.

$$y=\int\frac{1}{e^x(e^{-x}+2)}dx$$

$$y = \int \frac{e^{-x}}{(e^{-x} + 2)} dx$$

Let, $e^{-x} + 2 = t$

Differentiating both sides with respect to x

$$\frac{dt}{dx} = -e^{-x}$$

$\Rightarrow$ $-dt = e^{-x}\,dx$

$$y = \int \frac{-dt}{t}$$

Use formula $\int \frac{1}{t} dt = \ln t$

Y = -ln t + c

Again, put $e^{-x} + 2 = t$

Y = $-\ln(e^{-x} + 2) + c$

Note: Don't forget to replace t with the function of x at the end of solution. Always put constant c with indefinite integral.

17. Question

Write a value of $\int \frac{(\tan^{-1} x)^3}{1+x^2} dx$

Answer

Let, $\tan^{-1}x = t$

Differentiating both sides with respect to x

$$\frac{dt}{dx} = \frac{1}{1 + x^2}$$

$$\Rightarrow dt = \frac{dx}{1 + x^2}$$

$y= \int t^3\, dt$

Use formula $\int t^n dt = \frac{t^{n+1}}{n+1}$

$$y = \frac{t^4}{4} + c$$

Again, put $t = \tan^{-1}x$

$$y = \frac{(\tan^{-1} x)^4}{4} + c$$

18. Question

Write a value of $\int \frac{\sec^2 x}{(5 + \tan x)^4} dx$

Answer

Let, tan x = t

Differentiating both side with respect to x

$\frac{dt}{dx} = (\sec x)^2 \Rightarrow dt = \sec^2 x\, dx$

$$y = \int \frac{dt}{(5+t)^4}$$

Use formula $\int \frac{1}{(a+t)^n} dt = \frac{(a+t)^{-n+1}}{-n+1}$

$$y = \frac{(5+t)^{-3}}{-3} + c$$

Again, put t = tan x

$$y = -\frac{1}{3(5+\tan x)^3} + c$$

19. Question

Write a value of $\int \frac{\sin x + \cos x}{\sqrt{1+\sin 2x}} dx$

Answer

We know that

$1 + \sin 2x = \sin^2 x + \cos^2 x + 2\sin x\cos x = (\sin x + \cos x)^2$

$$y = \int \frac{\sin x + \cos x}{\sqrt{(\sin x + \cos x)^2}} dx$$

$$y = \int \frac{(\sin x + \cos x)}{(\sin x + \cos x)} dx$$

y= ∫dx

Use formula ∫c dx=cx, where c is constant

y = x + c

20. Question

Write a value of $\int \log_e x dx$

Answer

$y = \int 1 \times \log_e x\, dx$

By using integration by parts

Let, $\log_e x$ as Ist function and 1 as IInd function

Use formula $\int I \times II\, dx = I \int II\, dx - \int \left(\frac{d}{dx} I\right)\left(\int II\, dx\right) dx$

$$y = \log_e x \int dx - \int \left(\frac{d}{dx}\log_e x\right)\left(\int dx\right) dx$$

$$y = (\log_e x)x - \int \left(\frac{1}{x}\right)(x)\, dx$$

$y = x \log_e x - \int dx$

$y = x \log_e x - x + c$

21. Question

Write a value of $\int a^x e^x dx$

Answer

We know that a and e are constant so, $a^x e^x = (ae)^x$

$$y = \int (ae)^x \, dx$$

Use formula $\int c^x = \frac{c^x}{\log c}$ where c is constant

$$y = \frac{(ae)^x}{\log(ae)} + c$$

$$y = \frac{a^x e^x}{\log a \; + 1} + c$$

22. Question

Write a value of $\int e^{2x^2 + \ln x} dx$

Answer

We know that $e^{a+b} = e^a e^b$

$$y = \int e^{2x^2} e^{\ln x} dx$$

$$y = \int e^{2x^2} x \, dx$$

Let, $x^2 = t$

Differentiating both sides with respect to x

$$\frac{dt}{dx} = 2x$$

$$\Rightarrow \frac{1}{2} dt = x \, dx$$

$$y = \int \frac{1}{2} e^{2t} \, dt$$

Use formula $\int e^{a+bt} = \frac{e^{a+bt}}{b}$

$$y = \frac{1}{2} \frac{e^{2t}}{2} + c$$

Again, put $t = x^2$

$$y = \frac{e^{2x^2}}{4} + c$$

23. Question

Write a value of $\int \left(e^{x \log_e a} + e^{a \log_e x} \right) dx$

Answer

We know that by using property of logarithm

$$e^{x \log_e a} = e^{\log_e a^x} = a^x \text{ and } e^{a \log_e x} = e^{\log_e x^a} = x^a$$

$y = \int a^x + x^a\, dx$

$y = \int a^x\, dx + \int x^a\, dx$

Use formula $\int a^x\, dx = \frac{a^x}{\log a}$ and $\int x^a\, dx = \frac{x^{a+1}}{a+1}$

$$y = \frac{a^x}{\log a} + \frac{x^{a+1}}{a+1} + c$$

24. Question

Write a value of $\int \frac{\cos x}{\sin x \log \sin x} dx$

Answer

Let log(sin x) = t

Differentiating both sides with respect to x

$$\frac{dt}{dx} = \frac{\cos x}{\sin x} \Rightarrow dt = \frac{\cos x}{\sin x} dx$$

$$y = \int \frac{1}{t} dt$$

Use formula $\int \frac{1}{t} dt = \log t$

y = log t + c

Again, put t = log(sin x)

y = log(log(sin x)) + c

25. Question

Write a value of $\int \frac{\sin 2x}{a^2 \sin^2 x + b^2 \cos^2 x} dx$

Answer

We know that $\cos^2 x = 1 - \sin^2 x$

$(a^2 \sin^2 x + b^2 \cos^2 x) = a^2 \sin^2 x + b^2(1 - \sin^2 x)$

$= (a^2 - b^2)\sin^2 x + b^2$

$$y = \int \frac{\sin 2x}{(a^2 - b^2)(\sin x)^2 + b^2} dx$$

Let, $\sin^2 x = t$

Differentiating both sides with respect to x

$$\frac{dt}{dx} = 2 \sin x \cos x$$

= sin 2x

⇒ dt = sin2x dx

$$y = \int \frac{dt}{(a^2 - b^2)t + b^2}$$

Use formula $\int \frac{1}{ct+d} dt = \frac{\log(ct+d)}{c}$

$$y = \frac{\log[(a^2 - b^2)t + b^2]}{(a^2 - b^2)} + c$$

Again, put t = $\sin^2 x$

$$y = \frac{\log[(a^2 - b^2)(\sin x)^2 + b^2]}{(a^2 - b^2)} + c$$

26. Question

Write a value of $\int \frac{a^x}{3 + a^x} dx$

Answer

Let, 3 + a^x = t

Differentiating both sides with respect to x

$$\frac{dt}{dx} = a^x \log a$$

$$\Rightarrow \frac{dt}{\log a} = a^x\, dx$$

$$y = \int \frac{1}{(\log a)t} dt$$

Use formula $\int \frac{1}{t} dt = \log t$

$$y = \frac{\log t}{\log a} + c$$

Again, put t = 3 + a^x

$$y = \frac{\log(3 + a^x)}{\log a} + c$$

27. Question

Write a value of $\int \frac{1 + \log x}{3 + x \log x} dx$

Answer

Let, x(log x) = t

Differentiating both sides with respect to x

$$\frac{dt}{dx} = x\frac{1}{x} + \log x = 1 + \log x$$

⇒ dt = (1 + log x)dx

$$y = \int \frac{1}{3 + t} dt$$

Use formula $\int \frac{1}{a+t} dt = \log(a + t)$

y = log(3 + t) + c

Again, put t = x(log x)

y = log(3 + x(log x)) + c

28. Question

Write a value of $\int \frac{\sin x}{\cos^3 x} dx$

Answer

Let, cos x = t

Differentiating both sides with respect to x

$$\frac{dt}{dx} = -\sin x$$

⇒ -dt = sin x dx

$$y = \int \frac{-1}{t^3} dt$$

Use formula $\int \frac{1}{t^n} dt = \frac{t^{-n+1}}{-n+1}$

$$y = -\frac{t^{-2}}{-2} + c$$

Again, put t = cos x

$$y = \frac{1}{2(\cos x)^2} + c$$

29. Question

Write a value of $\int \frac{\sin x - \cos x}{\sqrt{1+\sin 2x}} dx$

Answer

We know that

1 + sin2x = $\sin^2$x + $\cos^2$x + 2sinxcosx

= $(\sin x + \cos x)^2$

$$y = \int \frac{\sin x - \cos x}{\sqrt{(\sin x + \cos x)^2}} dx$$

$$y = \int \frac{(\sin x - \cos x)}{(\sin x + \cos x)} dx$$

Let, sin x + cos x = t

Differentiating both sides with respect to x

$$\frac{dt}{dx} = \cos x - \sin x$$

⇒ -dt = (sin x – cos x)dx

$$y = \int \frac{-1}{t} dt$$

Use formula $\int \frac{1}{t} = \log t$

y = -log t + c

Again, put t = sin x + cos x

y = -log(sin x + cos x) + c

30. Question

Write a value of $\int \frac{1}{x(\log x)^n} dx$

Answer

Let, log x = t

Differentiating both sides with respect to x

$$\frac{dt}{dx} = \frac{1}{x}$$

$$\Rightarrow dt = \frac{1}{x}dx$$

$$y = \int \frac{1}{t^n} dt$$

Use formula $\int \frac{1}{t^n} dt = \frac{t^{-n+1}}{-n+1}$

$$y = \frac{t^{-n+1}}{-n+1} + c$$

Again, put t = log x

$$y = \frac{(\log x)^{-n+1}}{-n+1} + c$$

31. Question

Write a value of $\int e^{ax} \sin bx dx$

Answer

we know ∫f(x)g(x)= f(x) ∫g(x)-∫f' (x) ∫g(x)

Let $\int e^{ax} \sin bx\, dx = i$

Given that $\int e^{ax} \sin bx\, dx$

$$i = \sin bx \int e^{ax} - \int b \cos bx \int e^{ax}$$

$$\mathrm{i} = \sin bx \frac{e^{ax}}{a} - \int b \cos bx \frac{e^{ax}}{a}$$

$$\mathrm{i} = \sin bx \frac{e^{ax}}{a} - \frac{1}{a}\left[b \cos bx \frac{e^{ax}}{a} - \frac{b^2}{a} \int e^{ax} \sin bx\, dx\right]$$

$$\mathrm{i} = \sin bx \frac{e^{ax}}{a} - \frac{b}{a^2} \cos bx\, e^{ax} + \frac{b^2}{a^2}\mathrm{i}$$

$$\mathrm{i}\left(1 - \frac{b^2}{a^2}\right) = \frac{a \sin bx\ e^{ax} - b \cos bx\ e^{ax}}{a^2}$$

$$\mathrm{i} = \frac{a \sin bx\ e^{ax} - b \cos bx\ e^{ax}}{a^2}\left(\frac{a^2}{a^2 - b^2}\right)$$

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 - b^2}$$

32. Question

Write a value of $\int e^{ax} \cos bx \, dx$.s

Answer

we know ∫f(x)g(x)= f(x) ∫g(x)-∫f' (x) ∫g(x)

Let $\int e^{ax} \cos bx \, dx = i$

Given that $\int e^{ax} \cos bx \, dx$

$$i = \cos bx \int e^{ax} - \int -b \sin bx \int e^{ax}$$

$$\mathrm{i} = \cos bx \frac{e^{ax}}{a} + \int b \sin bx \frac{e^{ax}}{a}$$

$$\mathrm{i} = \cos bx \frac{e^{ax}}{a} + \frac{1}{a}\left[b \sin bx \frac{e^{ax}}{a} - \frac{b^2}{a} \int e^{ax} \cos bx \, dx\right]$$

$$\mathrm{i} = \cos bx \frac{e^{ax}}{a} + \frac{b}{a^2} \sin bx \, e^{ax} - \frac{b^2}{a^2} \mathrm{i}$$

$$\mathrm{i}\left(1 + \frac{b^2}{a^2}\right) = \frac{a \cos bx \;\; e^{ax} + b \sin bx \;\; e^{ax}}{a^2}$$

$$\mathrm{i} = \frac{a \cos bx \;\; e^{ax} + b \sin bx \;\; e^{ax}}{a^2}\left(\frac{a^2}{a^2 + b^2}\right)$$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}(a \sin bx + b \cos bx)}{a^2 + b^2}$$

33. Question

Write a value of $\int e^{x}\left(\frac{1}{x} - \frac{1}{x^2}\right) dx$.

Answer

given $\int e^x \left(\frac{1}{x} - \frac{1}{x^2}\right) dx$

$$= \int \frac{e^x}{x} dx - \int \frac{e^x}{x^2} dx$$

$$= \int \frac{e^x}{x} dx - \left[\frac{e^x}{x^2} - \int -\frac{e^x}{x}\right] + c$$

$$= -\frac{e^x}{x^2} + c$$

34. Question

Write a value of $\int e^{ax} \, | a f(x) + f'(x) | \, dx$.

Answer

given $\int e^{ax} |af(x) + f'(x)| dx$

$$= a \int e^{ax} f(x) dx + \int e^{ax} f'(x) dx$$

$$= a\left[f(x)\frac{e^{ax}}{a} - \int f'(x)\frac{e^{ax}}{a}dx\right] + \int e^{ax}f'(x)dx$$

$$= f(x)e^{ax} + c$$

35. Question

Write a value of $\int\sqrt{4-x^2}\ dx.$

Answer

we know that $\int\sqrt{a^2-x^2}dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{x^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$

Given $\int\sqrt{4-x^2}$

$$= \int\sqrt{2^2-x^2}$$

$$= \frac{x\sqrt{2^2-x^2}}{2} + \frac{x^2}{2}\sin^{-1}\left(\frac{x}{2}\right)$$

$$= \frac{x\sqrt{4-x^2}}{2} + \frac{x^2}{2}\sin^{-1}\left(\frac{x}{2}\right) + c$$

36. Question

Write a value of $\int\sqrt{9+x^2}\ dx.$

Answer

we know that $\int\sqrt{x^2+a^2}dx = \frac{x\sqrt{x^2-a^2}}{2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + c|$

Given $\int x^2+9$

$$= \int x^2+3^2$$

$$= \frac{x\sqrt{x^2+3^2}}{2} + \frac{3^2}{2}\log\left|x+\sqrt{x^2+3^2}\right|$$

$$= \frac{x\sqrt{x^2+9}}{2} + \frac{9}{2}\log\left|x+\sqrt{x^2+9}\right| + c$$

37. Question

Write a value of $\int\sqrt{x^2-9}dx$

Answer

we know that $\int\sqrt{x^2-a^2}dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c$

Given $\int\sqrt{x^2-9}\,dx$

$$= \int\sqrt{x^2-3^2}\,dx$$

$$= \frac{x\sqrt{x^2-3^2}}{2} - \frac{3^2}{2}\log\left|x+\sqrt{x^2-3^2}\right|$$

$$= \frac{x\sqrt{x^2-9}}{2} - \frac{9}{2}\log\left|x+\sqrt{x^2-9}\right| + c$$

38. Question

Evaluate: $\int \frac{x^2}{1+x^3}$

Answer

let $1 + x^3 = t$

Differentiating on both sides we get,

$3x^2 dx = dt$

$x^2 dx = \frac{1}{3} dt$

substituting it in $\int \frac{x^2}{1+x^3} dx$ we get,

$= \int \frac{1}{3t} dt$

$= \frac{1}{3} \log t + c$

$= \frac{1}{3} \log(1 + x^3) + c$

39. Question

Evaluate: $\int \frac{x^2 + 4x}{x^3 + 6x^2 + 5} dx$

Answer

let $x^3 + 6x^2 + 5 = t$

Differentiating on both sides we get,

$(3x^2 + 12x)dx = dt$

$3(x^2 + 4x)dx = dt$

$(x^2 + 4x)dx = \frac{1}{3} dt$

Substituting it in $\int \frac{x^2+4x}{x^3+6x^2+5} dx$ we get,

$= \int \frac{1}{3t} dt$

$= \frac{1}{3\log(x^3 + 6x^2 + 5)} + c$

40. Question

Evaluate: $\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}} dx$

Answer

let $\sqrt{x} = t$

Differentiating on both sides we get,

$$\frac{1}{2\sqrt{x}}dx = dt$$

$$\frac{1}{\sqrt{x}}dx = 2dt$$

substituting it in $\int \frac{sec^2\sqrt{x}}{\sqrt{x}}dx$ we get,

$$= \int 2sec^2 t\, dt$$

=2 tan t+c

$$= 2\tan\sqrt{x} + c$$

41. Question

Evaluate: $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx.$

Answer

let $\sqrt{x} = t$

Differentiating on both sides we get,

$$\frac{1}{2\sqrt{x}}dx = dt$$

$$\frac{1}{\sqrt{x}}dx = 2dt$$

substituting it in $\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx$ we get,

=∫2 sin t dt

=-2 cos t+c

$$= -2\cos\sqrt{x} + c$$

42. Question

Evaluate: $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx.$

Answer

let $\sqrt{x} = t$

Differentiating on both sides we get,

$$\frac{1}{2\sqrt{x}}dx = dt$$

$$\frac{1}{\sqrt{x}}dx = 2\mathrm{dt}$$

substituting it in $\int \frac{cos\sqrt{x}}{\sqrt{x}}dx$ we get,

=∫2cos t dt

=2 sin t+c

$= 2 \sin \sqrt{x} + c$

43. Question

Evaluate: $\int \frac{(1+\log x)^2}{x} dx.$

Answer

let 1 + log x = t

Differentiating on both sides we get,

$\frac{1}{x} dx = dt$

Substituting it in $\int \frac{(1+\log x)^2}{x}$ we get,

$= \int t^2 \, dt$

$= \frac{t^3}{3} + c$

$= \frac{(1+\log x)^3}{3} + c$

44. Question

Evaluate: $\int \sec^2(7-4x) \, dx.$

Answer

let 7 – 4x = t

Differentiating on both sides we get,

-4 dx = dt

$dx = -\frac{1}{4} dt$

substituting it in $\int sec^2(7-4x) dx$ we get,

$= \int -\frac{1}{4} sec^2 t \, dt$

=tan t+c

=tan (7-4x)+c

45. Question

Evaluate: $\int \frac{\log x^x}{x} dx.$

Answer

given $\int \frac{\log x^x}{x} dx$

$= \int \frac{x \log x}{x} dx$

$= \int \log x$

=x log x-x + c

1. Question

Write a value of $\int \frac{1+\cot x}{x+\log \sin x} dx.$

Answer

let x + log sin x= t

Differentiating it on both sides we get,

(1+cot x) dx=dt - i

Given that $\int \frac{1+\cot x}{x+\log \sin x} dx$

Substituting i in above equation we get,

$$= \int \frac{dt}{t}$$

=log t + c

= log(x + log sin x) + c

2. Question

Write a value of $\int e^{3\log x} x^4 dx.$

Answer

Consider $\int e^{3 \log x} x^4$

$e^{3\log x} = e^{\log x^3}$

$= x^3$

$\int e^{3 \log x} x^4 = \int x^3 x^4 dx$

$= \int x^7 \, dx$

$$= \frac{x^8}{8} + c$$

3. Question

Write a value of $\int x^2 \sin x^3 dx.$

Answer

let $x^3 = t$

Differentiating on both sides we get,

$3x^2 \, dx = dt$

$$x^2 dx = \frac{1}{3} dt$$

substituting above equation in $\int x^2 \sin x^3 \, dx$ we get,

$$= \int \frac{1}{3} \sin t \, dt$$

$$= -\frac{1}{3} \cos t + c$$

$= -\frac{1}{3}\cos x^3 + c$

4. Question

Write a value of $\int \tan^3 x \sec^2 x \, dx.$

Answer

let tan x = t

Differentiating on both sides we get,

$\sec^2 x \, dx = dt$

Substituting above equation in $\int \tan^3 x \sec^2 x \, dx$ we get,

$= \int t^3 \, dt$

$= \frac{t^4}{4} + c$

$= \frac{\tan^4 x}{4} + c$

5. Question

Write a value of $\int e^x (\sin x + \cos x) dx.$

Answer

we know $\int e^x (f(x) + f'(x))dx = e^x f(x) + c$

Given, $\int e^x (\sin x + \cos x) \, dx$

Here $f(x) = \sin x$ and $f'(x) = \cos x$

Therefore $\int e^x(\sin x + \cos x)\, dx = e^x \sin x + c$

6. Question

Write a value of $\int \tan^6 x \sec^2 x \, dx.$

Answer

let tan x=t

Differentiating on both sides we get,

$\sec^2 x \, dx = dt$

Substituting above equation in $\int \tan^3 x \sec^2 x \, dx$ we get,

$= \int t^6 \, dt$

$= \frac{t^7}{7} + c$

$= \frac{\tan^7 x}{7} + c$

7. Question

Write a value of $\int \frac{\cos x}{3 + 2\sin x} dx.$

Answer

let 3+2sin x=t

Differentiating on both sides we get,

2cos x dx=dt

$\cos x\, dx = \frac{1}{2} dt$

Substituting above equation in $\int \frac{\cos x}{3+2\sin x} dx$ we get,

$\int \frac{1}{2t} dt$

$= \frac{1}{2} \log t + c$

$= \frac{1}{2} \log(3 + 2\sin x) + c$

8. Question

Write a value of $\int e^{x} \sec x(1 + \tan x) dx.$

Answer

given,

$\int e^x \sec x(1 + \tan x)\, dx = \int e^x (\sec x + \sec x \tan x)\, dx$

$= e^x \sec x + c$

$\because \int e^x (f(x) + f'(x))\, dx = e^x f(x) + c$

9. Question

Write a value of $\int \frac{\log x^n}{x} dx.$

Answer

let $\log x^n = t$

Differentiating on both sides we get,

$\frac{1}{x^n} n x^{n-1} dx = dt$

$\frac{n}{x} dx = dt$

$\frac{1}{x} dx = \frac{1}{n} dt$

Substituting above equations in $\int \frac{\log x^n}{x} dx$ we get,

$\int \frac{1}{n} t\, dt$

$= \frac{1}{n} \frac{t^2}{2} + c$

$= \frac{(\log x^n)^2}{2n} + c$

10. Question

Write a value of $\int \frac{(\log x)^n}{x} dx.$

Answer

let log x=t

Differentiating on both sides we get,

$\frac{1}{x} dx = dt$

Substituting above equations in $\int \frac{(\log x)^n}{x} dx$ we get,

$\int t^n dt$

$= \frac{t^{n+1}}{n+1} + c$

$= \frac{(\log x)^{n+1}}{n+1} + c$

11. Question

Write a value of $\int e^{\log \sin x} \cos x \, dx.$

Answer

given $\int e^{\log \sin x} \cos x \, dx$

$= \int \sin x \cos x \, dx$ ($\because e^{\log x} = x$)

Let $\sin x = t$

Differentiating on both sides we get,

Cos x dx=dt

Substituting above equations in given equation we get,

$= \int t \, dt$

$= \frac{t^2}{2} + c$

$= \frac{\sin^2 x}{2} + c$

12. Question

Write a value of $\int \sin^3 x \cos x \, dx.$

Answer

let sin x=t

Differentiating on both sides we get,

Cos x dx=dt

Substituting above equation in $\int \sin^3 x \cos x \, dx$ we get,

$= \int t^3 \, dt$

$= \frac{t^4}{4} + c$

$= \frac{sin^4 x}{4} + c$

13. Question

Write a value of $\int \cos^4 x \sin x \, dx.$

Answer

let cos x=t

Differentiating on both sides we get,

-sin x dx=dt

Substituting above equation in $\int \cos^4 x \sin x \, dx$ we get,

$= \int -t^4 \, dt$

$= -\frac{t^5}{5} + c$

$= -\frac{cos^5 x}{5} + c$

14. Question

Write a value of $\int \tan x \sec^3 x \, dx.$

Answer

given $\int \tan x \sec^3 x \, dx$

$= \int (\tan x \sec x) \sec^2 x \, dx$

Let sec x=t

Differentiating on both sides we get,

tan x sec x dx=dt

Substituting above equation in $\int \tan x \sec^3 x \, dx$ we get,

$= \int t^2 \, dt$

$= \frac{t^3}{3} + c$

$= \frac{sec^3 x}{3} + c$

15. Question

Write a value of $\int \frac{1}{1+e^x} dx.$

Answer

given $\int \frac{1}{1+e^x} dx$

$= \int \left(1 - \frac{e^x}{1+e^x}\right) dx$

Let $1+e^x = t$

Differentiating on both sides we get,

$E^x\ dx = dt$

Substituting above equation in given equation we get,

$= \int \left(1 - \frac{1}{t}\right) dt$

$= t - \log t + c$

$= 1+e^x - \log(1+e^x) + c$

46. Question

Evaluate: $\int 2^x\ dx.$

Answer

Given, $\int 2^x\ dx.$

$= \frac{2^x}{\log 2} + c$ [since, $\int a^x\ dx = \frac{a^x}{\log a}$]

47. Question

Evaluate: $\int \frac{1-\sin x}{\cos^2 x}\ dx.$

Answer

Given, $\int \frac{1-\sin x}{\cos^2 x}\ dx.$

$= \int \frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\ dx$

$= \int \sec^2 x - \tan x.\sec x\ dx$ [since, $\cos x = \frac{1}{\sec x}$]

$= \tan x - \sec x + c$

48. Question

Evaluate: $\int \frac{x^3 - 1}{x^2}\ dx.$

Answer

Given, $\int \frac{x^3-1}{x^2}\ dx.$

$= \int \frac{x^3}{x^2} - \frac{1}{x^2}\ dx$

$= \int x - \frac{1}{x^2}\ dx$

[since, $\int x^n\ dx = \frac{x^{n+1}}{n+1}$]

$= \frac{x^2}{2} - \frac{x^{-2+1}}{-2+1} + c$

$=\frac{x^2}{2}-\frac{x^{-1}}{-1}+c$

$=\frac{x^2}{2}+\frac{1}{x}+c$

49. Question

Evaluate: $\int \frac{x^3-x^2+x-1}{x-1}\,dx.$

Answer

Given, $\int \frac{x^3-x^2+x-1}{x-1}\,dx.$

$=\int \frac{x^2(x-1)+x-1}{x-1}\,dx$

$=\int \frac{(x-1)[x^2+1]}{x-1}\,dx$

$=\int(x^2+1)dx$ [since, $\int x^n\,dx=\frac{x^{n+1}}{n+1}$]

$=\frac{x^3}{3}+x+c$

50. Question

Evaluate: $\int \frac{e^{\tan^{-1}}}{1+x^2}\,dx.$

Answer

Given, $\int \frac{e^{tan^{-1}}}{1+x^2}\,dx.$

Let $\tan^{-1}x=t$

ð $\frac{dy}{dx}(Tan^{-1}x)$ =dt

ð $\frac{1}{1+x^2}\,dx=dt$

Now, $\int \frac{e^{tan^{-1}}}{1+x^2}\,dx.$

$=\int e^t\,dt$

$=e^t+c$

$=e^{\tan^{-1}x}+c$

51. Question

Evaluate: $\int \frac{1}{\sqrt{1-x^2}}\,dx.$

Answer

Given,

$\int \frac{1}{\sqrt{1-x^2}} dx.$

$=\sin^{-1}x + c$

(It is a standard formula).

52. Question

Evaluate: $\int \sec x(\sec x + \tan x)\, dx.$

Answer

Given, $\int \sec x (\sec x + \tan x)\, dx$

$=\int (\sec^2 x + \sec x. \tan x)\, dx$

$= \tan x + \sec x + c$

53. Question

Evaluate: $\int \frac{1}{x^2+16} dx.$

Answer

Given, $\int \frac{1}{x^2+16} dx.$

We know that, $\int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$

By comparison, a=4

$= \frac{1}{4} tan^{-1}\frac{x}{4} + c$

54. Question

Evaluate: $\int (1-x)\sqrt{x}\, dx.$

Answer

Given, $\int (1-x)\sqrt{x}$ dx

$= \int (\sqrt{x} - x\sqrt{x})\, dx$

$= \int (x^{\frac{1}{2}} - x.x^{\frac{1}{2}})dx$

$= \int x^{\frac{1}{2}} - x^{\frac{3}{2}}\, dx$

$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + c$ [since, $\int x^n\, dx = \frac{x^{n+1}}{n+1}$]

$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + c$

$= \frac{2}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}} + c$

55. Question

Evaluate: $\int \frac{x+\cos 6x}{3x^2+\sin 6x}dx.$

Answer

Given,

$$\int \frac{x+\cos 6x}{3x^2+\sin 6x}dx.$$

Let $3x^2 + \sin 6x = t$

$\Rightarrow \frac{d}{dx}(3x^2 + sin6x)$ =dt

$\Rightarrow$ 6x + cos 6x. 6=dt

$\Rightarrow x + \cos 6x = \frac{dt}{6}$

Substituting the values,

$= \int \frac{1}{6t}\, dt$

$= \frac{1}{6}\log t + c$

$= \frac{1}{6}\log(3x^2 + \sin 6x) + c$

56. Question

If $\int\left(\frac{x-1}{x^2}\right)e^x\,dx = f(x)e^x + C,$ then write the value of f(x).

Answer

Consider, $\int \frac{x-1}{x^2}\, e^x\, dx$

$= \int \frac{x}{x^2} - \frac{1}{x^2}\, e^x\, dx$

$= \int \frac{1}{x} - \frac{1}{x^2}\, e^x\, dx$

It is clearly of the form,

$$\int e^x[f(x) + f^I(x)]dx = e^x f(x) + c$$

By comparison, $f(x)=\frac{1}{x}$; $f^I(x)= -\frac{1}{x^2}$

$$= e^x\,\frac{1}{x} + c$$

Therefore, the value of $f(x)=\frac{1}{x}$

57. Question

If $\int e^x(\tan x + 1)\sec x\, dx = e^x f(x) + C,$ then write the value f(x).

Answer

Given, $\int e^x(tanx + 1)secx\ dx$

It is clearly of the form,

$$\int e^x[f(x) + f^I(x)]dx = e^x f(x) + c$$

By comparison, f(x)=1+$\tan x$; f^I(x)=$\sec x$

= e^x (1+tanx) +C

Therefore, the value of f(x)=1+tanx

58. Question

Evaluate: $\int \frac{2}{1-\cos 2x} dx$

Answer

Given, $\int \frac{2}{1-cos2x}\ dx$

We Know that, cos2x=1-2$\sin^2$x

⇒ 1-cos2x=2$\sin^2$x

Substitute this in the given,

= $\int \frac{2}{2\sin^2 x}$ dx

= $\int \frac{1}{\sin^2 x}$ dx

= $\int$cosec^2 x dx

= -cotx +c

59. Question

Write the anti-derivative of $\left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right)$.

Answer

Anti-derivative is nothing but integration

Therefore its Anti-derivative can be found by integrating the above given equation.

$$= \int 3\sqrt{x} + \frac{1}{\sqrt{x}}\ dx$$

$$= \int 3x^{\frac{1}{2}} + x^{-\frac{1}{2}}\ dx$$

$$= 3\ \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c\ [since, \int x^n\ dx = \frac{x^{n+1}}{n+1}]$$

$$= 3\ \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= 2\ x^{\frac{3}{2}}\ +\ 2\ x^{\frac{1}{2}}\ + c$$

$= 2(x^{\frac{3}{2}} + x^{\frac{1}{2}}) + c$

60. Question

Evaluate: $\int \cos^{-1}(\sin x)dx$

Answer

Given, $\int \cos^{-1}(\sin x)\, dx$

Let us consider, $\int \cos^{-1} dx$

We know that, $\int f(x).g(x)\, dx = f(x) \int g(x)\, dx - \int [f^{I}(x) \int g(x)]\, dx$

By comparison, $f(x) = \cos^{-1} x$; $g(x)=1$

$= \cos^{-1} x\, x \int 1\, dx - \int -\frac{1}{\sqrt{1-x^2}}.x\, dx$

$= x\cos^{-1} x - \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}(-2x)\, dx$

$= x\cos^{-1} x - \frac{1}{2}\int (1-x^2)^{-\frac{1}{2}}(-2x)\, dx$

$= x\cos^{-1} x - \frac{1}{2}\frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$ (since, $\int [f(x)^n.f^{I}(x)]dx = \frac{f(x)^{n+1}}{n+1}$)

$= x\cos^{-1} x - (1-x^2)^{1/2} + c$

$= x\cos^{-1} x - \sqrt{1-x^2} + c$

Therefore, $\int \cos^{-1} x\, dx = x\cos^{-1} x - \sqrt{1-x^2} + c$

Replace 'x' with '$\sin x$' :-

ð $\int \cos^{-1}(sinx)\, dx = sinx.\cos^{-1}(sinx) - \sqrt{1-(sinx)^2} + c$

$= sinx.\cos^{-1} x\,(sinx) - \sqrt{cos^2 x} + c$

$= sinx.\cos^{-1} x\,(sinx) - cosx + c$

61. Question

Evaluate: $\int \frac{1}{\sin^2 x \cos^2 x}dx$

Answer

Given, $\int \frac{1}{\sin^2 x.\cos^2 x}dx$

$=\int \frac{\sin^2 x+\cos^2 x}{\sin^2 x.\cos^2 x}dx$ [since, $\sin^2 x+\cos^2 x=1$]

$=\int \frac{\sin^2 x}{\sin^2 x.\cos^2 x} + \frac{\cos^2 x}{\sin^2 x.\cos^2 x}\, dx$

$=\int \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}\, dx$

$=\int (\sec^2 x + cosec^2 x)dx$

$= \tan x - \cot x + c$

62. Question

Evaluate: $\int \frac{1}{x(1+\log x)} dx$

Answer

Given, $\int \frac{1}{x(1+logx)}\, dx$

Let 1+log x=t

$\Rightarrow \frac{d}{dx}(1+logx) = dt$

$\Rightarrow \frac{1}{x}\, dx \;=\; dt$

$= \int \frac{1}{t} dt$

=logt +c

=log (1+logx)+c

MCQ

18. Question

Mark the correct alternative in each of the following:

Evaluate $\int \frac{x+3}{(x+4)^2} e^x dx =$

A. $\frac{e^x}{x+4} + C$

B. $\frac{e^x}{x+3} + C$

C. $\frac{1}{(x+4)^2} + C$

D. $\frac{e^x}{(x+4)^2} + C$

Answer

$\int \frac{x+3}{(x+4)^2} e^x dx$

$= \int \frac{x+4}{(x+4)^2} e^x dx - \int \frac{1}{(x+4)^2} e^x dx$

$= \int e^x \left(\frac{1}{x+4} dx - \frac{1}{(x+4)^2} dx\right)$

$\left[\because f(x) = \frac{1}{x+4}\, ; f'(x) = -\frac{1}{(x+4)^2}\right]$

$= e^x \left(\frac{1}{x+4}\right) + c$

$\because \{\int e^{x} f(x)+f'x]=e^{x} f(x)\}$

18. Question

Mark the correct alternative in each of the following:

Evaluate $\int \frac{x+3}{(x+4)^2} e^x dx =$

A. $\frac{e^x}{x+4}+C$

B. $\frac{e^x}{x+3}+C$

C. $\frac{1}{(x+4)^2}+C$

D. $\frac{e^x}{(x+4)^2}+C$

Answer

$\int \frac{x+3}{(x+4)^2} e^x dx$

$= \int \frac{x+4}{(x+4)^2} e^x dx - \int \frac{1}{(x+4)^2} e^x dx$

$= \int e^x \left(\frac{1}{x+4} dx - \frac{1}{(x+4)^2} dx\right)$

$\left[\because f(x) = \frac{1}{x+4}\ ; f'(x) = -\frac{1}{(x+4)^2}\right]$

$= e^x \left(\frac{1}{x+4}\right) + c$

$\because \{\int e^{x} f(x)+f'x]=e^{x} f(x)\}$

19. Question

Mark the correct alternative in each of the following:

Evaluate $\int \frac{\sin x}{3+4\cos^2 x} dx$

A. $\log\left(3+4\cos^x x\right)+C$

B. $\frac{1}{2\sqrt{3}} \tan^{-1}\left(\frac{\cos x}{\sqrt{3}}\right)+C$

C. $-\frac{1}{2\sqrt{3}} \tan^{-1}\left(\frac{2\cos x}{\sqrt{3}}\right)+C$

D. $\frac{1}{2\sqrt{3}} \tan^{-1}\left(\frac{2\cos x}{\sqrt{3}}\right)+C$

Answer

$\int \frac{\sin x}{3+4(\cos x)^2} dx$

⇒ cos x=t then ;

⇒-sin (x)dx=dt

$= -\int \frac{dt}{3+4t^2}$ ($\int \frac{dt}{a+bt^2} = \frac{1}{\sqrt{ab}}\tan^{-1}\sqrt{\frac{b}{a}}$)

$= -\frac{1}{2\sqrt{3}}\tan^{-1}\sqrt{\frac{4}{3}}t$ put $(\cos x = t)$

$\Rightarrow -\frac{1}{2\sqrt{3}} tan^{-1}\left(\frac{2\cos x}{\sqrt{3}}\right) + C$

19. Question

Mark the correct alternative in each of the following:

Evaluate $\int \frac{\sin x}{3+4\cos^2 x} dx$

A. $\log\left(3+4\cos^{x} x\right)+C$

B. $\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{\cos x}{\sqrt{3}}\right)+C$

C. $-\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{2\cos x}{\sqrt{3}}\right)+C$

D. $\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{2\cos x}{\sqrt{3}}\right)+C$

Answer

$\int \frac{\sin x}{3+4(\cos x)^2} dx$

⇒ cos x=t then ;

⇒-sin (x)dx=dt

$= -\int \frac{dt}{3+4t^2}$ ($\int \frac{dt}{a+bt^2} = \frac{1}{\sqrt{ab}}\tan^{-1}\sqrt{\frac{b}{a}}$)

$= -\frac{1}{2\sqrt{3}}\tan^{-1}\sqrt{\frac{4}{3}}t$ put $(\cos x = t)$

$\Rightarrow -\frac{1}{2\sqrt{3}} tan^{-1}\left(\frac{2\cos x}{\sqrt{3}}\right) + C$

20. Question

Mark the correct alternative in each of the following:

Evaluate $\int e^x \left(\frac{1-\sin x}{1-\cos x}\right) dx$

A. $-e^x \tan\frac{x}{2}+C$

B. $-e^{x}\cot\frac{x}{2}+C$

C. $-\frac{1}{2}e^{x}\tan\frac{x}{2}+C$

D. $-\frac{1}{2}e^{x}\cot\frac{x}{2}+C$

Answer

Given, $\int e^{x}(\frac{1-\sin x}{1-\cos x})dx$

$=-\int e^{x}(\frac{\sin x}{1-\cos x}-\frac{1}{1-\cos x})dx$ $\{\int e^{x}[f(x)+f'(x)]=e^{x}f(x)\}$

$\Rightarrow f(x)=\frac{\sin x}{1-\cos x};\ f'(x)=-\frac{1}{1-\cos x}$

$=-e^{x}\left(\frac{\sin x}{1-\cos x}\right)$

$\because\left[\frac{\sin x}{1-\cos x}=\cot\frac{x}{2}\right]$

$=-e^{x}\cot\frac{x}{2}+c$

20. Question

Mark the correct alternative in each of the following:

Evaluate $\int e^{x}\left(\frac{1-\sin x}{1-\cos x}\right)dx$

A. $-e^{x}\tan\frac{x}{2}+C$

B. $-e^{x}\cot\frac{x}{2}+C$

C. $-\frac{1}{2}e^{x}\tan\frac{x}{2}+C$

D. $-\frac{1}{2}e^{x}\cot\frac{x}{2}+C$

Answer

Given, $\int e^{x}(\frac{1-\sin x}{1-\cos x})dx$

$=-\int e^{x}(\frac{\sin x}{1-\cos x}-\frac{1}{1-\cos x})dx$ $\{\int e^{x}[f(x)+f'(x)]=e^{x}f(x)\}$

$\Rightarrow f(x)=\frac{\sin x}{1-\cos x};\ f'(x)=-\frac{1}{1-\cos x}$

$=-e^{x}\left(\frac{\sin x}{1-\cos x}\right)$

$\because\left[\frac{\sin x}{1-\cos x}=\cot\frac{x}{2}\right]$

$= -e^{x} cot\frac{x}{2} + c$

21. Question

Mark the correct alternative in each of the following:

Evaluate $\int \frac{2}{\left(e^{x}+e^{-x}\right)^{2}} dx$

A. $\frac{-e^{-x}}{e^{x}+e^{-x}}+C$

B. $-\frac{1}{e^{x}+e^{-x}}+C$

C. $\frac{-1}{\left(e^{x}+1\right)^{2}}+C$

D. $\frac{1}{e^{x}-e^{-x}}+C$

Answer

Given $\int \frac{2}{(e^{x}+e^{-x})^{2}} dx$

$= \int \frac{2e^{2x}}{(e^{2x}+1)^{2}} dx$

if $t=e^{2x}+1$

;then $\frac{dt}{dx} = 2e^{2x}$

$\Rightarrow \int \frac{dt}{t^{2}} = -\frac{1}{t} + c$

$\Rightarrow -\frac{1}{e^{2x}+1} + c$

$= \frac{-e^{-x}}{e^{x}+e^{-x}} + C$

21. Question

Mark the correct alternative in each of the following:

Evaluate $\int \frac{2}{\left(e^{x}+e^{-x}\right)^{2}} dx$

A. $\frac{-e^{-x}}{e^{x}+e^{-x}}+C$

B. $-\frac{1}{e^{x}+e^{-x}}+C$

C. $\frac{-1}{\left(e^{x}+1\right)^{2}}+C$

D. $\frac{1}{e^{x}-e^{-x}}+C$

Answer

Given$\int \frac{2}{(e^x+e^{-x})^2}dx$

$= \int \frac{2e^{2x}}{(e^{2x}+1)^2}dx$

if t=e^{2x} +1

;then $\frac{dt}{dx} = 2e^{2x}$

$\Rightarrow \int \frac{dt}{t^2} = -\frac{1}{t} + c$

$\Rightarrow -\frac{1}{e^{2x}+1} + c$

$= \frac{-e^{-x}}{e^x + e^{-x}} + C$

22. Question

Mark the correct alternative in each of the following:

Evaluate $\int \frac{e^{x}(1+x)}{\cos^{2}\left(xe^{x}\right)}dx =$

A. 2 $\log_e$ cos (xe^x) + C

B. sec (xe^x) + C

C. tan (xe^x) + C

D. tan (x + e^x) +C

Answer

let (t)=xe^x;

$\frac{dt}{dx} = e^x(1+x)$

$\Rightarrow \int \frac{dt}{(\cos t)^2} = \int (\sec t)^2\, dt$

= tan t

(put (t)= xe^x)

= tan (xe^x) + c

22. Question

Mark the correct alternative in each of the following:

Evaluate $\int \frac{e^x(1+x)}{\cos^2(xe^x)}dx =$

A. 2 $\log_e \cos(xe^x) + C$

B. $\sec(xe^x) + C$

C. $\tan(xe^x) + C$

D. $\tan(x + e^x) + C$

Answer

let (t)=xe^x;

$\frac{dt}{dx} = e^x(1+x)$

$\Rightarrow \int \frac{dt}{(\cos t)^2} = \int (\sec t)^2\, dt$

= tan t

(put (t)= xe^x)

= tan (xe^x) + c

23. Question

Mark the correct alternative in each of the following:

Evaluate $\int \frac{\sin^2 x}{\cos^4 x}dx =$

A. $\frac{1}{3}\tan^2 x + C$

B. $\frac{1}{2}\tan^2 x + C$

C. $\frac{1}{3}\tan^3 x + C$

D. none of these

Answer

I $= \int (\tan x)^2 (\sec x)^2\, dx$

$\Rightarrow$ tanx =t $\left[\frac{dt}{dx} = (\sec x)^2\right]$

$\Rightarrow \int t^2\, dt = \frac{t^3}{3} + c$

$\Rightarrow I = \frac{1}{3}(\tan x)^3$+c

23. Question

Mark the correct alternative in each of the following:

Evaluate $\int \frac{\sin^2 x}{\cos^4 x}dx =$

A. $\frac{1}{3}\tan^2 x + C$

B. $\frac{1}{2}\tan^2 x + C$

C. $\frac{1}{3}\tan^3 x + C$

D. none of these

Answer

I $=\int (\tan x)^2 (\sec x)^2\, dx$

$\Rightarrow$ tanx =t $\left[\frac{dt}{dx} = (\sec x)^2\right]$

$\Rightarrow \int t^2\, dt \; = \frac{t^3}{3} + c$

$\Rightarrow I = \frac{1}{3}(\tan x)^3$+c

24. Question

Mark the correct alternative in each of the following:

The primitive of the function $f(x) = \left(1 - \frac{1}{x^2}\right)a^{x+\frac{1}{x}}, a > 0$ is

A. $\frac{a^{x+\frac{1}{x}}}{\log_e a}$

B. $\log_e a . a^{x+\frac{1}{x}}$

C. $\frac{a^{x+\frac{1}{x}}}{x}\log_e a$

D. $x\frac{a^{x+\frac{1}{x}}}{\log_e a}$

Answer

I$=\int \left(1 - \frac{1}{x^2}\right) a^{x+\frac{1}{x}} dx$

$\Rightarrow let\ x + \frac{1}{x} = t;$

$1 - \frac{1}{x^2} = \frac{dt}{dx}$

$=\int a^t\, dt$

$\Rightarrow I = \frac{a^t}{\log_e a}\left(put\ t = x + \frac{1}{x}\right)$

$\Rightarrow I = \frac{a^{x+\frac{1}{x}}}{lo\,g_e\,a} + C$

24. Question

Mark the correct alternative in each of the following:

The primitive of the function $f(x) = \left(1 - \frac{1}{x^2}\right) a^{x+\frac{1}{x}}, a > 0$ is

A. $\frac{a^{x+\frac{1}{x}}}{\log_e a}$

B. $\log_e a . a^{x+\frac{1}{x}}$

C. $\frac{a^{x+\frac{1}{x}}}{x} \log_e a$

D. $x \frac{a^{x+\frac{1}{x}}}{\log_e a}$

Answer

$I = \int \left(1 - \frac{1}{x^2}\right) a^{x+\frac{1}{x}} dx$

$\Rightarrow let\ x + \frac{1}{x} = t;$

$1 - \frac{1}{x^2} = \frac{dt}{dx}$

$= \int a^t\, dt$

$\Rightarrow I = \frac{a^t}{\log_e a} \left(put\ t = x + \frac{1}{x}\right)$

$\Rightarrow I = \frac{a^{x+\frac{1}{x}}}{lo\,g_e\,a} + C$

25. Question

Mark the correct alternative in each of the following:

The value of $\int \frac{1}{x + x \log x} dx$ is

A. 1 + logx

B. x + logx

C. x log(1 + logx)

D. log (1 + logx)

Answer

$I = \int \frac{1}{x(1+\log_e x)} dx$

⇒let$(1+\log_e x)=t \left[\frac{dt}{dx}=\frac{1}{x}\right]$

$\Rightarrow \int \frac{1}{t} dt = \log_e t$

⇒I=log(1+log x)+C

25. Question

Mark the correct alternative in each of the following:

The value of $\int \frac{1}{x + x\log x} dx$ is

A. 1 + logx

B. x + logx

C. x log(1 + logx)

D. log (1 + logx)

Answer

$I = \int \frac{1}{x(1+\log_e x)} dx$

⇒let$(1+\log_e x)=t \left[\frac{dt}{dx}=\frac{1}{x}\right]$

$\Rightarrow \int \frac{1}{t} dt = \log_e t$

⇒I=log(1+log x)+C

26. Question

Mark the correct alternative in each of the following:

$\int \sqrt{\frac{x}{1-x}} dx$ is equal to

A. $\sin^{-1}\sqrt{x} + C$

B. $\sin^{-1}\left(\sqrt{x} - \sqrt{x(1-x)}\right) + C$

C. $\sin^{-1}\left\{\sqrt{x(1-x)}\right\} + C$

D. $\sin^{-1}\sqrt{x} - \sqrt{x(1-x)} + C$

Answer

let x=$(\sin t)^2$; (dx=2$\sin t \cos t \, dt$)

$$I = \int \sqrt{\frac{(\sin t)^2}{1-(\sin t)^2}} \times 2 \sin t \cos t \, dt$$

I= ∫ $(\sin t)^2$ dt

I=∫(1-cos 2t)dt

I=∫1dt -∫cos 2t dt

$$I = t - \frac{\sin 2t}{2} + c\ [t = \sin^{-1}\sqrt{x}](\cos t = \sqrt{1-x})$$

$$I = \sin^{-1}(\sqrt{x}) - (\sqrt{x}\sqrt{1-x}) + c$$

26. Question

Mark the correct alternative in each of the following:

$\int \sqrt{\frac{x}{1-x}}\,dx$ is equal to

A. $\sin^{-1}\sqrt{x} + C$

B. $\sin^{-1}\left(\sqrt{x} - \sqrt{x(1-x)}\right) + C$

C. $\sin^{-1}\left\{\sqrt{x(1-x)}\right\} + C$

D. $\sin^{-1}\sqrt{x} - \sqrt{x(1-x)} + C$

Answer

let x=$(\sin t)^2$; (dx=2$\sin t \cos t\, dt$)

$$I = \int \sqrt{\frac{(\sin t)^2}{1-(\sin t)^2}} \times 2 \sin t \cos t\, dt$$

I= ∫ (sin t)^2 dt

I=∫(1-cos 2t)dt

I=∫1dt -∫cos 2t dt

$$I = t - \frac{\sin 2t}{2} + c\ [t = \sin^{-1}\sqrt{x}](\cos t = \sqrt{1-x})$$

$$I = \sin^{-1}(\sqrt{x}) - (\sqrt{x}\sqrt{1-x}) + c$$

27. Question

Mark the correct alternative in each of the following:

$$\int e^x\{f(x) + f'(x)\}dx =$$

A. e^x f(x) + C

B. e^x + f(x) + C

C. 2e^x f(x) + C

D. e^x – f(x) + C

Answer

let I=$\int e^x(f(x) + f'(x))$dx

Open the brackets, we get

I={∫ e^x f(x) dx + ∫ e^x f′(x) dx}

=U+∫ e^x f′(x) dx

U=∫e^x f(x)dx

To solve U using integration by parts

$U = f(x) \int e^x dx - \int [f'(x) \int e^x]$

$= f(x) e^x - \int f'(x) e^x$

$= U + \int e^x f'(x) dx$

$I = e^x f(x) + \int f'(x) e^x dx - \int e^x f'(x) dx$

$I = e^x f(x) + c$

27. Question

Mark the correct alternative in each of the following:

$$\int e^x \{f(x) + f'(x)\} dx =$$

A. $e^x f(x) + C$

B. $e^x + f(x) + C$

C. $2e^x f(x) + C$

D. $e^x - f(x) + C$

Answer

let $I = \int e^x (f(x) + f'(x)) dx$

Open the brackets, we get

$I = \{\int e^x f(x) dx + \int e^x f'(x) dx\}$

$= U + \int e^x f'(x) dx$

$U = \int e^x f(x) dx$

To solve U using integration by parts

$U = f(x) \int e^x dx - \int [f'(x) \int e^x]$

$= f(x) e^x - \int f'(x) e^x$

$= U + \int e^x f'(x) dx$

$I = e^x f(x) + \int f'(x) e^x dx - \int e^x f'(x) dx$

$I = e^x f(x) + c$

28. Question

Mark the correct alternative in each of the following:

The value of $\int \frac{\sin x + \cos x}{\sqrt{1 - \sin 2x}} dx$ is equal to

A. $\sqrt{\sin 2x} + C$

B. $\sqrt{\cos 2x} + C$

C. $\pm (\sin x - \cos x) + C$

D. $\pm \log (\sin x - \cos x) + C$

Answer

I= $\int \frac{\sin x+\cos x}{\sin x-\cos x} dx \; (\sqrt{1-\sin 2x} = \pm\{\sin x - \cos x\})$

Let t=sin x-cos x $\left(\frac{dt}{dx} = sin\,x + cos\,x\right)$

$I = \int \frac{dt}{t}$

I=±log(sin x-cos x)+c

28. Question

Mark the correct alternative in each of the following:

The value of $\int \frac{\sin x + \cos x}{\sqrt{1-\sin 2x}} dx$ is equal to

A. $\sqrt{\sin 2x} + C$

B. $\sqrt{\cos 2x} + C$

C. ± (sinx – cosx) + C

D. ±log (sinx – cosx) +C

Answer

I= $\int \frac{\sin x+\cos x}{\sin x-\cos x} dx \; (\sqrt{1-\sin 2x} = \pm\{\sin x - \cos x\})$

Let t=sin x-cos x $\left(\frac{dt}{dx} = sin\,x + cos\,x\right)$

$I = \int \frac{dt}{t}$

I=±log(sin x-cos x)+c

29. Question

Mark the correct alternative in each of the following:

If $\int x \sin x \, dx = -x\cos x + \alpha$, then α is equal to

A. sin x + C

B. cos x + C

C. C

D. none of these

Answer

using integration by parts

I=∫x sin x d

$= x \int sin\,x\,dx - \int \frac{dx}{dx}(x) \int sin\,x)$

I = x cos x + ∫ cos x dx

(∵ ∫sin x=-cos x)

= x cos x + sin x + c

29. Question

Mark the correct alternative in each of the following:

If $\int x \sin x \, dx = -x\cos x + \alpha$, then α is equal to

A. sin x + C

B. cos x + C

C. C

D. none of these

Answer

using integration by parts

$I = \int x \sin x \, d$□

$$= x \int \sin x \, dx - \int \frac{dx}{dx}(x) \int \sin x)$$

$I = x \cos x + \int \cos x \, dx$

$(\because \int \sin x = -\cos x)$

$= x \cos x + \sin x + c$

30. Question

Mark the correct alternative in each of the following:

$$\int \frac{\cos 2x - 1}{\cos 2x + 1} dx =$$

A. tan x – x + C

B. x + tan x + C

C. x – tan x + C

D. –x – cot x + C

Answer

$$I = \int \frac{1-2(\sin x)^2-1}{2(\cos x)^2-1+1}$$

$$I = -\int \frac{(\sin x)^2}{(\cos x)^2} dx$$

$I = - \int (\tan x)^2 \, dx$

$I = - \int (-1 + (\sec x)^2 \, dx$

$= (x - \tan x) + c$

30. Question

Mark the correct alternative in each of the following:

$$\int \frac{\cos 2x - 1}{\cos 2x + 1} dx =$$

A. tan x – x + C

B. x + tan x + C

C. x – tan x + C

D. –x – cot x + C

Answer

$I = \int \frac{1-2(\sin x)^2-1}{2(\cos x)^2-1+1}$

$$I = -\int \frac{(\sin x)^2}{(\cos x)^2} dx$$

I = - ∫ (tan x)^2 dx

I = - ∫ (-1 + (sec x)^2 dx

= (x-tan x) + c

31. Question

Mark the correct alternative in each of the following:

$\int \frac{\cos 2x - \cos 2\theta}{\cos x - \cos \theta} dx$ is equal to

A. 2(sinx + x cosθ) + C

B. 2(sinx – x cosθ) + C

C. 2(sinx + 2x cosθ) + C

D. 2(sinx – 2x cosθ) + C

Answer

I= $\int \frac{\{2(\cos x)^2-1\}-\{2(\cos\theta)^2-1\}}{\cos x-\cos\theta} dx$

$$I = 2\int \frac{(\cos x)^2 - (\cos\theta)^2}{\cos x - \cos\theta} dx$$

$$I = 2\int \frac{(\cos x - \cos\theta)(\cos x + \cos\theta)}{\cos x - \cos\theta} dx$$

I=2∫(cos x+ cos θ) dx

I= 2(sin x+x cos θ) + c

31. Question

Mark the correct alternative in each of the following:

$\int \frac{\cos 2x - \cos 2\theta}{\cos x - \cos \theta} dx$ is equal to

A. 2(sinx + x cosθ) + C

B. 2(sinx – x cosθ) + C

C. 2(sinx + 2x cosθ) + C

D. 2(sinx – 2x cosθ) + C

Answer

I= $\int \frac{\{2(\cos x)^2-1\}-\{2(\cos\theta)^2-1\}}{\cos x-\cos\theta} dx$

$$I = 2\int \frac{(\cos x)^2 - (\cos\theta)^2}{\cos x - \cos\theta} dx$$

$$I = 2\int \frac{(\cos x - \cos\theta)(\cos x + \cos\theta)}{\cos x - \cos\theta} dx$$

I=2∫(cos x+ cos θ) dx

I= 2(sin x+x cos θ) + c

32. Question

Mark the correct alternative in each of the following:

$\int \frac{x^9}{\left(4x^2+1\right)^6} dx$ is equal to

A. $\frac{1}{5x}\left(4+\frac{1}{x^2}\right)^{-5}+C$

B. $\frac{1}{5}\left(4+\frac{1}{x^2}\right)^{-5}+C$

C. $\frac{1}{10x}\left(\frac{1}{x^2}+4\right)^{-5}+C$

D. $\frac{1}{10}\left(\frac{1}{x^2}+4\right)^{-5}+C$

Answer

I= $\int \frac{x^9}{(4x^2+1)^6} dx$

$I = \int \frac{x^9}{x^{12}(4+\frac{1}{x^2}\,{}^{6}} dx$

$I = \int \frac{1}{x^3(4+\frac{1}{x^2})^6} dx$

Let$\left(4+\frac{1}{x^2}\right) = t\ ;\ \frac{-2}{x^3}dx = dt$

$I = \int \frac{dt}{-2t^6}$

$I = \frac{1}{10}\left[\frac{1}{t^5}\right]$

$I = \frac{1}{10}\left([4+\frac{1}{x^2}]^{-5}\right) + c$

32. Question

Mark the correct alternative in each of the following:

$\int \frac{x^9}{\left(4x^2+1\right)^6} dx$ is equal to

A. $\frac{1}{5x}\left(4+\frac{1}{x^2}\right)^{-5}+C$

B. $\frac{1}{5}\left(4+\frac{1}{x^2}\right)^{-5}+C$

C. $\frac{1}{10x}\left(\frac{1}{x^2}+4\right)^{-5}+C$

D. $\frac{1}{10}\left(\frac{1}{x^2}+4\right)^{-5}+C$

Answer

I= $\int \frac{x^9}{(4x^2+1)^6} dx$

$I = \int \frac{x^9}{x^{12}(4+\frac{1}{x^2}{}^6} dx$

$I = \int \frac{1}{x^3(4+\frac{1}{x^2})^6} dx$

Let$\left(4+\frac{1}{x^2}\right) = t$; $\frac{-2}{x^3}dx = dt$

$I = \int \frac{dt}{-2t^6}$

$I = \frac{1}{10}\left[\frac{1}{t^5}\right]$

$I = \frac{1}{10}\left([4+\frac{1}{x^2}]^{-5}\right) + c$

33. Question

Mark the correct alternative in each of the following:

$\int \frac{x^3}{\sqrt{1+x^2}} dx = a\left(1+x^2\right)^{3/2} + b\sqrt{1+x^2} + C$, then

A. $a = \frac{1}{3}, b = 1$

B. $a = -\frac{1}{3}, b = 1$

C. $a = -\frac{1}{3}, b = -1$

D. $a = \frac{1}{3}, b = -1$

Answer

let $(\sqrt{1+x^2})$=t

$\frac{x}{\sqrt{1+x^2}} dx = dt;$

$I = \int \frac{x^3}{\sqrt{1+x^2}} dx = \int x^2 dt = \int (t^2-1) dt$

$I = \frac{t^3}{3} - t \; [put(t) = \sqrt{1+x^2}\,]$

$I = \frac{(1+x^2)^{\frac{3}{2}}}{3} - \sqrt{1+x^2} + C$

[a= $\frac{1}{3}$]; [b=-1]

33. Question

Mark the correct alternative in each of the following:

$\int \frac{x^3}{\sqrt{1+x^2}} dx = a\left(1+x^2\right)^{3/2} + b\sqrt{1+x^2} + C$, then

A. $a = \frac{1}{3}, b = 1$

B. $a = -\frac{1}{3}, b = 1$

C. $a = -\frac{1}{3}, b = -1$

D. $a = \frac{1}{3}, b = -1$

Answer

let $(\sqrt{1+x^2})$=t

$\frac{x}{\sqrt{1+x^2}} dx = dt;$

$I = \int \frac{x^3}{\sqrt{1+x^2}} dx = \int x^2 dt = \int (t^2 - 1) dt$

$I = \frac{t^3}{3} - t\ [put(t) = \sqrt{1+x^2}\ ]$

$I = \frac{(1+x^2)^{\frac{3}{2}}}{3} - \sqrt{1+x^2} + C$

[a= $\frac{1}{3}$]; [b=-1]

34. Question

Mark the correct alternative in each of the following:

$\int \frac{x^3}{x+1} dx$

A. $x + \frac{x^2}{2} + \frac{x^3}{3} - \log|1-x| + C$

B. $x + \frac{x^2}{2} - \frac{x^3}{3} - \log|1-x| + C$

C. $x - \frac{x^2}{2} - \frac{x^3}{3} - \log|1+x| + C$

D. $x-\frac{x^2}{2}+\frac{x^3}{3}-\log|1+x|+C$

Answer

$$=\int \frac{x^3+1}{x+1}\,dx-\int \frac{1}{x+1}dx$$

$$=\int \frac{(x+1)(x^2-x+1)}{x+1}dx\ -\int \frac{1}{x+1}dx$$

$$=\int (x^2-x+1)dx-\int \frac{1}{x+1}dx$$

$$=\frac{x^3}{3}-\frac{x^2}{2}+x-log(1+x)+c$$

34. Question

Mark the correct alternative in each of the following:

$$\int \frac{x^3}{x+1}dx$$

A. $x+\frac{x^2}{2}+\frac{x^3}{3}-\log|1-x|+C$

B. $x+\frac{x^2}{2}-\frac{x^3}{3}-\log|1-x|+C$

C. $x-\frac{x^2}{2}-\frac{x^3}{3}-\log|1+x|+C$

D. $x-\frac{x^2}{2}+\frac{x^3}{3}-\log|1+x|+C$

Answer

$$=\int \frac{x^3+1}{x+1}\,dx-\int \frac{1}{x+1}dx$$

$$=\int \frac{(x+1)(x^2-x+1)}{x+1}dx\ -\int \frac{1}{x+1}dx$$

$$=\int (x^2-x+1)dx-\int \frac{1}{x+1}dx$$

$$=\frac{x^3}{3}-\frac{x^2}{2}+x-log(1+x)+c$$

35. Question

Mark the correct alternative in each of the following:

If $\int \frac{1}{(x+2)(x^2+1)}dx$ a log |1 + x^2 +b $\tan^{-1}$

$x+\frac{1}{5}\log|x+2|+C$, then

A. $a=-\frac{1}{10}, b=-\frac{2}{5}$

B. $a=\frac{1}{10}, b=-\frac{2}{5}$

C. $a=-\frac{1}{10}, b=\frac{2}{5}$

D. $a=\frac{1}{10}, b=\frac{2}{5}$

Answer

$U = \int \frac{1}{(x+2)(x^2+1)} dx$

$U = \int \frac{A}{x+2} dx + \int \frac{Bx+c}{x^2+1} dx$

$\frac{1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+c}{x^2+1}$ (compare coefficient of x^2, *and* x both side)

$\left[A = \frac{1}{5} ; B = -\frac{1}{5} ; C = \frac{2}{5}\right]$ put the value of A,B,C in U

$U = \int \frac{\frac{1}{5}}{x+2} dx + \int \frac{-\frac{1}{5}x + \frac{2}{5}}{x^2+1} dx$

$U = \frac{1}{5}\left[\int \frac{1}{x+2} dx + \int \frac{-x}{x^2+1} dx + \int \frac{2}{x^2+1} dx\right]$

$U = \frac{1}{5}\left[log(X+2) - \frac{1}{2}log(x^2+1) + 2\tan^{-1} X\right] + C$

35. Question

Mark the correct alternative in each of the following:

If $\int \frac{1}{(x+2)(x^2+1)} dx$ a log |1 + x^2 +b $\tan^{-1}$

$x + \frac{1}{5}\log|x+2| + C$, then

A. $a=-\frac{1}{10}, b=-\frac{2}{5}$

B. $a=\frac{1}{10}, b=-\frac{2}{5}$

C. $a=-\frac{1}{10}, b=\frac{2}{5}$

D. $a=\frac{1}{10}, b=\frac{2}{5}$

Answer

$U = \int \frac{1}{(x+2)(x^2+1)} dx$

$$U = \int \frac{A}{x+2}dx + \int \frac{Bx+c}{x^2+1}dx$$

$\frac{1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+c}{x^2+1}$ (compare coefficient of $x^2, and\ x$ both side)

$\left[A = \frac{1}{5}\ ;\ B = -\frac{1}{5}\ ;\ C = \frac{2}{5}\right]$ put the value of A,B,C in U

$$U = \int \frac{\frac{1}{5}}{x+2}dx + \int \frac{-\frac{1}{5}x + \frac{2}{5}}{x^2+1}dx$$

$$U = \frac{1}{5}\left[\int \frac{1}{x+2}dx + \int \frac{-x}{x^2+1}dx + \int \frac{2}{x^2+1}dx\right]$$

$$U = \frac{1}{5}\left[log(X+2) - \frac{1}{2}log(x^2+1) + 2\,tan^{-1}X\right] + C$$

Revision exercise

106. Question

$$\int \frac{1}{x\sqrt{1+x^3}}dx$$

Answer

Let $x = sin^{\frac{2}{3}}t$

Differentiate both side with respect to t

$$\frac{dx}{dt} = \frac{2}{3}sin^{-\frac{1}{3}}t\cos t \Rightarrow dx = \frac{2}{3}sin^{-\frac{1}{3}}t\cos t\ dt$$

$$y = \int \frac{1}{sin^{\frac{2}{3}}t\sqrt{1+sin^2 t}}\frac{2}{3}sin^{-\frac{1}{3}}t\cos t\ dt$$

$$y = \frac{2}{3}\int cosec\ t\ dt$$

$$y = \frac{2}{3}\ln(\operatorname{cosec} t - \cot t) + c$$

Again, put $t = \sin^{-1} x^{\frac{3}{2}}$

$$y = \frac{2}{3}\ln(\operatorname{cosec}\sin^{-1} x^{\frac{3}{2}} - \cot\sin^{-1} x^{\frac{3}{2}}) + c$$

$$y = \frac{2}{3}\ln\left(x^{\frac{-3}{2}} - \frac{\sqrt{1-x^3}}{x^{\frac{3}{2}}}\right) + c$$

$$y = -\ln x + \frac{2}{3}\ln(1 - \sqrt{1-x^3}) + c$$

107. Question

Evaluate $\int \frac{\sin x + \cos x}{\sin^4 x + \cos^4 x} dx$

Answer

$$\int \frac{(sinx + cosx)}{\sin^4 x + \cos^4 x}dx$$

$$= \int \frac{(sinx + cosx)}{(sin^2 x + cos^2 x)^{\,2} - 2sin^2 xcos^2 x} dx$$

$$= \int \frac{(sinx + cosx)}{1 - 2sin^2 xcos^2 x} dx$$

$$= \int \frac{2(sinx + cosx)}{2 - 4sin^2 xcos^2 x} dx$$

$$= \int \frac{2(sinx + cosx)}{2 - sin^2 2x} dx$$

Let sinx – cosx=t ,

(cosx+sinx)dx=dt

$$= \int \frac{2}{2 - (1 - t^2)^2} dt$$

$$= \int \frac{2}{(\sqrt{2} - 1 + t^2)(\sqrt{2} + 1 - t^2)} dt$$

$$= \frac{1}{\sqrt{2}} \int (\frac{1}{(\sqrt{2} + 1 + t^2)} - \frac{1}{(\sqrt{2} - 1 - t^2)}) dt$$

$$= \frac{1}{\sqrt{2}} \int \left(\frac{1}{(\sqrt{2} + 1 + t^2)}\right) dt - \frac{1}{\sqrt{2}} \int (\frac{1}{(\sqrt{2} - 1 - t^2)}) \, dt$$

$$= \frac{1}{\sqrt{2}} \int \left(\frac{1}{((\sqrt{\sqrt{2} + 1}))^2 + t^2)}\right) dt - \frac{1}{\sqrt{2}} \int (\frac{1}{((\sqrt{\sqrt{2} - 1}))^2 - t^2)}) \, dt$$

$$= \frac{1}{\sqrt{2}} \left[\frac{1}{2\sqrt{\sqrt{2} + 1}} log \left|\frac{t - \sqrt{\sqrt{2} + 1}}{t + \sqrt{\sqrt{2} + 1}}\right|\right] - \frac{1}{\sqrt{2}} [\frac{1}{\sqrt{\sqrt{2} - 1}} tan^{-1}(\frac{t}{\sqrt{\sqrt{2} - 1}})] + c$$

108. Question

Evaluate $\int x^2 \tan^{-1} x \, dx$

Answer

$\int x^2 \tan^{-1} x \, dx$

Here we will use integration by parts,

$$\int u.\,dv = uv - \int vdu$$

Choose u in these oder LIATE(L-LOGS,I-INVERSE,A-ALGEBRAIC,T-TRIG,E-EXPONENTIAL)

So here,u=tan^{-1}x

$= \tan^{-1} x \int x^2 dx - \frac{1}{3} \int x^3 (d(\tan^{-1} x))/$ ....(

$dx \;+ c$

$\int x^2 dx = \left(\frac{x^3}{3}\right) + c)$

$$= \left(\frac{x^3}{3}\right) \tan^{-1} x - \frac{1}{3} \int \frac{x^3}{1 + x^2} dx$$

Putting 1+x^2 =t,

2xdx=dt ,

$x\,dx = \frac{dt}{2}$

$= \left(\frac{x^3}{3}\right)\tan^{-1}x - \frac{1}{3}\int \frac{x x^2}{1+x^2}dx$

$= \left(\frac{x^3}{3}\right)\tan^{-1}x - \frac{1}{3}\int \frac{(t-1)}{t}\frac{dt}{2}$

$= \left(\frac{x^3}{3}\right)\tan^{-1}x - \frac{1}{6}\int \frac{(t-1)}{t}dt$

$= \left(\frac{x^3}{3}\right)\tan^{-1}x - \frac{1}{6}[\int 1\,dt - \int \frac{1}{t}dt]$

$= \left(\frac{x^3}{3}\right)\tan^{-1}x - \frac{1}{6}[-logt + t] + c$

Resubstituting t

$= \left(\frac{x^3}{3}\right)\tan^{-1}x - \frac{1}{6}[-\log(1+x^2) + (1+x^2)] + c$

109. Question

Evaluate $\int \tan^{-1}\sqrt{x}\,dx$

Answer

$\int \tan^{-1}\sqrt{x}\,dx$

∫u. dv=uv-∫v du

Choose u in these odder

LIATE(L-LOGS,I-INVERSE,A-ALGEBRAIC,T-TRIG,E-EXPONENTIAL)

Here u=$\tan^{-1}\sqrt{x}$ and v=1.

$\therefore \int \tan^{-1}\sqrt{x}\,dx$

$\therefore x\tan^{-1}\sqrt{x} - \int x.\frac{d(\tan^{-1}\sqrt{x})}{dx}$

$= x\tan^{-1}\sqrt{x} - \frac{1}{2}\int \frac{\sqrt{x}}{1+x}dx$

Put $\sqrt{x} = t;$

$\frac{1}{2\sqrt{x}}dx = dt;$

dx=2tdt

and x=t^2

$= x\tan^{-1}\sqrt{x} - \int \frac{t^2}{1+t^2}dt$

$= x\tan^{-1}\sqrt{x} - [\int \frac{1+t^2}{1+t^2}dt - \int \frac{1}{1+t^2}dt]$

$= x\tan^{-1}\sqrt{x} - [\sqrt{x} - \tan^{-1}\sqrt{x}] + c$

110. Question

Evaluate $\int \sin^{-1}\sqrt{x}\,dx$

Answer

$\int \sin^{-1}\sqrt{x}\,dx$

$\int u.dv = uv - \int vdu$

Choose u in these order LIATE(L-LOGS,I-INVERSE,A-ALGEBRAIC,T-TRIG,E-EXPONENTIAL)

u=$\sin^{-1}\sqrt{x}$ v=1

$\therefore \int \sin^{-1}\sqrt{x} = x.\sin^{-1}\sqrt{x} - \frac{1}{2}\int \frac{\sqrt{x}}{\sqrt{1-x}}\,dx$

Put $\sqrt{x} = t;$

dx=2tdt

$= x.\sin^{-1}\sqrt{x} - \int \frac{t^2}{\sqrt{1-t^2}}\,dt$

Now put t=sinu;

dt=cos u du;

$\sqrt{1-t^2} = \sqrt{1-sin^2 u}$

=cos u

$= x.\sin^{-1}\sqrt{x} - \int \frac{\sin^2 u\, cosu\, du}{\sqrt{1-\sin^2 u}}$

$= x.\sin^{-1}\sqrt{x} - \int \frac{\sin^2 u\, cosu\, du}{cosu}$

$= x.\sin^{-1}\sqrt{x} - \int \sin^2 udu$...(Here we can substitute $\sin^2 x=(1-\cos 2u)/2$)

$= x.\sin^{-1}\sqrt{x} - \int \frac{1-\cos 2u}{2}du$

$= x.\sin^{-1}\sqrt{x} - \left[\int \frac{1-\cos 2u}{2}du\right]$

$= x.\sin^{-1}\sqrt{x} - \left[\frac{u}{2} - \frac{1}{4}sin2u\right] + c$

Put $u = \sin^{-1}\sqrt{x},$

I= $x.\sin^{-1}\sqrt{x} - \left[\frac{\sin^{-1}\sqrt{x}}{2} - \frac{\sqrt{x}\sqrt{(1-x)}}{2}\right] + c$

111. Question

Evaluate $\int \sec^{-1}\sqrt{x}\,dx$

Answer

$\int \sec^{-1}\sqrt{x}dx$

$\int u.dv = uv - \int vdu$

Choose u in these order LIATE(L-LOGS,I-INVERSE,A-ALGEBRAIC,T-TRIG,E-EXPONENTIAL)

Here u=$\sec^{-1}\sqrt{x}$ and v=1.

$$\int \sec^{-1}\sqrt{x}dx = xsec^{-1}x - \int \frac{xdx}{2x\sqrt{x-1}}$$

$$= xsec^{-1}x - \int \frac{dx}{2\sqrt{x-1}}$$

Put x-1=t dx=dt

$$= xsec^{-1}x - \int \frac{dt}{2\sqrt{t}}$$

$$= xsec^{-1}x - \frac{2}{2}(\sqrt{t}) + c$$

$$= xsec^{-1}x - (\sqrt{x-1}) + c$$

112. Question

Evaluate $\int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx$

Answer

Put x=cos2t;dx=-2sin2t

$$= \int \tan^{-1}\sqrt{\frac{1-x}{1+x}}dx = \int \tan^{-1}\sqrt{\frac{1-cos2t}{1+cos2t}}\,(-2sin2t)dt$$

$$= \int \tan^{-1}\sqrt{\frac{1-cos2t}{1+cos2t}}\,(-2sin2t)dt$$

$$= -2\int \tan^{-1} tant\ sin2t\ dt$$

$$= -2\int tsin2t\ \ dt$$

$$= -2[-\frac{tcos2t}{2} + \frac{1}{2}\int cos2t\ dt]$$

$$= tcos2t - \frac{sin2t}{2} + c$$

$$= \frac{xcos^{-1}x}{2} - \frac{\sqrt{1-x^2}}{2} + c$$

113. Question

Evaluate $\int \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$

Answer

$$\int \sin^{-1}\sqrt{\frac{x}{a+x}}dx$$

Put x=atan^{2}t;dx=2a.tant.sec^{2}t dt

$$= \int \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx = \int sin^{-1}\sqrt{\frac{\mathrm{a}\tan^2 t}{a+\mathrm{a}\tan^2 t}}\,2\mathrm{a.tant.sec}^2 t\ \mathrm{dt} = \int \mathrm{t.\,2a.tant.sec}^2 t\ \mathrm{dt}$$

$$= 2a\int \mathrm{t.tant.sec}^2 t\ \mathrm{dt}$$

$$= 2a\left[\frac{t(\tan^2 t)}{2} - \int \frac{\tan^2 t}{2}dt\right] + c$$

$$= 2a[\frac{t(\tan^2 t)}{2} - \frac{tant}{2} + \frac{t}{2}] + c$$

$$= a[t(\tan^2 t) - tant + t] + c$$

$$= xtan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax} + \mathrm{a}\tan^{-1}\sqrt{\frac{x}{a}} + c.$$

114. Question

Evaluate $\int \sin^{-1}\left(3x - 4x^3\right) dx$

Answer

Put x=sint ;dx=costdt

$$\int \sin^{-1}(3x - 4x^3)dx = \int \sin^{-1}(3sint - 4\sin^3 t)costdt \ \ldots\ldots(3sint - 4\sin^3 t) = sin3t.$$

$$= \int \sin^{-1}(sin3t)costdt = \int 3t\ cost\ dt$$

$$= 3\int t\ cost\ dt$$

By by parts,

=3[t sin t-∫sin t dt]+c

=3[t sin t + cos t]+c

$$= 3\,\mathrm{x}\sin^{-1} x + 3\sqrt{1-x^2} + c.$$

115. Question

Evaluate $\int \left(\sin^{-1} x\right)^3 dx$

Answer

$$\int (\sin^{-1} x)^3\ dx$$

Put x=sin t;

dx=cos t dt

$$\int (\sin^{-1} x)^3\ dx = \int (\sin^{-1}(sint))^3 cost\ dt$$

$$\int t^3 cost\ d = [t^3 sint - 3\int t^2 sint\ dt] = [t^3 sint - 3[-t^2 cost + 2\int tcost\ dt]]$$

$$= \left[t^3 sint + 3t^2 cost - 6\int t\ cost\ dt\right] = [t^3 sint + 3t^2 cost - 6[tsint + cost]] + c$$

$= [t^3 sint + 3t^2 cost - 6tcost - 6cost] + c$

$= [(\sin^{-1} x)^3 x + 3(\sin^{-1} x)^2 \sqrt{1-x^2} - 6x\sin^{-1} x - 6\sqrt{1-x^2}] + c$

116. Question

Evaluate $\int \cos^{-1}\left(1-2x^2\right) dx$

Answer

Put x=sin t

;dx=cos t dt;

$\int \cos^{-1}(1-2x^2)\, dx = \int \cos^{-1}(1-2\sin^2 t) costdt = \int \cos^{-1}(1-\sin^2 t - \sin^2 t) costdt$

$\int \cos^{-1}(\cos^2 t - \sin^2 t) costdt = \int \cos^{-1}(cos2t) costdt$

$2\int tcost\, dt = 2[tsint + cost]$+c

$Ans = 2x\sin^{-1} x + 2\sqrt{1-x^2} + c$

117. Question

Evaluate $\int \frac{x\sin^{-1} x}{\left(1-x^2\right)^{3/2}} dx$

Answer

$\int \frac{x \sin^{-1} x}{(1-x^2)\sqrt{1-x^2}} dx$

we can put $\sin^{-1}x=t; dx/(1-x^2)^{1/2}=dt; (1-x^2)=\cos^2 t$ and x=sint.

$\int \frac{tsint}{\cos^2 t} dt = \int t\, tant\, sect\, dt$

By by parts,

$\int t\, tant\, sect\, dt = t\, sect - \int sect\, dt$.......

$\because \int sect\, tant\, dt = \int \frac{sint}{\cos^2 t} dt$

=t sec t-log (tan t + sec t) + C'

Put cost=u;

-sin t dt=du

$= \sin^{-1} x \sec(\sin^{-1} x) - \log(\tan(\sin^{-1} x) + \sec(\sin^{-1} x)) + c' \int -u^{-2} du$

$=-(-u^{-1})+c$

=sec t + C

118. Question

Evaluate $\int e^{2x}\left(\frac{1+\sin 2x}{1+\cos 2x}\right) dx$

Answer

Put 2x=t dx=dt/2

$$\frac{1}{2}\int e^t\left(\frac{1+sint}{1+cost}\right)dt = \frac{1}{2}\int (e^t tan\frac{t}{2} + \frac{1}{2}e^t \sec^2\frac{t}{2})dt$$

$$= \frac{1}{2}\int (e^t \tan\frac{t}{2})dt + \frac{1}{4}\int e^t \sec^2\frac{t}{2}dt$$

$$= \frac{1}{2}\int (e^t \tan\frac{t}{2})dt + \frac{1}{4}[2e^t \tan\frac{t}{2} - \int 2e^t \tan\frac{t}{2}] = e^t\frac{tan\frac{t}{2}}{2} + c$$

119. Question

Evaluate $\int \frac{\sqrt{1-\sin x}}{1+\cos x} e^{-x/2} dx$

Answer

$$= \int e^{-\frac{x}{2}} \frac{\sqrt{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}}{2\cos^2\frac{x}{2}} =$$

$$\int e^{-\frac{x}{2}}\frac{(\sin\frac{x}{2}-\cos\frac{x}{2})}{2\cos^2\frac{x}{2}}dx$$

$$= \int e^{-\frac{x}{2}}(\frac{\sin\frac{x}{2}}{2\cos^2\frac{x}{2}} - \frac{\cos\frac{x}{2}}{2\cos^2\frac{x}{2}})\,dx$$

$$= \int \left[\frac{1}{2}\tan\frac{x}{2}\sec\frac{x}{2}e^{-\frac{x}{2}} - \frac{1}{2}\sec\frac{x}{2}e^{-\frac{x}{2}}\right]dx$$

$$= \frac{1}{2}\int \tan\frac{x}{2}\sec\frac{x}{2}e^{-\frac{x}{2}}dx - \frac{1}{2}\int \sec\frac{x}{2}e^{-\frac{x}{2}}dx$$

$$= \frac{1}{2}\int \tan\frac{x}{2}\sec\frac{x}{2}e^{-\frac{x}{2}}dx - \frac{1}{2}[\sec\frac{x}{2}\int e^{-\frac{x}{2}}dx - \int \frac{d}{dx}(\sec\frac{x}{2})\int(e^{-\frac{x}{2}}dx)\,dx$$

$$= \frac{1}{2}\int \tan\frac{x}{2}\sec\frac{x}{2}e^{-\frac{x}{2}}dx + e^{-\frac{x}{2}}\sec\frac{x}{2} + \frac{1}{2}\int \frac{1}{2}\tan\frac{x}{2}\sec\frac{x}{2}\,(\frac{e^{-\frac{x}{2}}}{-\frac{1}{2}})$$

$$= \sec\frac{x}{2}\,(e^{\frac{-x}{2}}) + c$$

120. Question

Evaluate $\int e^x \frac{(1-x)^2}{(1+x^2)^2} dx$

Answer

$$= \int e^x\frac{(1+x^2-2x)}{(1+x^2)^2}$$

$$= \int e^x\frac{dx}{1+x^2} - \int \frac{2xe^x dx}{(1+x^2)^2}$$

$$= \int e^x[\frac{1}{1+x^2} - \frac{2x}{(1+x^2)^2}]dx \ldots\ldots(\int e^x(f(x)+f'(x)) = e^x f(x) + c)$$

$$= e^x\frac{1}{1+x^2} + c$$

121. Question

Evaluate $\int \frac{e^{m\tan^{-1}x}}{\left(1+x^2\right)^{3/2}} dx$

Answer

$$= e^m \int \frac{\tan^{-1} x}{(1+x^2)\sqrt{1+x^2}} dx$$

Put $\tan^{-1}x=t, dx/(1+x^2)=dt$, $1+x^2=\sec^2 x$;

$$= e^m \int \frac{tdt}{sect} = e^m \int tcostdt$$

$$= e^m \left[tsint - \int sintdt\right]$$

$$= e^m[tsint + cost] + c$$

$$= e^m \left[\frac{xtan^{-1}x}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+x^2}}\right] + c$$

122. Question

Evaluate $\int \frac{x^2}{(x-1)^3(x+1)} dx$

Answer

$$= \int \frac{x^2}{(x-1)^3(x+1)} \mathrm{dx}$$

By using partial differentiation,

$$= \frac{x^2}{(x-1)^3(x+1)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$$

$$x^2 = A(x-1)^3 + B(x-1)^2(x+1) + C(x-1)^1(x+1) + D(x+1)$$

By substituting the x^2 coefficients and other coefficients we can get,

A=-1/8;B=1/8;C=3/4;D=1/2;

$$= \int \frac{-dx}{8(x+1)} + \int \frac{dx}{8(x-1)} + \int \frac{3dx}{4(x-1)^2} + \int \frac{dx}{2(x-1)^3}$$

$$= -\frac{1}{8}\log(1+x) + \frac{1}{8}\log(x-1) - \frac{3}{4(x-1)} - \frac{1}{4}\left(\frac{1}{1-x^2}\right) + c$$

123. Question

Evaluate $\int \frac{x}{x^3-1} dx$

Answer

$$= \int \frac{x}{(x^3-1)} dx = \int \frac{x}{(x-1)(x^2+x+1)} dx$$

$$= \int (\frac{1}{3(x-1)} - \frac{x-1}{3(x^2+x+1)})$$

$= \frac{1}{3}\int \frac{1}{x-1}dx - \frac{1}{3}\int \frac{x-1}{x^2+x+1}dx$

$= \frac{1}{3}\log(x-1) - \frac{1}{3}[\int \frac{(2x+1)}{2(x^2+x+1)}dx - \int \frac{3}{2((x^2+x+1))}dx]$

$= \frac{1}{3}\ log(x-1) - \frac{1}{3}[I1 + I2]$

$I_1 = \frac{1}{2}\int \frac{(2x+1)}{(x^2+x+1)}dx$

put $x^2+x+1=t$;

$(2x+1)dx=dt$

$I_1 = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}logt + c = \frac{1}{2}\log(x^2+x+1) + c$

Now, $I_2 = \frac{3}{2}\int \frac{dx}{x^2+x+1} = \frac{3}{2}\int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}$

put $(2x+1)/\sqrt{3} = u$;

$2dx/\sqrt{3} = dt$;

$dx = \sqrt{3}dt/2$

$= \frac{3}{2}.\frac{2}{\sqrt{3}}\int \frac{du}{u^2+1} = \frac{3}{2}.\frac{2}{\sqrt{3}}\tan^{-1}u + c = \sqrt{3}\tan^{-1}\frac{2x+1}{\sqrt{3}} + c$

So, answer is $= \frac{1}{3}\log(x-1) - \frac{1}{3}[\frac{1}{2}\log(x^2+x+1) - \sqrt{3}\tan^{-1}\frac{2x+1}{\sqrt{3}}$]+c

124. Question

Evaluate $\int \frac{1}{1+x+x^2+x^3}dx$

Answer

$= \int \frac{dx}{1+x+x^2+x^3} = \int \frac{dx}{(1+x)(1+x^2)}$

We can write the integral as follows,

$= \int [\frac{dx}{2(x+1)}] - \int \left[\frac{x-1}{2(x^2+1)}dx\right] = \frac{1}{2}\log(x+1) - \frac{1}{2}[\int \frac{xdx}{x^2+1} - \int \frac{dx}{x^2+1}]$

$= \frac{1}{2}\log(x+1) - \frac{1}{2}[\log\frac{(x^2+1)}{2} - \tan^{-1}x] + c$

125. Question

Evaluate $\int \frac{1}{(x^2+2)(x^2+5)}dx$

Answer

$\int \frac{dx}{(x^2+5)(x^2+2)}$

By partial fractions, $\frac{1}{(x^2+5)(x^2+2)} = \frac{A}{x^2+5} + \frac{B}{x^2+2}$

Solving these two equations,2A+5B=1 and A+B=0

We get A=-1/3 and B=1/3

$$= -\frac{1}{3}\int \frac{dx}{(x^2+5)} + \frac{1}{3}\int \frac{dx}{(x^2+2)} = -\frac{1}{3}.\frac{1}{\sqrt{5}}\tan^{-1}\frac{x}{\sqrt{5}} + \frac{1}{3}.\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + c$$

126. Question

$$\int \frac{x^2-2}{x^5-x}\,dx$$

Answer

By partial fractions,

$$= \frac{x^2-2}{x^2-5} = \frac{x^2-2}{(x-1)x(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x} + \frac{C}{x+1} + \frac{D}{x^2+1}$$

So by solving, A=- � ;B=2; C=- � ;D = -3/2

$$= \int -\frac{dx}{4(x-1)} + \int \frac{2}{x}dx - \frac{1}{4}\int \frac{dx}{x+1} - \frac{3}{2}\int \frac{xdx}{x^2+1}$$

$$= -\frac{1}{4}\log(x-1) + 2\log x - \frac{1}{4}\log(x+1) - \frac{3}{4}\log(x^2+1) + c$$

127. Question

Evaluate $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$

Answer

Let, x = $\sin^2$t

Differentiating both side with respect to t

$$\frac{dx}{dt} = 2\sin t\cos t \Rightarrow dx = 2\sin t\cos t\,dt$$

$$y = \int \sqrt{\frac{1-\sin t}{1+\sin t}}\,2\sin t\,\cos t\,dt$$

$$y = \int \sqrt{\frac{(1-\sin t)}{(1+\sin t)} \times \frac{(1-\sin t)}{(1-\sin t)}}\,2\sin t\,\cos t\,dt$$

$$y = 2\int (1-\sin t)\sin t\,dt$$

$$y = 2\int \sin t - \frac{1-\cos 2t}{2}dt$$

$$y = 2\left(-\cos t - \frac{t}{2} + \frac{\sin 2t}{4}\right) + c$$

Again, put $t = sin\sqrt{x}$

$$y = 2\left(-\cos sin\sqrt{x} - \frac{sin\sqrt{x}}{2} + \frac{\sin(2sin\sqrt{x})}{4}\right) + c$$

$$y = 2\left(-\sqrt{1-x} - \frac{sin\sqrt{x}}{2} + \frac{1}{2}\sqrt{x - x^2}\right) + c$$

128. Question

$$\int \frac{x^2+x+1}{(x+1)^2(x+2)} dx$$

Answer

$$= \int \frac{x^2+x+1}{(x+1)^2(x+2)} dx$$

by partial fraction,

$$\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2}$$

So we get these three equations ,

2A+2B+C=1

3A+B+2C=1

A+C=1

So the values are A=-2;C=3;B=1

$$\int \frac{x^2+x+1}{(x+1)^2(x+2)} dx = \int \left(-\frac{2dx}{x+1}\right) + \int \frac{dx}{(x+1)^2} + \int \frac{3dx}{x+2}$$

$$= -2\log(x+1) + 3\log(x+2) - \frac{1}{x+1} + c$$

129. Question

$$\int \frac{\sin 4x - 2}{1 - \cos 4x} e^{2x} dx$$

Answer

Put 2x=t;

2dx=dt;dx=dt/2

$$= -\int \frac{sin4x - 2}{cos4x - 1} dx = -\frac{1}{2}\int \frac{e^t(sin2t - 2)}{cos2t - 1} dt = \frac{1}{4}\int \frac{e^t(2sintcost - 2)}{cos^2 t} dt$$

$$= \frac{2}{4}\int e^t cottdt - \frac{2}{4}\int e^t cosec^2 tdt = \frac{1}{2}[\int e^t cottdt - \int e^t cosec^2 tdt]$$

$$= \frac{1}{2}[e^t cott + \int e^t cosec^2 tdt - \int e^t cosec^2 tdt]$$

$$= \frac{1}{2}\left[\frac{e^{2x} cot2x}{2}\right] + c$$

130. Question

Evaluate $\int \frac{\{\cot x + \cot^3\} x}{1 + \cot^3 x} dx$

Answer

$$= \int \frac{cotx(1+\cot^2 x)}{1+\cot^3 x}dx = \int \frac{cotxcosec^2 x}{1+\cot^3 x}dx$$

Put cot x=t, -cosec^{2}x dx = dt;

$$= -\int \frac{tdt}{t^3+1} = -\int \frac{tdt}{(t+1)(t^2-t+1)}$$

By partial fractions it’s a remembering thing

That if you see the above integral just apply the below return result,

$$= -\int [\frac{(t+1)}{3(t^2-t+1)} - \frac{1}{3(t+1)}]dt$$

$$= \frac{1}{3}\log(t+1) - \frac{1}{3}\int [\frac{2t-1}{2(t^2-t+1)} + \frac{3}{2(t^2-t+1)}]dt$$

$$= \frac{1}{3}\log(t+1) - \frac{1}{6}\log(t^2-t+1) - \frac{1}{2}\int \frac{dt}{\left(t-\frac{1}{2}\right)^2 + \frac{3}{4}}$$

$$= \frac{1}{3}\log(t+1) - \frac{1}{6}\log(t^2-t+1) - \frac{1}{2}[\frac{2}{\sqrt{3}}\tan^{-1}\frac{(2t-1)}{\sqrt{3}}] + c$$

$$= \frac{1}{3}\log(cotx+1) - \frac{1}{6}\log(\cot^2 x - cotx + 1) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2cotx-1}{\sqrt{3}}\right) + c$$

16. Question

Evaluate $\int \frac{1}{e^x+1}dx$

Answer

$\int \frac{1}{e^x+1}dx$

We can write above integral as

$$\Rightarrow \int \frac{1+e^x-e^x}{e^x+1}dx$$

$$\Rightarrow \underbrace{\int \frac{1+e^x}{e^x+1}dx}_{(1)} + \underbrace{\int \frac{-e^x}{e^x+1}dx}_{(2)}$$

Considering first integral:

$$\int \frac{1+e^x}{1+e^x}dx$$

Since the numerator and denominator are exactly same, our integrand simplifies to 1 and integrand becomes:

$\Rightarrow \int dx$

$\Rightarrow x$

$\therefore \int \frac{1+e^x}{1+e^x}dx = x$ --- (3)

Considering second integral:

$$\int \frac{-e^x}{e^x+1}dx$$

Let u = 1 + e^x, du = e^xdx

Apply u – substitution:

$$\int \frac{1}{u}(-du) = -ln|u|$$

Replacing the value of u we get,

$$\int \frac{-e^x}{e^x+1}dx = -ln|1+e^x| + C \text{ ---(4)}$$

From (3) and (4) we get,

$$\Rightarrow \int \frac{1+e^x}{e^x+1}dx + \int \frac{-e^x}{e^x+1}dx = x - ln|1+e^x| + C$$

$$\therefore \int \frac{1}{e^x+1}dx = x - ln|1+e^x| + C$$

17. Question

Evaluate $\int \frac{e^x - 1}{e^x + 1}dx$

Answer

$$\int \frac{e^x-1}{e^x+1}dx$$

We can write above integrand as:

$$\int \left(\frac{e^x}{e^x+1} - \frac{1}{e^x+1}\right)dx$$

$$\Rightarrow \underbrace{\int \frac{e^x}{e^x+1}dx}_{(A)} - \underbrace{\int \frac{1}{e^x+1}dx}_{(B)}$$

Considering integrand (A)

$$A = \int \frac{e^x}{e^x+1}dx$$

Put e^x+1 = t

Differentiating w.r.t x we get,

e^xdx = dt

Substituting values we get

$$A = \int \frac{e^x}{e^x+1}dx = \int \frac{dt}{t}dx = \ln|t| + C$$

Substituting the value of t we get,

$$A = \ln|e^x+1| + C$$

$$\therefore A = \int \frac{e^x}{e^x+1}dx = \ln|e^x+1| + C \text{ --(i)}$$

Considering integrand (B)

$$B = \int \frac{1}{e^x + 1} dx$$

We can write above integral as

$$\Rightarrow \int \frac{1 + e^x - e^x}{e^x + 1} dx$$

$$\Rightarrow \int \frac{1+e^x}{e^x+1} dx + \int \frac{-e^x}{e^x+1} dx$$

(1) (2)

Considering first integral:

$$\int \frac{1 + e^x}{1 + e^x} dx$$

Since the numerator and denominator are exactly same, our integrand simplifies to 1 and integrand becomes:

$\Rightarrow \int dx$

$\Rightarrow x$

$\therefore \int \frac{1+e^x}{1+e^x} dx = x$ --- (3)

Considering second integral:

$$\int \frac{-e^x}{e^x + 1} dx$$

Let $u = 1 + e^x$, $du = e^x dx$

Apply u – substitution:

$$\int \frac{1}{u} (-du) = -ln|u|$$

Replacing the value of u we get,

$\int \frac{-e^x}{e^x+1} dx = -ln|1 + e^x| + C$ ---(4)

From (3) and (4) we get,

$$\Rightarrow \int \frac{1 + e^x}{e^x + 1} dx + \int \frac{-e^x}{e^x + 1} dx = x - ln|1 + e^x| + C$$

$\therefore B = \int \frac{1}{e^x+1} dx = x - ln|1 + e^x| + C$ --(ii)

From (i) and (ii) we get,

$$\int \frac{e^x}{e^x + 1} dx - \int \frac{1}{e^x + 1} dx = (ln|e^x + 1| - (x - ln|1 + e^x|)) + C$$

$$= 2\, ln|e^x + 1| - x + C$$

$$\therefore \int \frac{e^x - 1}{e^x + 1} dx = 2\, ln|e^x + 1| - x + C$$

18. Question

Evaluate $\int \frac{1}{e^x + e^{-x}} dx$

Answer

$\int \frac{1}{e^x+e^{-x}} dx$

We can write above integral as:

$$= \int \frac{1}{e^x + \frac{1}{e^x}} dx$$

$= \int \frac{e^x}{e^{2x}+1} dx$ --(1)

Let $e^x = t$

Differentiating w.r.t x we get,

$e^x\, dx = dt$

∴ integral (1) becomes,

$$= \int \frac{1}{t^2+1} dt$$

$= \tan^{-1}(t) + C \left(\because \int \frac{1}{x^2+1} dx = \tan^{-1}(x)\right)$

Putting value of t we get,

$= \tan^{-1}(e^x) + C$

$$\therefore \int \frac{1}{e^x + e^{-x}} dx = \tan^{-1}(e^x) + C$$

19. Question

Evaluate $\int \frac{\cos^7 x}{\sin x} dx$

Answer

$\int \frac{\cos^7 x}{\sin x} dx$

We can write above integral as:

$\int \frac{(\cos^2 x)^3 . \cos x}{\sin x} dx$ --(1)

Put Sinx = t

Differentiting w.r.t x we get,

Cosx.dx = dt

∴ integral (1) becomes,

$$= \int \frac{(\cos^2 x)^3}{t} dt$$

$= \int \frac{(1-\sin^2 x)^3}{t} dt$ -- ($\because \sin^2(x) + \cos^2(x) = 1$)

$$= \int \frac{(1-t^2)^3}{t} dt$$

$$= \int \frac{(1)^3 - (t^2)^3 - 3(1)(t^2)(1-t^2)}{t} dt = \int \frac{1 - t^6 - 3t^2 + 3t^4}{t} dt$$

$$= \int \frac{1}{t} dt - \int \frac{t^6}{t} dt - \int \frac{3t^2}{t} dt + \int \frac{3t^4}{t} dt$$

$= \log|t| - \frac{t^6}{6} - \frac{3t^2}{2} + \frac{3t^4}{4} + C$

Putting value of t = Sin(x) we get,

$= \log|\sin x| - \frac{\sin^6 x}{6} - \frac{3\sin^2 x}{2} + \frac{3\sin^4 x}{4} + C$

$\therefore \int \frac{\cos^7 x}{\sin x} dx = \log|\sin x| - \frac{\sin^6 x}{6} - \frac{3\sin^2 x}{2} + \frac{3\sin^4 x}{4} + C$

20. Question

Evaluate $\int \sin x \sin 2x \sin 3x \, dx$

Answer

$\int \sin x \sin 2x \sin 3x \, dx$

We can write above integral as:

$= \frac{1}{2} \int (2 \sin x \sin 2x) \sin 3x \, dx$ --(1)

We know that,

2 sinA.sinB = cos(A-B) – cos(A+B)

Now, considering A as x and B as 2x we get,

= 2 sinx.sin2x = cos(x-2x) – cos(x+2x)

= 2 sinx.sin2x = cos(-x) – cos(3x)

= 2 sinx.sin2x = cos(x) – cos(3x) [∵ cos(-x) = cos(x)]

∴ integral (1) becomes,

$= \frac{1}{2} \int (\cos x - \cos 3x) \sin 3x \, dx$

$= \frac{1}{2} \int (\cos x . \sin 3x - \cos 3x . \sin 3x) dx$

$= \frac{1}{2} \left[\int (\cos x . \sin 3x) \, dx - \int (\cos 3x . \sin 3x) \, dx \right]$

$= \frac{1}{4} \left[\int 2(\cos x . \sin 3x) \, dx - \int 2(\cos 3x . \sin 3x) \, dx \right]$

Cosidering $\int 2 (\cos x . \sin 3x) \, dx$

We know,

2 sinA.cosB = sin(A+B) + sin(A-B)

Now, considering A as 3x and B as x we get,

2 sin3x.cosx = sin(4x) + sin(2x)

$\therefore \int 2 (\cos x . \sin 3x) \, dx = \int \sin 4x + \sin 2x \, dx$ --(2)

Again, Cosidering $\int 2(\cos 3x . \sin 3x) \, dx$

We know,

2 sinA.cosB = sin(A+B) + sin(A-B)

Now, considering A as 3x and B as 3x we get, 2 sin3x.cos3x = sin(6x) + sin(0)

= sin(6x)

$\therefore \int 2\,(\cos 3x.\sin 3x)\,dx = \int \sin 6x\,dx$ --(3)

∴ integral becomes,

$= \frac{1}{4}\left[\int 2(\cos x.\sin 3x)\,dx - \int 2(\cos 3x.\sin 3x)\,dx\right]$

$= \frac{1}{4}[\int (\sin 4x + \sin 2x)dx - \int \sin 6x\,dx\ ]$ [From (2) and (3)]

$= \frac{1}{4}\left[\int \sin 4x\,dx + \int \sin 2x\,dx\ - \int \sin 6x\,dx\right]$

$= \frac{1}{4}\left[\frac{-\cos 4x}{4} + \left(\frac{-\cos 2x}{2}\right) - \left(\frac{-\cos 6x}{6}\right)\right] + C$

$\left[\because \int \sin(ax+b)\,dx = -\frac{\cos(ax+b)}{a} + C\right]$

$= \frac{1}{4}\left[\frac{\cos 6x}{6} - \frac{\cos 4x}{4} - \frac{\cos 2x}{2}\right] + C$

$\therefore \int \sin x \sin 2x \sin 3x\,dx = \frac{1}{4}\left[\frac{\cos 6x}{6} - \frac{\cos 4x}{4} - \frac{\cos 2x}{2}\right] + C$

21. Question

Evaluate $\int \cos x \cos 2x \cos 3x\, dx$

Answer

$\int \cos x \cos 2x \cos 3x\,dx$

We can write above integral as:

$= \frac{1}{2}\int (2\cos x \cos 2x) \cos 3x\,dx$ --(1)

We know that,

2 cosA.cosB = cos(A+B) + cos(A-B)

Now, considering A as x and B as 2x we get,

= 2 cosx.cos2x = cos(x+2x) + cos(x-2x)

= 2 cosx.cos2x = cos(3x) + cos(-x)

= 2 cosx.cos2x = cos(3x) + cos(x) [∵ cos(-x) = cos(x)]

∴ integral (1) becomes,

$= \frac{1}{2}\int (\cos 3x + \cos x)\cos 3x\,dx$

$= \frac{1}{2}\int (\cos 3x.\cos 3x + \cos x.\cos 3x)\,dx$

$= \frac{1}{2}\left[\int (\cos^2 3x)\,dx + \int (\cos x.\cos 3x)\,dx\right]$

$= \frac{1}{4}\left[\int 2(\cos^2 3x) + \int 2(\cos x.\cos 3x)\,dx\right]$

Cosidering $\int 2\,(\cos x.\cos 3x)\,dx$

We know,

2 cosA.cosB = cos(A+B) + cos(A-B)

Now, considering A as x and B as 3x we get,

2 cosx.cos3x = cos(4x) + cos(-2x)

2 cosx.cos3x = cos(4x) + cos(2x) [∵ cos(-x) = cos(x)]

$\therefore \int 2\,(\cos x.\cos 3x)\,dx = \int (\cos 4x + \cos 2x)\,dx$ --(2)

Cosidering ∫ $2\cos^2 3x$

We know,

$\cos 2A = 2\cos^2 A - 1$

$2\cos^2 A = 1 + \cos 2A$

Now, considering A as 3x we get,

$\int 2\cos^2 3x = \int 1 + \cos 2(3x) = \int 1 + \cos(6x)$

$\therefore \int 2\,(\cos^2 3x)dx = \int 1 + \cos 6x\,dx$ --(3)

∴ integral becomes,

$$= \frac{1}{4}\left[\int 2(\cos^2 3x) + \int 2(\cos x.\cos 3x)\,dx\right]$$

$= \frac{1}{4}[\int (1 + \cos 6x)\,dx + \int (\cos 4x + \cos 2x)\,dx\ ]$ [From (2) and (3)]

$$= \frac{1}{4}\left[\int (1 + \cos 6x)\,dx + \int \cos 4x\,dx + \int \cos 2x\,dx\right]$$

$$= \frac{1}{4}\left[x + \frac{\sin 6x}{6}\right] + \frac{1}{4}\left[\frac{\sin 4x}{4}\right] + \frac{1}{4}\left[\frac{\sin 2x}{2}\right] + C$$

$$= \frac{1}{4}\left[x + \frac{\sin 6x}{6} + \frac{\sin 4x}{4} + \frac{\sin 2x}{2}\right] + C$$

$$\therefore \int \cos x \cos 2x \cos 3x\,dx = \frac{1}{4}\left[x + \frac{\sin 6x}{6} + \frac{\sin 4x}{4} + \frac{\sin 2x}{2}\right] + C$$

22. Question

$$\int \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\,dx$$

Answer

$\int \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\,dx$

We can write above integral as

$= \int \frac{\sin x + \cos x}{\sqrt{1 - 1 + \sin 2x}}\,dx$ [Adding and subtracting 1 in denominator]

$$= \int \frac{\sin x + \cos x}{\sqrt{1 - (1 - \sin 2x)}}\,dx$$

$= \int \frac{\sin x + \cos x}{\sqrt{1 - (\sin^2 x + \cos^2 x - 2\sin x \cos x)}}\,dx$ ∵ $\sin^2 x + \cos^2 x = 1$ and

sin2x = 2 sinx cosx

$= \int \frac{(\sin x + \cos x)}{\sqrt{1 - (\sin x - \cos x)^2}}\,dx$ ∵ $\sin^2 x + \cos^2 x - 2 \sin x \cos x = (\sin x - \cos x)^2$

Put sinx – cosx = t

Differentiating w.r.t x we get,

(cosx + sinx)dx = dt

Putting values we get,

$$= \int \frac{(\sin x + \cos x)}{\sqrt{1 - (\sin x - \cos x)^2}} dx = \int \frac{dt}{\sqrt{1 - t^2}}$$

$$= \int \frac{dt}{\sqrt{1 - t^2}} = \sin^{-1} t + C$$

Putting value of t we get,

$$\therefore \int \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx = \sin^{-1}(\sin x - \cos x) + C$$

23. Question

$$\int \frac{\sin x - \cos x}{\sqrt{\sin 2x}} dx$$

Answer

$$\int \frac{\sin x - \cos x}{\sqrt{\sin 2x}} dx$$

We can write above integral as

$= \int \frac{\sin x - \cos x}{\sqrt{1 + \sin 2x - 1}} dx$ [Adding and subtracting 1 in denominator]

$$= \int \frac{\sin x - \cos x}{\sqrt{(1 + \sin 2x) - 1}} dx$$

$= \int \frac{\sin x - \cos x}{\sqrt{(\sin^2 x + \cos^2 x + 2 \sin x \cos x) - 1}} dx \because \sin^2 x + \cos^2 x = 1$ and

$\sin 2x = 2 \sin x \cos x$

$= \int \frac{(\sin x - \cos x)}{\sqrt{(\sin x + \cos x)^2 - 1}} dx \because \sin^2 x + \cos^2 x + 2 \sin x \cos x = (\sin x + \cos x)^2$

Taking minus (-) common from numerator we get,

$$= -\int \frac{(-\sin x + \cos x)}{\sqrt{(\sin x + \cos x)^2 - 1}} dx$$

Put sinx + cosx = t

Differentiating w.r.t x we get,

(cosx - sinx)dx = dt

Putting values we get,

$$= -\int \frac{(\cos x - \sin x)}{\sqrt{(\sin x + \cos x)^2 - 1}} dx = -\int \frac{dt}{\sqrt{t^2 - 1}}$$

We know that,

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + C$$

Here x = t and a = 1

$$\therefore -\int \frac{dt}{\sqrt{t^2-1}} = -\log\left|t+\sqrt{t^2-1}\right| + C$$

Putting value of t we get,

$$\int \frac{\sin x - \cos x}{\sqrt{\sin 2x}} dx = -\log\left|\sin x + \cos x + \sqrt{(\sin x + \cos x)^2 - 1}\right| + C$$

∴ from (1) we get,

$$\therefore \int \frac{\sin x - \cos x}{\sqrt{\sin 2x}} dx = -\log\left|\sin x + \cos x + \sqrt{\sin 2x}\right| + C$$

24. Question

Evaluate $\int \frac{1}{\sin(x-a)\sin(x-b)} dx$

Answer

$$Let\ I = \int \frac{1}{\sin(x-a)\sin(x-b)} dx$$

Multiply and divide $\frac{1}{\sin(a-b)}$ in R.H.S we get,

$$I = \frac{1}{\sin(a-b)} \int \frac{\sin(a-b)}{\sin(x-a)\sin(x-b)} dx$$

We can write above integral as:

$$= \frac{1}{\sin(a-b)} \int \frac{\sin(a-b+x-x)}{\sin(x-a)\sin(x-b)} dx$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin[(x-b)-(x-a)]}{\sin(x-a)\sin(x-b)} dx$$

$$= \frac{1}{\sin(a-b)} \int \left[\frac{\sin(x-b)\cos(x-a) - \cos(x-b)\sin(x-a)}{\sin(x-a)\sin(x-b)}\right] dx$$

[∵ sin(A+B) = sinA.cosB – cosA.sinB]

$$= \frac{1}{\sin(a-b)} \int \left[\frac{\sin(x-b)\cos(x-a)}{\sin(x-a)\sin(x-b)} - \frac{\cos(x-b)\sin(x-a)}{\sin(x-a)\sin(x-b)}\right] dx$$

By simplifying we get,

$$= \frac{1}{\sin(a-b)} \int \left[\frac{\cos(x-a)}{\sin(x-a)} - \frac{\cos(x-b)}{\sin(x-b)}\right] dx$$

$$= \frac{1}{\sin(a-b)} \int [\cot(x-a) - \cot(x-b)] dx$$

$$= \frac{1}{\sin(a-b)} [\log|\sin(x-a)| - \log|\sin(x-b)|] + C$$

[∵ ∫ cotx dx = log|sinx| + C]

$$= \frac{1}{\sin(a-b)} \left[\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right|\right] + C$$

$$\therefore I = \int \frac{1}{\sin(x-a)\sin(x-b)} dx = \frac{1}{\sin(a-b)} \left[\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right|\right] + C$$

25. Question

Evaluate $\int \frac{1}{\cos(x-a)\cos(x-b)} dx$

Answer

$Let\ I = \int \frac{1}{\cos(x-a)\cos(x-b)} dx$

Multiply and divide $\frac{1}{\sin(a-b)}$ in R.H.S we get,

$$I = \frac{1}{\sin(a-b)} \int \frac{\sin(a-b)}{\cos(x-a)\cos(x-b)} dx$$

We can write above integral as:

$$= \frac{1}{\sin(a-b)} \int \frac{\sin(a-b+x-x)}{\cos(x-a)\cos(x-b)} dx$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin[(x-b)-(x-a)]}{\cos(x-a)\cos(x-b)} dx$$

$$= \frac{1}{\sin(a-b)} \int \left[\frac{\sin(x-b)\cos(x-a) - \cos(x-b)\sin(x-a)}{\cos(x-a)\cos(x-b)}\right] dx$$

[∵ sin(A+B) = sinA.cosB – cosA.sinB]

$$= \frac{1}{\sin(a-b)} \int \left[\frac{\sin(x-b)\cos(x-a)}{\cos(x-a)\cos(x-b)} - \frac{\cos(x-b)\sin(x-a)}{\cos(x-a)\cos(x-b)}\right] dx$$

By simplifying we get,

$$= \frac{1}{\sin(a-b)} \int \left[\frac{\sin(x-b)}{\cos(x-b)} - \frac{\sin(x-a)}{\cos(x-a)}\right] dx$$

$$= \frac{1}{\sin(a-b)} \int [\tan(x-b) - \tan(x-a)] dx$$

$$= \frac{1}{\sin(a-b)} [-\log|\cos(x-b)| + \log|\cos(x-a)|]$$

[∵ ∫ tanx dx = -log|cosx| + C]

$$= \frac{1}{\sin(a-b)} [\log|\cos(x-a)| - \log|\cos(x-b)|]$$

$$= \frac{1}{\sin(a-b)} \left[\log\left|\frac{\cos(x-a)}{\cos(x-b)}\right|\right] + C$$

$$\therefore I = \int \frac{1}{\cos(x-a)\cos(x-b)} dx = \frac{1}{\sin(a-b)} \left[\log\left|\frac{\cos(x-a)}{\cos(x-b)}\right|\right] + C$$

26. Question

Evaluate $\int \frac{\sin x}{\sqrt{1+\sin x}} dx$

Answer

$\int \frac{\sin x}{\sqrt{1+\sin x}} dx$

We can write above integral as:

$= \int \frac{1+\sin x-1}{\sqrt{1+\sin x}} dx$ (Adding and subtracting 1 in numerator)

$$= \int \frac{1+\sin x}{\sqrt{1+\sin x}}dx - \int \frac{1}{\sqrt{1+\sin x}}dx$$

$$= \int \sqrt{1+\sin x}\, dx - \int \frac{1}{\sqrt{1+\sin x}}dx$$

Consider

$$\sqrt{1+\sin x} = \sqrt{\sin^2\frac{x}{2} + \cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}} = \sqrt{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2}$$

($\because$ $\sin^2 x + \cos^2 x = 1$ and $\sin 2x = 2 \sin x.\cos x$)

$\therefore \sqrt{1+\sin x} = \sin\frac{x}{2} + \cos\frac{x}{2}$ --- (1)

$$\therefore \int \sqrt{1+\sin x}\, dx - \int \frac{1}{\sqrt{1+\sin x}}dx$$
$$= \int \left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)dx - \int \frac{1}{\sin\frac{x}{2} + \cos\frac{x}{2}}dx$$

[From (1)]

Considering,

$$\int \left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)dx - \int \frac{1}{\sin\frac{x}{2} + \cos\frac{x}{2}}dx$$

$$= -2\cos\frac{x}{2} + 2\sin\frac{x}{2} - \int \frac{1}{\frac{2\tan\frac{x}{4}}{1+\tan^2\frac{x}{4}} + \frac{1-\tan^2\frac{x}{4}}{1+\tan^2\frac{x}{4}}}dx$$

$$\because \sin\frac{x}{2} = \frac{2\tan\frac{x}{4}}{1+\tan^2\frac{x}{4}} \textit{ and } \cos\frac{x}{2} = \frac{1-\tan^2\frac{x}{4}}{1+\tan^2\frac{x}{4}}$$

$$= -2\cos\frac{x}{2} + 2\sin\frac{x}{2} - \int \frac{1+\tan^2\frac{x}{4}}{\left(2\tan\frac{x}{4} + 1 - \tan^2\frac{x}{4}\right) + (1-1)}\, dx$$

(Adding and subtracting 1 in denominator)

$$= -2\cos\frac{x}{2} + 2\sin\frac{x}{2} + \int \frac{1+\tan^2\frac{x}{4}}{-\left[\left(-2\tan\frac{x}{4} + 1 + \tan^2\frac{x}{4}\right) - 2\right]}\, dx$$

$$= -2\cos\frac{x}{2} + 2\sin\frac{x}{2} - \int \frac{\sec^2\frac{x}{4}}{\left(\tan\frac{x}{4} - 1\right)^2 - 2}\, dx \text{ --(2)}$$

$$\because -2\tan\frac{x}{4} + 1 + \tan^2\frac{x}{4} = \left(\tan\frac{x}{4} - 1\right)^2$$

$$\textit{Put } \tan\frac{x}{4} - 1 = u$$

$$\sec^2\frac{x}{4}dx = 4du$$

Putting values in (2) we get,

$$= -2\cos\frac{x}{2} + 2\sin\frac{x}{2} - 4\int \frac{du}{(u)^2 - (\sqrt{2})^2}$$

$$We\ know\ \int \frac{du}{(x)^2-(a)^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$$

$$= -2\cos\frac{x}{2} + 2\sin\frac{x}{2} - 4\frac{1}{2\sqrt{2}}\log\left|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right| + C$$

Substituting value of u we get,

$$= -2\cos\frac{x}{2} + 2\sin\frac{x}{2} - \sqrt{2}\log\left|\frac{\tan\frac{x}{4}-1-\sqrt{2}}{\tan\frac{x}{4}-1+\sqrt{2}}\right| + C$$

$$\therefore \int \frac{\sin x}{\sqrt{1+\sin x}}dx = -2\cos\frac{x}{2} + 2\sin\frac{x}{2} - \sqrt{2}\log\left|\frac{\tan\frac{x}{4}-1-\sqrt{2}}{\tan\frac{x}{4}-1+\sqrt{2}}\right| + C$$

27. Question

Evaluate $\int \frac{\sin x}{\cos 2x}\,dx$

Answer

$$Let\ I = \int \frac{\sin x}{\cos 2x}dx$$

We know cos 2x = $2\cos^2 x - 1$

Putting values in I we get,

$$I = \int \frac{\sin x}{\cos 2x}dx = \int \frac{\sin x}{2\cos^2 x - 1}dx$$

Put cosx = t

Differentiating w.r.t to x we get,

sinx dx = -dt

Putting values in integral we get,

$$I = -\int \frac{dt}{2t^2-1} = -\int \frac{dt}{\left(\sqrt{2}\,t\right)^2 - (1)^2}$$

Again put √2× t = u

Differentiating w.r.t to t we get,

$$dt = \frac{du}{\sqrt{2}}$$

Putting values in integral we get,

$$I = \frac{1}{\sqrt{2}}\int \frac{du}{(1)^2-(u)^2}$$

$$We\ know\ \int \frac{dx}{(1)^2-(x)^2} = \sin^{-1}x + C$$

$$I = \frac{1}{\sqrt{2}}\sin^{-1}u + C$$

Substituting value of u we get,

$$I = \frac{1}{\sqrt{2}}\sin^{-1}\sqrt{2}\,t\ + C$$

Substituting value of t we get,

$$I = \frac{1}{\sqrt{2}}\sin^{-1}(\sqrt{2}\cos x) + C$$

$$\therefore I = \int \frac{\sin x}{\cos 2x}dx = \frac{1}{\sqrt{2}}\sin^{-1}(\sqrt{2}\cos x) + C$$

28. Question

Evaluate $\int \tan^3 x \, dx$

Answer

$\int \tan^3 x \, dx$

We can write above integral as:

$\int \tan^3 x \, dx = \int (\tan^2 x)(\tan x)\, dx$ ----(Splitting $\tan^3 x$)

$= \int (\sec^2 x - 1)(\tan x)\, dx$ (Using $\tan^2 x = \sec^2 x - 1$)

$$= \underbrace{\int \sec^2 x \,(\tan x)\, dx}_{(1)} - \underbrace{\int (\tan x)\, dx}_{(2)}$$

Considering integral (1)

Let u = tanx

du = $\sec^2 x$ dx

Substituting values we get,

$$\int \sec^2 x\,(\tan x)\, dx = \int u\, du = \frac{u^2}{2} + C$$

Substituting value of u we get,

$$\int \sec^2 x\,(\tan x)\, dx = \frac{\tan^2 x}{2} + C$$

∴ integral becomes,

$$\int \sec^2 x\,(\tan x)\, dx - \int (\tan x)\, dx = \frac{\tan^2 x}{2} - \int (\tan x)\, dx$$

$= \frac{\tan^2 x}{2} - (-\log|\cos x|) + C$ [∵ ∫ tanx dx = -log|cosx| + C]

$$\therefore \int \tan^3 x\, dx = \frac{\tan^2 x}{2} + \log|\cos x| + C$$

29. Question

$\int \tan^4 x \, dx$

Answer

$\int \tan^4 x \, dx$

We can write above integral as:

$\int \tan^4 x \, dx = \int (\tan^2 x)\,(\tan^2 x) dx$ ----(Splitting $\tan^4 x$)

$= \int (\sec^2 x - 1) \tan^2 x \, dx$ (Using $\tan^2 x = \sec^2 x - 1$)

$$= \underbrace{\int \sec^2 x \, (\tan^2 x) \, dx}_{(1)} - \underbrace{\int (\tan^2 x) \, dx}_{(2)}$$

Considering integral (1)

Let u = tanx

du = $\sec^2 x$ dx

Substituting values we get,

$$\int \sec^2 x \, (\tan^2 x) \, dx = \int u^2 \, du = \frac{u^3}{3} + C$$

Substituting value of u we get,

$$\int \sec^2 x \, (\tan^2 x) \, dx = \frac{\tan^3 x}{3} + C$$

Considering integral (2)

$$\int (\tan^2 x) \, dx = \int (\sec^2 x - 1) dx$$

$$= \int (\sec^2 x) dx - \int 1 dx$$

$$= \tan x - x + C$$

∴ integral becomes,

$$\int \sec^2 x \, (\tan^2 x) \, dx - \int (\tan^2 x) \, dx = \frac{\tan^3 x}{3} + C - (\tan x - x + C)$$

$= \frac{\tan^3 x}{3} - \tan x + x + C$ [∵ C+C is a constant]

$$\therefore \int \tan^4 x \, dx = \frac{\tan^3 x}{3} - \tan x + x + C$$

30. Question

$$\int \tan^5 x \, dx$$

Answer

$$\int \tan^5 x \, dx$$

We can write above integral as:

$\int \tan^5 x \, dx = \int (\tan^3 x)(\tan^2 x) dx$ ----(Splitting $\tan^5 x$)

$= \int \tan^3 x (\sec^2 x - 1) dx$ (Using $\tan^2 x = \sec^2 x - 1$)

$$= \int \sec^2 x \, (\tan^3 x) \, dx - \int (\tan^3 x) \, dx$$

$= \int \sec^2 x \, (\tan^3 x) \, dx - \int (\tan^2 x)(\tan x) \, dx$ ----(Splitting $\tan^3 x$)

$$= \int \sec^2 x \, (\tan^3 x) \, dx - \int (\sec^2 x - 1)(\tan x) \, dx$$

(Using $\tan^2 x = \sec^2 x - 1$)

$$= \underbrace{\int \sec^2 x\,(\tan^3 x)\;dx}_{(1)} - \underbrace{\int \sec^2 x\,(\tan x)\,dx}_{(2)} - \underbrace{\int (\tan x)\,dx}_{(3)}$$

Considering integral (1)

Let u = tanx

du = $\sec^2$x dx

Substituting values we get,

$$\int \sec^2 x\,(\tan^3 x)\;dx = \int u^3\;du = \frac{u^4}{4} + C$$

Substituting value of u we get,

$$\int \sec^2 x\,(\tan^3 x)\,dx = \frac{\tan^4 x}{4} + C$$

Considering integral (2)

Let t = tanx

dt = $\sec^2$x dx

Substituting values we get,

$$\int \sec^2 x\,(\tan x)\,dx = \int t\;dt = \frac{t^2}{2} + C$$

Substituting value of t we get,

$$\int \sec^2 x\,(\tan x)\,dx = \frac{\tan^2 x}{2} + C$$

Considering integral (3)

$\int (\tan x)\,dx = -\log|\cos x|$ [∵ ∫ tanx dx = -log|cosx| + C]

∴ integral becomes,

$$\int \sec^2 x\,(\tan^3 x)\;dx - \int \sec^2 x\,(\tan x)\,dx - \int (\tan x)\,dx = \frac{\tan^4 x}{4} + C - \left(\frac{\tan^2 x}{2} + C\right) - (-\log|\cos x|)$$

$= \left(\frac{\tan^4 x}{4}\right) + \left(\frac{\tan^2 x}{2}\right) + (\log|\cos x|) + C$ [∵ C+C+C is a constant]

$$\therefore \int \tan^5 x\,dx = \left(\frac{\tan^4 x}{4}\right) + \left(\frac{\tan^2 x}{2}\right) + (\log|\cos x|) + C$$

86. Question

Evaluate $\int \sqrt{a^2 - x^2}\;dx$

Answer

Let, x = a sin t

Differentiate both side with respect to t

$\frac{dx}{dt} = a\cos t \Rightarrow dx = a \cos t\, dt$

$$y = \int \sqrt{a^2 - (a\sin t)^2}\; a\cos t\; dt$$

$$y = \int (a\cos t)(a\cos t)dt$$

$y = \int a^2 \cos^2 t\, dt$

$$y = \int a^2 \left(\frac{1+\cos 2t}{2}\right) dt$$

$$y = \frac{a^2}{2}\int 1 + \cos 2t\; dt$$

$$y = \frac{a^2}{2}\left(t + \frac{\sin 2t}{2}\right) + c$$

Again, put $t = \sin^{-1}\frac{x}{a}$

$$y = \frac{a^2}{2}\left(\sin^{-1}\frac{x}{a} + \frac{\sin\left(2\sin^{-1}\frac{x}{a}\right)}{2}\right) + c$$

$$y = \frac{a^2}{2}\left(\sin^{-1}\frac{x}{a} + \frac{2 \times \frac{x}{a} \times \sqrt{1 - \frac{x^2}{a^2}}}{2}\right) + c$$

$$y = \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + c$$

87. Question

Evaluate $\int \sqrt{3x^2 + 4x + 1}\, dx$

Answer

Make perfect square of quadratic equation

$3x^2 + 4x + 1 = 3\left(x^2 + \frac{4}{3}x + \frac{1}{3}\right)$

$$= 3\left(x^2 + 2\left(\frac{2}{3}\right)(x) + \left(\frac{2}{3}\right)^2 - \frac{1}{9}\right)$$

$$= 3\left[\left(x + \frac{2}{3}\right)^2 - \frac{1}{9}\right]$$

$$y = \int \sqrt{3\left[\left(x + \frac{2}{3}\right)^2 - \frac{1}{9}\right]}\, dx$$

$$y = \sqrt{3}\int \sqrt{\left[\left(x + \frac{2}{3}\right)^2 - \frac{1}{9}\right]}\, dx$$

Using formula, $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ln\left(x + \sqrt{x^2 - a^2}\right)$

$$y = \sqrt{3}\frac{\left(x + \frac{2}{3}\right)}{2}\sqrt{\left(x + \frac{2}{3}\right)^2 - \frac{1}{9}} - \frac{\sqrt{3}}{18}\ln\left(\left(x + \frac{2}{3}\right) + \sqrt{\left(x + \frac{2}{3}\right)^2 - \frac{1}{9}}\right) + c$$

$$y=\frac{3x+2}{6}\sqrt{3x^2+4x+1}-\frac{\sqrt{3}}{18}\ln\left(\left(x+\frac{2}{3}\right)+\sqrt{x^2+\frac{4x}{3}+\frac{1}{3}}\right)+c$$

88. Question

Evaluate $\int\sqrt{1+2x-3x^2}\,dx$

Answer

Make perfect square of quadratic equation

$1 + 2x - 3x^2 = 3\left[-\left(x^2-\frac{2}{3}x-\frac{1}{3}\right)\right]$

$$=3\left[\frac{4}{9}-\left(x^2-2\left(\frac{1}{3}\right)(x)+\left(\frac{1}{3}\right)^2\right)\right]$$

$$=3\left[\left(\frac{2}{3}\right)^2-\left(x-\frac{1}{3}\right)^2\right]$$

$$y=\sqrt{3}\int\left[\left(\frac{2}{3}\right)^2-\left(x-\frac{1}{3}\right)^2\right]dx$$

Using formula, $\int\sqrt{a^2-x^2}\,dx=\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}$

$$y=\sqrt{3}\left(\frac{\left(\frac{2}{3}\right)^2}{2}\sin^{-1}\frac{\left(x-\frac{1}{3}\right)}{\left(\frac{2}{3}\right)}+\frac{\left(x-\frac{1}{3}\right)}{2}\sqrt{\left(\frac{2}{3}\right)^2-\left(x-\frac{1}{3}\right)^2}\right)+c$$

$$y=\frac{2\sqrt{3}}{9}\sin^{-1}\frac{(3x-1)}{2}+\frac{(3x-1)}{6}\sqrt{1+2x-3x^2}+c$$

89. Question

Evaluate $\int x\sqrt{1+x-x^2}\,dx$

Answer

Make perfect square of quadratic equation

$1 + x - x^2 = \frac{5}{4}-\left(x^2-2\left(\frac{1}{2}\right)(x)+\left(\frac{1}{2}\right)^2\right)$

$$=\left(\frac{\sqrt{5}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2$$

$$y=\int x\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}\,dx$$

Let, $x-\frac{1}{2}=t\Rightarrow x=t+\frac{1}{2}$

Differentiate both side with respect to t

$\frac{dx}{dt}=1\Rightarrow dx=dt$

$$y=\int\left(t+\frac{1}{2}\right)\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-t^2}\,dt$$

$$y = \int t\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - t^2} + \frac{1}{2}\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - t^2}\ dt$$

$$A = \int t\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - t^2}\ dt$$

Let, $t^2 = z$

Differentiate both side with respect to z

$$2t\frac{dt}{dz} = 1\ \Rightarrow tdt = \frac{1}{2}dz$$

$$A = \frac{1}{2}\int \sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - z}\ dz$$

$$A = \frac{1}{4}\int \sqrt{5-4z}\ dz$$

$$A = \frac{-1}{24}(5-4z)^{\frac{3}{2}} + c_1$$

Put z = t^2 and $t = x - \frac{1}{2}$

$$A = \frac{-1}{24}\left(5 - 4\left(x-\frac{1}{2}\right)^2\right)^{\frac{3}{2}} + c_1$$

$$A = \frac{-1}{3}(1+x-x^2)^{\frac{3}{2}} + c_1$$

$$B = \int \frac{1}{2}\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - t^2}\,dt$$

$$B = \frac{1}{2}\left(\frac{\left(\frac{\sqrt{5}}{2}\right)^2}{2}\sin^{-1}\frac{t}{\left(\frac{\sqrt{5}}{2}\right)} + \frac{t}{2}\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - t^2}\right) + c_2$$

$$B = \frac{5}{16}\sin^{-1}\left(\frac{2t}{\sqrt{5}}\right) + \frac{t}{8}\sqrt{5-4t^2} + c_2$$

Put $t = x - \frac{1}{2}$

$$B = \frac{5}{16}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + \frac{\left(x-\frac{1}{2}\right)}{8}\sqrt{5 - 4\left(x-\frac{1}{2}\right)^2} + c_2$$

$$B = \frac{5}{16}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + \frac{(2x-1)}{8}\sqrt{1+x-x^2} + c_2$$

The final answer is y = A + B

$$y = \frac{-1}{3}(1+x-x^2)^{\frac{3}{2}} + \frac{5}{16}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + \frac{(2x-1)}{8}\sqrt{1+x-x^2} + c$$

$$y = \frac{1}{24}(8x^2-2x-11)\sqrt{1+x-x^2} + \frac{5}{16}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + c$$

90. Question

Evaluate $\int (2x+3)\sqrt{4x^2+5x+6}\,dx$

Answer

Make perfect square of quadratic equation

$4x^2 + 5x + 6 = 4\left[\left(x+\frac{5}{8}\right)^2 + \frac{71}{64}\right]$

$$y = 2\int (2x+3)\sqrt{\left[\left(x+\frac{5}{8}\right)^2 + \left(\frac{\sqrt{71}}{8}\right)^2\right]}\,dx$$

Let, $x + \frac{5}{8} = t \Rightarrow x = t - \frac{5}{8}$

Differentiate both side with respect to t

$\frac{dx}{dt} = 1 \Rightarrow dx = dt$

$$y = 2\int \left(2t+\frac{7}{4}\right)\sqrt{\left[t^2 + \left(\frac{\sqrt{71}}{8}\right)^2\right]}\,dt$$

$$A = 4\int t\sqrt{\left(\frac{\sqrt{71}}{8}\right)^2 + t^2}\,dt$$

Let, $t^2 = z$

Differentiate both side with respect to z

$$2t\frac{dt}{dz} = 1 \Rightarrow tdt = \frac{1}{2}dz$$

$$A = 2\int \sqrt{\left(\frac{\sqrt{71}}{8}\right)^2 + z}\,dz$$

$$A = \frac{1}{4}\int \sqrt{71+64z}\,dz$$

$$A = \frac{1}{384}(71+64z)^{\frac{3}{2}} + c_1$$

Put $z = t^2$ and $t = x + \frac{5}{8}$

$$A = \frac{1}{384}\left(71 + 64\left(x+\frac{5}{8}\right)^2\right)^{\frac{3}{2}} + c_1$$

$$A = \frac{1}{6}(4x^2+5x+6)^{\frac{3}{2}} + c_1$$

$$B = \int \frac{7}{2}\sqrt{\left(\frac{\sqrt{71}}{8}\right)^2 + t^2}\,dt$$

$$B = \frac{7}{2}\left(\frac{t}{2}\sqrt{\left(\frac{\sqrt{71}}{8}\right)^2 + t^2} + \frac{\left(\frac{\sqrt{71}}{8}\right)^2}{2}\ln\left(t + \sqrt{\left(\frac{\sqrt{71}}{8}\right)^2 + t^2}\right)\right) + c_2$$

Put $t = x + \frac{5}{8}$

$$B = \frac{7}{2}\left(\frac{\left(x+\frac{5}{8}\right)}{2}\sqrt{\left(\frac{\sqrt{71}}{8}\right)^2 + \left(x+\frac{5}{8}\right)^2}\right) +$$

$$\frac{7\left(\frac{\sqrt{71}}{8}\right)^2}{4}\ln\left(\left(x+\frac{5}{8}\right) + \sqrt{\left(\frac{\sqrt{71}}{8}\right)^2 + \left(x+\frac{5}{8}\right)^2}\right) + c_2$$

$$B = \frac{7}{2}\left(\frac{(8x+5)}{32}\sqrt{4x^2+5x+6}\right) +$$

$$\frac{497}{256}\ln\left(\left(x+\frac{5}{8}\right) + \sqrt{\left(\frac{\sqrt{71}}{8}\right)^2 + \left(x+\frac{5}{8}\right)^2}\right) + c_2$$

The final answer is y = A + B

$$y = \frac{1}{6}(4x^2+5x+6)^{\frac{3}{2}} + \frac{7}{2}\left(\frac{(8x+5)}{32}\sqrt{4x^2+5x+6}\right) +$$

$$\frac{497}{256}\ln\left(\left(x+\frac{5}{8}\right) + \sqrt{x^2+\frac{5}{4}x+\frac{3}{2}}\right) + c$$

$$y = \frac{1}{192}(128x^2+328x+297)\sqrt{4x^2+5x+6} +$$

$$\frac{497}{256}\ln\left(\left(x+\frac{5}{8}\right) + \sqrt{x^2+\frac{5}{4}x+\frac{3}{2}}\right) + c$$

91. Question

Evaluate $\int\left(1+x^2\right)\cos 2x\, dx$

Answer

$$y = \int cos\, 2x \; + x^2 \cos 2x\, dx$$

$$A = \int \cos 2x\, dx$$

$$A = \frac{\sin 2x}{2} + c_1$$

$$B = \int x^2 \cos 2x\, dx$$

Use the method of integration by parts

$$B = x^2\int \cos 2x\, dx - \int \frac{d}{dx}(x^2)\left(\int \cos 2x\, dx\right) dx$$

$$B = x^2\frac{\sin 2x}{2} - \int x \sin 2x\, dx$$

$$B = x^2\frac{\sin 2x}{2} - (x\int \sin 2x\; dx - \int \frac{d}{dx}(x)\left(\int \sin 2x\, dx\right)$$

$$B = x^2\frac{\sin 2x}{2} + x\frac{\cos 2x}{2} - \frac{\sin 2x}{4} + c_2$$

The final answer is y = A + B

$$y = \frac{\sin 2x}{2} + x^2\frac{\sin 2x}{2} + x\frac{\cos 2x}{2} - \frac{\sin 2x}{4} + c$$

$$y = \frac{(1+x^2)}{2}\sin 2x + \frac{x}{2}\cos 2x - \frac{1}{4}\sin 2x + c$$

92. Question

Evaluate $\int \log_{10} x\, dx$

Answer

Use the method of integration by parts

$$y = \int 1 \times \log_{10} x\ dx$$

$$y = \log_{10} x \int dx - \int \frac{d}{dx}\log_{10} x\left(\int dx\right)dx$$

$$y = x\log_{10} x - \int x\frac{1}{x\log_e 10}dx$$

$$y = x\log_{10} x - \frac{x}{\log_e 10} + c$$

$$y = x(\log_e x - 1)\log_{10} e + c$$

93. Question

Evaluate $\int \frac{\log(\log x)}{x} dx$

Answer

Let, log x = t

Differentiating both side with respect to t

$$\frac{1}{x}\frac{dx}{dt} = 1 \Rightarrow \frac{dx}{x} = dt$$

Note:- Always use direct formula for ∫log x dx

y = ∫log t dt

y = t log t – t + c

Again, put t = log x

y = (log x)log(log x) – log x + c

94. Question

Evaluate $\int x \sec^2 2x\, dx$

Answer

Use method of integration by parts

$$y = x\int sec^2 2x\, dx - \int \frac{d}{dx}x\left(\int sec^2 2x\, dx\right)dx$$

$$y = x\frac{\tan 2x}{2} - \int \frac{\tan 2x}{2} dx$$

Use formula ∫tan x dx = log secx

$$y = \frac{x}{2}\tan 2x - \frac{\log(\sec 2x)}{4} + c$$

95. Question

Evaluate $\int x \sin^3 x \, dx$

Answer

We know that $sin^3 x = \frac{3\sin x - \sin 3x}{4}$

$$y = \int x\left(\frac{3\sin x - \sin 3x}{4}\right) dx$$

$$y = \frac{3}{4}\int x \sin x \, dx - \frac{1}{4}\int x \sin 3x \, dx$$

Use method of integration by parts

$$y = \frac{3}{4}\left(x\int \sin x \, dx - \int \frac{d}{dx}x\left(\int \sin x \, dx\right) dx\right)$$

$$-\frac{1}{4}\left(x\int \sin 3x \, dx - \int \frac{d}{dx}x\left(\int \sin 3x \, dx\right) dx\right) y$$

$$= \frac{3}{4}\left(-x\cos x + \int \cos x \, dx\right) - \frac{1}{4}\left(-x\frac{\cos 3x}{3} + \int \frac{\cos 3x}{3} dx\right)$$

$$y = \frac{3}{4}(-x\cos x + \sin x) - \frac{1}{4}\left(-x\frac{\cos 3x}{3} + \frac{\sin 3x}{9}\right) + c$$

$$y = \frac{1}{4}\left(-3x\cos x + 3\sin x + \frac{x}{3}\cos 3x - \frac{\sin 3x}{9}\right) + c$$

96. Question

Evaluate $\int (x+1)^2 e^x \, dx$

Answer

y = $\int(x^2 + 2x + 1)\, e^x$ dx

y = $\int(x^2 + 2x)e^x$ dx + $\int e^x$ dx

We know that $\int(f(x) + f'(x))e^x \, dx = f(x)\, e^x$

Here, f(x) = x^2 then f'(x) = 2x

y = $x^2e^x + e^x$ + c

y = $(x^2 + 1)e^x$ + c

97. Question

Evaluate $\int \log\left(x + \sqrt{x^2 + a^2}\right) dx$

Answer

Use method of integration by parts

$$y=\log(x+\sqrt{x^2+a^2})\int dx-\int\frac{d}{dx}\log\left(x+\sqrt{x^2+a^2}\right)(\int dx)dx$$

$$y=x\log\left(x+\sqrt{x^2+a^2}\right)-\int\frac{1+\frac{2x}{2\sqrt{x^2+a^2}}}{x+\sqrt{x^2+a^2}}x\,dx$$

$$y=\ x\log\left(x+\sqrt{x^2+a^2}\right)-\int\frac{x}{\sqrt{x^2+a^2}}dx$$

Let, $x^2 + a^2 = t$

Differentiating both side with respect to t

$$2x\frac{dx}{dt}=1\ \Rightarrow x\,dx=\frac{dt}{2}$$

$$y=\ x\log\left(x+\sqrt{x^2+a^2}\right)-\frac{1}{2}\int\frac{1}{\sqrt{t}}dt$$

$$y=x\log\left(x+\sqrt{x^2+a^2}\right)-\sqrt{t}+c$$

Again, put $t = x^2 + a^2$

$$y=x\log\left(x+\sqrt{x^2+a^2}\right)-\sqrt{x^2+a^2}+c$$

98. Question

Evaluate $\int\frac{\log x}{x^3}dx$

Answer

Use method of integration by parts

$$y=\log x\int\frac{1}{x^3}dx-\int\frac{d}{dx}\log x\left(\int\frac{1}{x^3}dx\right)dx$$

$$y=-\log x\frac{1}{2x^2}+\int\frac{1}{2x^3}dx$$

$$y=-\frac{1}{2x^2}\log x-\frac{1}{4x^2}+c$$

$$y=-\frac{1}{4x^2}(2\log x+1)+c$$

99. Question

Evaluate $\int\frac{\log(1-x)}{x^2}dx$

Answer

Use method of integration by parts

$$y=\log(1-x)\int\frac{1}{x^2}dx-\int\frac{d}{dx}\log(1-x)\left(\int\frac{1}{x^2}dx\right)dx$$

$$y=-\log(1-x)\frac{1}{x}-\int\frac{1}{(1-x)x}dx$$

$$y=-\frac{1}{x}\log(1-x)-\int\frac{x+(1-x)}{(1-x)x}dx$$

$$y = -\frac{1}{x}\log(1-x) - \int \frac{1}{(1-x)} + \frac{1}{x}dx$$

$$y = -\frac{1}{x}\log(1-x) + \log(1-x) - \log x + c$$

$$y = \left(1 - \frac{1}{x}\right)\log(1-x) - \log x + c$$

100. Question

Evaluate $\int x^3 (\log x)^2 \, dx$

Answer

Use method of integration by parts

$$y = \ log^2 x \int x^3 dx - \int \frac{d}{dx} log^2 x \left(\int x^3 dx\right) dx$$

$$y = \ log^2 x \ \frac{x^4}{4} - \int \frac{2\log x}{x}\frac{x^4}{4}dx$$

$$y = \frac{x^4}{4}log^2 x - \frac{1}{2}(\log x \int x^3 dx - \int \frac{d}{dx}\log x \left(\int x^3 dx\right) dx$$

$$y = \frac{x^4}{4}log^2 x - \frac{1}{2}\left(\log x \frac{x^4}{4} - \int \frac{1}{x}\frac{x^4}{4}dx\right)$$

$$y = \frac{x^4}{4}log^2 x - \frac{x^4}{8}\log x + \frac{x^4}{32} + c$$

101. Question

Evaluate $\int \frac{1}{x\sqrt{1+x^n}} dx$

Answer

Let, $\sqrt{1+x^n} = t$

Differentiate both side with respect to t

$$\frac{nx^{n-1}}{2\sqrt{1+x^n}}\frac{dx}{dt} = 1 \Rightarrow \frac{dx}{x\sqrt{1+x^n}} = \frac{2dt}{n(t^2-1)}$$

$$y = \int \frac{2}{n(t^2-1)}dt$$

Use formula $\int \frac{1}{t^2-a^2}dt = \frac{1}{2a}\ln\left(\frac{t-a}{t+a}\right)$

$$y = \frac{1}{n}\ln\left(\frac{t-1}{t+1}\right) + c$$

Again put $t = \sqrt{1+x^n}$

$$y = \frac{1}{n}\ln\left(\frac{\sqrt{1+x^n}-1}{\sqrt{1+x^n}+1}\right) + c$$

102. Question

Evaluate $\int \frac{x^2}{\sqrt{1-x}} dx$

Answer

Let, x = $\sin^2 t$

Differentiate both side with respect to t

$\frac{dx}{dt} = 2\sin t\cos t\ dt$ ⇒ dx = 2 sin t cos t dt

$$y = \int \frac{\sin^4 t}{\cos t} 2\sin t\cos t\,dt$$

$$y = 2\int sin^5 t\,dt$$

$$y = 2\int (1 - cos^2 t)^2 \sin t\,dt$$

Let, cos t = z

Differentiate both side with respect to z

$-\sin t\frac{dt}{dz} = 1$ ⇒ sin t dt = -dz

$$y = -2\int (1 - z^2)^2 dz$$

$$y = -2\int 1 + z^4 - 2z^2 dz$$

$$y = -2\left(z + \frac{z^5}{5} - 2\frac{z^3}{3}\right) + c$$

Again put z = cos t and $t = \sin^{-1}\sqrt{x}$

$$y = -2\left(\cos(\sin^{-1}\sqrt{x}) + \frac{cos^5(\sin^{-1}\sqrt{x})}{5} - 2\frac{cos^3(\sin^{-1}\sqrt{x})}{3}\right) + c$$

$$y = -2\left(\sqrt{1-x} + \frac{(1-x)^2\sqrt{1-x}}{5} - \frac{2(1-x)\sqrt{1-x}}{3}\right) + c$$

$$y = \frac{-2}{15}\sqrt{1-x}(3x^2 + 4x + 8) + c$$

103. Question

Evaluate $\int \frac{x^5}{\sqrt{1+x^3}} dx$

Answer

Let, 1 + x^3 = t

Differentiate both side with respect to t

$$3x^2\frac{dx}{dt} = 1 \Rightarrow x^2 dx = \frac{dt}{3}$$

$$y = \frac{1}{3}\int \frac{(t-1)}{\sqrt{t}} dt$$

$$y = \frac{1}{3}\int \sqrt{t} - \frac{1}{\sqrt{t}} dt$$

$$y = \frac{1}{3}\left(\frac{2}{3}t^{\frac{3}{2}} - 2\sqrt{t}\right) + c$$

Again, put t = 1 + x^3

$$y = \frac{1}{3}\left(\frac{2}{3}(1+x^3)^{\frac{3}{2}} - 2\sqrt{1+x^3}\right) + c$$

$$y = \frac{2}{9}\sqrt{1+x^3}(x^3-2) + c$$

104. Question

Evaluate $\int \frac{1+x^2}{\sqrt{1+x^2}}\, dx$

Answer

$$y = \int \sqrt{1+x^2}\, dx$$

Use formula $\sqrt{a^2+x^2} = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right)$

$$y = \frac{x}{2}\sqrt{x^2+1} + \frac{1}{2}\ln\left(x+\sqrt{x^2+1}\right) + c$$

105. Question

Evaluate $\int x\sqrt{\frac{1-x}{1+x}}\, dx$

Answer

Let, x = sin t

Differentiate both side with respect to t

$\frac{dx}{dt} = \cos t \Rightarrow$ dx = cos t dt

$$y = \int \sin t \sqrt{\frac{1-\sin t}{1+\sin t}}\cos t\, dt$$

$$y = \int \sin t \sqrt{\frac{(1-\sin t)(1-\sin t)}{(1+\sin t)(1-\sin t)}}\cos t\, dt$$

$$y = \int \sin t\,(1-\sin t)dt$$

$$y = \int \sin t\, dt - \int \sin^2 t\, dt$$

$$y = -\cos t - \int \frac{1-\cos 2t}{2}dt$$

$$y = -\cos t - \left(\frac{t}{2} - \frac{\sin 2t}{4}\right) + c$$

Again put t = $\sin^{-1}x$

$$y = -\cos(\sin^{-1}x) - \left(\frac{(\sin^{-1}x)}{2} - \frac{\sin 2(\sin^{-1}x)}{4}\right) + c$$

$$y = -\sqrt{1-x^2} - \frac{\sin^{-1}x}{2} + \frac{x\sqrt{1-x^2}}{2} + c$$

$y = \left(\frac{x}{2} - 1\right)\sqrt{1 - x^2} - \frac{1}{2}\sin^{-1} x + c$

66. Question

Evaluate $\int \frac{1}{\sin x(2+3\cos x)} dx$

Answer

To solve this type of solution, we are going to substitute the value of sinx and cosx in terms of tan(x/2)

$$sin\ x = \frac{2\left[tan\frac{x}{2}\right]}{1+\tan^2\frac{x}{2}}$$

$$cos\ x = \frac{\left(1 - \frac{\tan^2 x}{2}\right)}{1 + \frac{\tan^2 x}{2}}$$

$$I = \int \frac{1}{\frac{2\tan\frac{x}{2}}{1+tan^2\frac{x}{2}}\left(2 + 3.\frac{1-tan^2\frac{x}{2}}{1+tan^2\frac{x}{2}}\right)} dx$$

$$I = \int \frac{sec^2\frac{x}{2}}{2\tan\frac{x}{2}(2 + 2tan^2\frac{x}{2} + 3 - 3tan^2\frac{x}{2})} dx$$

In this type of equations, we apply substitution method so that equation may be solve in simple way

Let $tan\left(\frac{x}{2}\right) = t$

$$\frac{1}{2}.sec^2\frac{x}{2}dx = dt$$

Put these terms in above equation,we get $I = \int \frac{dt}{t(5-t^2)}$

$$I = \int \frac{t^{-3}dt}{(5t^{-2} - 1)}$$

Let us now again apply the substitution method in above equation

Let $t^{-2} = k$

$-2.t^{-3}dt = dk$

Substitute these terms in above equation gives-

$$I = -\frac{1}{10}\int \frac{dk}{k}$$

$$I = \frac{1}{10k^2} = \frac{1}{10}.\left(\frac{5-t^2}{t^2}\right)^2$$

$$= \frac{1}{10}.\left(\frac{5}{t^2} - 1\right)^2$$

Now put the value of t, t=tan(x/2) in above equation gives us the finally solution

$$I = \frac{1}{10}.\left(\frac{5}{tan^2\frac{x}{2}} - 1\right)^2$$

67. Question

Evaluate $\int \frac{1}{\sin x + \sin 2x} dx$

Answer

To solve this type of solution ,we are going to substitute the value of sinx and cosx in terms of tan(x/2)

$$sin\ x = \frac{2\left[tan\frac{x}{2}\right]}{1+\tan^2\frac{x}{2}}$$

$$cos\ x = \frac{\left(1-\frac{\tan^2 x}{2}\right)}{1+\frac{\tan^2 x}{2}}$$

$$I = \int \frac{1}{\frac{2\tan x/2}{1+tan^2\frac{x}{2}}\left(1+2.\frac{1-tan^2\frac{x}{2}}{1+tan^2\frac{x}{2}}\right)} dx$$

$$I = \int \frac{sec^2\frac{x}{2}}{2\tan\frac{x}{2}(3-tan^2\frac{x}{2})} dx$$

In this type of equations we apply substitution method so that equation may be solve in simple way

Let $tan\left(\frac{x}{2}\right) = t$

$$\frac{1}{2}.\sec^2\frac{x}{2}dx = dt$$

Put these terms in above equation,we get $I = \int \frac{dt}{t(3-t^2)}$

$$I = \int \frac{t^{-3}dt}{(3t^{-2}-1)}$$

Let us now again apply the substitution method in above equation

Let $t^{-2} = k$

$-2.t^{-3}dt = dk$

Substitute these terms in above equation gives-

$$I = -\frac{1}{6}\int \frac{dk}{k}$$

$$I = \frac{1}{6k^2}$$

$$= \frac{1}{6}.\left(\frac{3-t^2}{t^2}\right)^2$$

$$= \frac{1}{6}.(\frac{3}{t^2}-1)^2$$

Now put the value of t, t=tan(x/2) in above equation gives us the finally solution

$$I = \frac{1}{6}.(\frac{3}{tan^2\frac{x}{2}}-1)^2$$

68. Question

Evaluate $\int \frac{1}{\sin^4 x + \cos^4 x} dx$

Answer

Consider $\int \frac{1}{sin^4 x+cos^4 x}\ dx,$

Divide num and denominator by $\cos^4 x$ to get,

$$\int \frac{1}{sin^4 x + cos^4 x}\ dx = \int \frac{\frac{1}{cos^4 x}}{\frac{sin^4 x}{cos^4 x}+\frac{cos^4 x}{cos^4 x}}\ dx$$

$$= \int \frac{\sec^4 x}{\tan^4 x + 1}\ dx$$

$$= \int \frac{\sec^2 x\ .\sec^2 x}{\tan^4 x + 1}\ dx$$

$$= \int \frac{\sec^2 x\,(1+\tan^2 x)}{\tan^4 x + 1}\ dx$$

Let tan x = t

$\sec^2 x$ dx = dt

$$= \int \frac{(1+t^2)}{t^4+1}\ dt$$

Now divide both numerator and denominator by $\frac{1}{t^2}$ to get,

$$= \int \frac{\left(\frac{1}{t^2}+1\right)}{\left(t^2+\frac{1}{t^2}\right)+2-2}\ dt$$

$$= \int \frac{\left(\frac{1}{t^2}+1\right)}{\left(1-\frac{1}{t}\right)^2+2}\ dt$$

Let $1-\frac{1}{t} = u$

$$\left(1+\frac{1}{t^2}\right)dt = du$$

$$= \int \frac{du}{u^2+2}$$

$$= \int \frac{du}{u^2+\left(\sqrt{2}\right)^2}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)+c$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1-\frac{1}{t}}{\sqrt{2}}\right)+c$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1-\frac{1}{\tan x}}{\sqrt{2}}\right)+c$$

69. Question

Evaluate $\int \frac{1}{5-4\sin x}\,dx$

Answer

in this integral we are going to put the value of sin (x) in terms of tan(x/2)-

$$I=\int \frac{2dt}{5+5t^2-8t}$$

$$I=\frac{2}{5}\int \frac{1}{\left(t-\frac{4}{5}\right)^2+\left(\frac{3}{5}\right)^2}dt$$

By applying the formula of $1/(x^2+a^2)$ in above equation yields the integral-

$$I=\frac{2}{5}\cdot\frac{1}{\frac{3}{5}}.\tan^{-1}\frac{\left(t-\frac{4}{5}\right)}{\left(\frac{3}{5}\right)}$$

$$I=\frac{2}{3}.\tan^{-1}\frac{5t-4}{3}$$

By putting the value of t in above equation ,

$$I=\frac{2}{3}.\tan^{-1}\left(\frac{5}{3}\tan\frac{x}{2}-\frac{4}{3}\right)$$

70. Question

Evaluate $\int \sec^4 x\ dx$

Answer

above equation can be solve by using one formula that is ($i+\tan^2x=\sec^2x$)

$I=\int \sec^4x\ dx$

$=\int \sec^2x\ \sec^2x\ dx$

$=\int \sec^2x\ (1+\tan^2x)\ dx$

$=\int \sec^2x\ dx+\int \sec^2x\ \tan^2x\ dx$

Put tanx=t in above equation so that $\sec^2xdx=dt$

$$I=tanx+\int t^2dt\ =tanx+\frac{t^3}{3}$$

$$=tanx+\frac{tan^3x}{3}$$

71. Question

Evaluate $\int \text{cosec}^4\ 2x\ dx$

Answer

above equation can we solve by the formula of ($1+\cot^2x=\text{cosec}^2x$)

$I=\int \text{cosec}^4 2x\ dx$

$=\int \text{cosec}^2 2x\ (1+\cot^2 2x)\ dx$

$=\int \text{cosec}^2 2x\ dx+\int \text{cosec}^2 2x\ \cot^2 2x\ dx$

Let us consider that cot2x=t then $-2.\text{cosec}^2 2xdx=dt$

$$I = -\frac{\cot(2x)}{2} - \frac{1}{2}.(t^2 dt)$$

$$I = -\frac{\cot(2x)}{2} - \frac{1}{6}.(cot2x)^3$$

72. Question

Evaluate $\int \frac{1+\sin x}{\sin x(1+\cos x)} dx$

Answer

first divide nominator by denominator –

$$I = \int \frac{1}{sinx(1+cosx)}dx + \int \frac{1}{1+cosx}dx$$

$$= \int \frac{1}{sinx(1+cosx)}dx + \int \frac{1}{1+2cos^2x-1}dx$$

: To solve this type of solution ,we are going to substitute the value of sinx and cosx in terms of tan(x/2)

$$sin\ x = \frac{2\left[tan\frac{x}{2}\right]}{1+\tan^2\frac{x}{2}}$$

$$cos\ x = \frac{\left(1-\frac{\tan^2 x}{2}\right)}{1+\frac{\tan^2 x}{2}}$$

$$I = \int \frac{1}{\frac{2\tan x/2}{1+tan^2\frac{x}{2}}\left(1+\frac{1-tan^2\frac{x}{2}}{1+tan^2\frac{x}{2}}\right)}dx$$

$$I = \int \frac{sec^2x/2}{2tanx/2(1+tan^2\frac{x}{2}+1-tan^2\frac{x}{2})}dx$$

In this type of equations we apply substitution method so that equation may be solve in simple way

Let tan(x/2)=t

$1/2.sec^2(x/2)dx=dt$

Put these terms in above equation,we get $I = \int \frac{dt}{2t}$

Substitute these terms in above equation gives-

$$I = \frac{1}{2}\int \frac{dt}{t}$$

$$I = \frac{-1}{2t^2}$$

Now put the value of t, t=tan(x/2) in above equation gives us the finally solution

$$I = \frac{-1}{2}.\left(\frac{1}{tan^2\frac{x}{2}}\right)$$

73. Question

Evaluate $\int \frac{1}{2+\cos x} dx$

Answer

To solve this type of solution ,we are going to substitute the value of sinx and cosx in terms of tan(x/2)

$$sin\ x = \frac{2\left[tan\frac{x}{2}\right]}{1+\tan^2\frac{x}{2}}$$

$$cos\ x = \frac{\left(1-\frac{\tan^2 x}{2}\right)}{1+\frac{\tan^2 x}{2}}$$

$$I = \int \frac{1}{\left(2+\frac{1-tan^2\frac{x}{2}}{1+tan^2\frac{x}{2}}\right)} dx$$

$$I = \int \frac{sec^2\frac{x}{2}}{(2+2tan^2\frac{x}{2}+1-tan^2\frac{x}{2})} dx$$

In this type of equations we apply substitution method so that equation may be solve in simple way

Let tan(x/2)=t

1/2.sec^2(x/2)dx=dt

Put these terms in above equation,we get $I = 2\int \frac{dt}{(3+t^2)}$

$$I = \frac{2.1}{(\sqrt{3})}\tan^{-1}\frac{t}{\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}}.\tan^{-1}\left(\frac{\mathrm{x}}{2\sqrt{3}}\right)$$

74. Question

Evaluate $\int\sqrt{\frac{\mathrm{a}+\mathrm{x}}{\mathrm{x}}}\,\mathrm{dx}$

Answer

to solve this integral we have to apply substitution method for which we are going to put x=a.tan^{2}k

This means dx = 2.a.tank.sec^{2}k.dk,then I will be,

$$I = \int\sqrt{\frac{asec^2k}{atan^2k}}.2a.\tan k.sec^2\ k.dk = 2a.cosec\ k.\tan k.sec^2k.dk$$

In this above integral let tank =t then sec^{2}kdk=dt ,put in above equation-

$$I = 2a\int\sqrt{(t^2+1)}.dt$$

Apply the formula of sqrt(x^2+a^2)=x/2.sqrt(a^2+x^2)+a^2/2ln|x+sqrt(a^2+x^2)|

$$I = 2a\left[\frac{t}{2}.\sqrt{1+t^2}+\frac{1}{2}.\ln\left|t+\sqrt{1+t^2}\right|\right]$$

Now put the value of t in above integral t=tank,then finally integral will be-

$$I = 2a[\frac{tank}{2}.\sqrt{1+tan^2k}+\frac{1}{2}.\ln|tank+\sqrt{1+tan^2k}\,]$$

Now put the value of k in terms of x that is tan^{2}k=x/a in above integral –

$I = 2a[\frac{1}{2}\sqrt{\frac{x}{a}}.\sqrt{1+\frac{x}{a}} + \frac{1}{2}.\ln|\frac{1}{2}\sqrt{\frac{x}{a}} + \sqrt{1+\frac{x}{a}}]$

75. Question

Evaluate $\int \frac{6x+5}{\sqrt{6+x-2x^2}}\ dx$

Answer

$y = 6\int \frac{x+\frac{5}{6}}{\sqrt{6+x-2x^2}} dx$

$y = \frac{6}{-4}\int \frac{-4\left(x+\frac{5}{6}\right)}{\sqrt{6+x-2x^2}} dx$

$y = -\frac{3}{2}\int \frac{-4x-\frac{10}{3}+1-1}{\sqrt{6+x-2x^2}} dx$

$y = -\frac{3}{2}\int \frac{-4x+1}{\sqrt{6+x-2x^2}} dx\ -\frac{3}{2}\int \frac{-\frac{10}{3}-1}{\sqrt{6+x-2x^2}} dx$

$A = -\frac{3}{2}\int \frac{-4x+1}{\sqrt{6+x-2x^2}} dx$

Let, 6 + x – $2x^2$ = t

Differentiating both side with respect to t

$(1-4x)\frac{dx}{dt} = 1$ ⇒ (1 – 4x)dx = dt

$A = -\frac{3}{2}\int \frac{1}{\sqrt{t}} dt$

$A = -\frac{3}{2}2\sqrt{t} + c_1$

Again, put t = 6 + x – $2x^2$

$A = -3\sqrt{6+x-2x^2} + c_1$

$B = -\frac{3}{2}\int \frac{-\frac{10}{3}-1}{\sqrt{6+x-2x^2}} dx$

$B = \frac{13}{2}\int \frac{1}{\sqrt{6+x-2x^2}} dx$

Make perfect square of quadratic equation

6 + x – $2x^2$= $2\left(\left(\frac{7}{4}\right)^2 - \left(x-\frac{1}{4}\right)^2\right)$

$B = \frac{13}{2\sqrt{2}}\int \frac{1}{\sqrt{\left(\frac{7}{4}\right)^2 - \left(x-\frac{1}{4}\right)^2}} dx$

Use formula $\int \frac{1}{\sqrt{a^2-x^2}} dx\ =\ \sin^{-1}\frac{x}{a}$

$B = \frac{13}{2\sqrt{2}}\sin^{-1}\frac{\left(x-\frac{1}{4}\right)}{\left(\frac{7}{4}\right)} + c_2$

$$B = \frac{13}{2\sqrt{2}}\sin^{-1}\frac{4x-1}{7} + c_2$$

The final solution of the question is y = A + B

$$y = -3\sqrt{6+x-2x^2} + \frac{13}{2\sqrt{2}}\sin^{-1}\left(\frac{4x-1}{7}\right) + C$$

76. Question

Evaluate $\int \frac{\sin^5 x}{\cos^4 x} dx$

Answer

to solve this type of integration we have to let cosx either sinx =t then manuplate them

Let cos x =t then -sinx dx =dt

Also apply the formula of ($\sin^2 t + \cos^2 t = 1$)

$$I = \int \frac{\sin^5 x}{\cos^4 x} dx = -\int \frac{(1-t^2)^2}{t^4} dt$$

$$= -\int \frac{1+t^4-2t^2}{t^4} dt$$

$$= -[\int t^{-4} dt + \int 1dt - \int \frac{2}{t^2} dt]$$

$$I = \frac{t^{-3}}{3} - t - \frac{2}{t}$$

Now put the value of t in above integral

$$I = \frac{1}{3\cos^3 x} - \cos x - \frac{2}{\cos x}$$

77. Question

Evaluate $\int \frac{\cos^5 x}{\sin x} dx$

Answer

to solve this type of integration we have to let cosx either sinx =t then manuplate them

Let sin x =t then cosx dx =dt

Also apply the formula of ($\sin^2 t + \cos^2 t = 1$)

$$I = \int \frac{\cos^5 x}{\sin x} dx$$

$$= \int \frac{(1-t^2)^2}{t} dt = \int \frac{1+t^4-2t^2}{t} dt = \int \frac{1}{t} dt + \int t^3 dt - \int 2t dt$$

$$I = -\frac{1}{t^2} + \frac{t^4}{4} - t^2$$

Now put the value of t in above integral

$$I = \frac{-1}{\sin^2 x} + (\sin^4 x)/4 - \sin^2 x$$

78. Question

Evaluate $\int \frac{sin^6 x}{cos x}\, dx$

Answer

$$y = \int \left(\frac{sin^4 x(1-cos^2 x)}{cos x}\right) dx$$

$$y = \int \left(\frac{sin^4 x}{\cos x} - \frac{sin^4 x\, cos^2 x}{\cos x}\right) dx$$

$$y = \int \left(\frac{sin^2 x(1-cos^2 x)}{\cos x} - sin^4 x \cos x\right) dx$$

$$y = \int \left(\frac{sin^2 x}{\cos x} - \frac{sin^2 x\, cos^2 x}{\cos x} - sin^4 x \cos x\right) dx$$

$$y = \int \left(\frac{sin^2 x}{\cos x} - sin^2 x \cos x - sin^4 x \cos x\right) dx$$

$$y = \int \left(\frac{1-cos^2 x}{\cos x}\right) dx - \int (sin^2 x \cos x + sin^4 x \cos x) dx$$

Let, sin x = t

Differentiating both side with respect to t

$\cos x \frac{dx}{dt} = 1 \Rightarrow$ cos x dx = dt

$$y = \int \left(\frac{1}{\cos x} - \cos x\right) dx - \int t^2 + t^4 dt$$

$$y = \ln(\sec x + \tan x) - \sin x - \frac{t^3}{3} - \frac{t^5}{5} + c$$

Again put t = sin x

$$y = \ln(\sec x + \tan x) - \sin x - \frac{sin^3 x}{3} - \frac{sin^5 x}{5} + c$$

$$y = \frac{1}{2}\ln\left(\frac{1+\sin x}{1-\sin x}\right) - \sin x - \frac{sin^3 x}{3} - \frac{sin^5 x}{5} + c$$

79. Question

Evaluate $\int \frac{\sin^2 x}{\cos^6 x} dx$

Answer

dividing by $\cos^6$x yields-

I=∫$\tan^2$x.$\sec^4$x dx

Let us consider tanx=t

Then $\sec^2$xdx=dt,put in above equation-

$$I = \int t^2(1+t^2)dt = \int (t^2+t^4)dt = \int t^2 dt + \int t^4 dt = \frac{t^3}{3} + \frac{t^5}{5}$$

Now reput the value of t,which is t=tanx

$$I = \frac{(tan^3 x)}{3} + \frac{tan^5 x}{5}$$

80. Question

Evaluate $\int \sec^6 x \, dx$

Answer

in this integral we will use the formula $1+\tan^2 x=\sec^2 x$,

$I = \int \sec^2 x \sec^4 x \, dx$

$= \int \sec^2 x \, (1 + \tan^2 x)^2 \, dx$

Now put tan x=t which means $\sec^2 x dx=dt$,

$I = \int (1+t^2)^2 \, dt$

$= \int (1+t^4+2t^2) \, dt$

Now put the value of t, which is t=tan x in above integral-

$$I = tanx + \frac{tan^5 x}{5} + 2.\frac{tan^3 x}{3}$$

81. Question

Evaluate $\int \tan^5 x \sec^3 x \, dx$

Answer

in this integral we will use the formula $1+\tan^2 x=\sec^2 x$,

Then equation will be transform in below form-

$I = \int \tan^5 x \sec^2 x \sec x \, dx$

$= \int \sec x \tan^5 x \sec^2 x \, dx$

Now put tan x=t which means $\sec^2 x dx=dt$,

$$I = \int t^5.\sqrt{1+t^2}\, dt$$

In this above integral put $1+t^2=k^2$

that is mean tdt=kdk

$I = \int (k^4 + 1- 2k) \, k^2 \, dk$

$= \int (k^6 + k^2 - 2k^3) dk$

$$= \frac{k^7}{7} + \frac{k^3}{3} - \frac{k^4}{2}$$

Now put the value of $k=(1+t^2)=\sec^2 x$ in above equation-

$$I = \frac{sec^{14}x}{7} + \frac{sec^6 x}{3} - \frac{sec^8 x}{2}$$

82. Question

Evaluate $\int \tan^3 x \sec^4 x \, dx$

Answer

in this integral we will use the formula $1+\tan^2 x=\sec^2 x$, Then equation will be transform in below form-

I = $\int \tan^3 x \sec^2 x \sec^2 x\, dx$

= $\int \tan^3 x (1 + \tan^2 x) \sec^2 x\, dx$

Now put tanx=t which means $\sec^2 x dx = dt$,

$$I = \int t^3(1+t^2)\,dt = \int (t^4 + t^5)dt$$

$$I = \frac{t^5}{5} + \frac{t^6}{6}$$

Now put the value of t,which is t=tanx in above integral-

$$I = \frac{tan^5 x}{5} + \frac{tan^6 x}{6}$$

83. Question

Evaluate $\int \frac{1}{\sec x + \operatorname{cosec} x}\, dx$

Answer

$$y = \int \frac{\sin x \cos x}{\sin x + \cos x} dx$$

$$y = \frac{1}{2}\int \frac{1 + 2\sin x \cos x - 1}{\sin x + \cos x} dx$$

Use 1 = $\sin^2 x + \cos^2 x$

$$y = \frac{1}{2}\int \frac{sin^2 x + cos^2 x + 2\sin x \cos x}{\sin x + \cos x} dx - \frac{1}{2}\int \frac{1}{\sin x + \cos x} dx$$

Use sin x + cos x $= \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)$

$$= \sqrt{2}\left(\sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4}\right)$$

$$= \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$$

$$y = \frac{1}{2}\int \frac{(\sin x + \cos x)^2}{\sin x + \cos x} dx - \frac{1}{2}\int \frac{1}{\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)} dx$$

$$y = \frac{1}{2}\int \sin x + \cos x\, dx - \frac{1}{2\sqrt{2}}\int cosec\left(x + \frac{\pi}{4}\right) dx$$

$$y = \frac{1}{2}(-\cos x + \sin x) - \frac{1}{2\sqrt{2}}\ln\left(\tan\left(\frac{x}{2} + \frac{\pi}{8}\right)\right) + c$$

84. Question

Evaluate $\int \sqrt{a^2 + x^2}\, dx$

Answer

in these type of problems we put the value of x=a tank

That is mean that dx=a $\sec^2 k$ dk

$$I = \int \sqrt{a^2 + a^2 tan^2 k}\; a.sec^2 k.dk$$

$= \int$ a. sec k. a. sec^2k dk

$=\int a^2 sec^3k$ dk

By upper solve questions we can find out the value of integration of sec^3x,whixh is equal to

$$i = \int sec^3 x dx = \frac{1 + secx.tanx}{2}$$

Put the value of integration of sec^3x in above equation we get our finally integral which is –

$$I = a^2.\frac{1 + seck.tank}{2}$$

Now put the value of k which is tan^{-1}(x/a) in above equation-

$$I = a^2.\left(\frac{1 + \frac{x}{a}.\sec(tan^{-1}\frac{x}{a})}{2}\right)$$

85. Question

Evaluate $\int \sqrt{x^2 - a^2}\, dx$

Answer

Consider $\int \sqrt{x^2 - a^2}\, dx$,

Let $I = \sqrt{x^2 - a^2}$ and II = 1

As ∫ I.II dx = I.∫ II dx - ∫ [d/dx(I). ∫ II dx]

So,

$$= \sqrt{x^2 - a^2} \int 1\, dx - \int \frac{d}{dx}\left(\sqrt{x^2 - a^2}\right).\int 1\, dx$$

$$= x\sqrt{x^2 - a^2} - \int \frac{1}{2\sqrt{x^2 - a^2}}.2x.x\, dx$$

$$= x\sqrt{x^2 - a^2} - \int \frac{x^2}{\sqrt{x^2 - a^2}} dx$$

$$= x\sqrt{x^2 - a^2} - \int \frac{x^2 - a^2 + a^2}{\sqrt{x^2 - a^2}} dx$$

$$= x\sqrt{x^2 - a^2} - \int \frac{x^2 - a^2}{\sqrt{x^2 - a^2}} dx - \int \frac{a^2}{\sqrt{x^2 - a^2}} dx$$

$$I = x\sqrt{x^2 - a^2} - \int \sqrt{x^2 - a^2} dx - \int \frac{a^2}{\sqrt{x^2 - a^2}} dx$$

$$I = x\sqrt{x^2 - a^2} - I - \int \frac{a^2}{\sqrt{x^2 - a^2}} dx$$

$$2I = x\sqrt{x^2 - a^2} - \int \frac{a^2}{\sqrt{x^2 - a^2}} dx$$

$$2I = x\sqrt{x^2 - a^2} - a^2 \log\left|x + \sqrt{x^2 - a^2}\right| + c$$

$$I = \frac{1}{2}\left(x\sqrt{x^2 - a^2} - a^2 \log\left|x + \sqrt{x^2 - a^2}\right| + c\right)$$

46. Question

Evaluate $\int \frac{1}{1-x-4x^2} dx$

Answer

Given, $\int \frac{1}{(1-x-4x^2)} dx$

$$= -\int \frac{1}{4x^2+x-1} dx$$

$$= -\int \frac{1}{4x^2+x+\frac{1}{16}-\frac{17}{16}} dx$$

$$= -\int \frac{1}{\left(2x+\frac{1}{4}\right)^2-\frac{17}{16}} dx$$

$$= -\int \frac{1}{\left(2x+\frac{1}{4}\right)^2-\left(\frac{\sqrt{17}}{4}\right)^2} dx$$

It is clearly of the form, $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a}\log\frac{x-a}{x+a} + c$

Where $x = 2x + \frac{1}{4}$; $a = \frac{\sqrt{17}}{4}$

$$= -\frac{1}{2(\frac{\sqrt{17}}{4})} log \frac{2x+\frac{1}{4}-\frac{\sqrt{17}}{4}}{2x+\frac{1}{4}+\frac{\sqrt{17}}{4}} + c$$

$$= -\frac{2}{\sqrt{17}} log \frac{2x+\frac{1}{4}-\frac{\sqrt{17}}{4}}{2x+\frac{1}{4}+\frac{\sqrt{17}}{4}} + c$$

47. Question

Evaluate $\int \frac{1}{3x^2+13x-10} dx$

Answer

Given, $\int \frac{1}{3x^2+13x-10} dx$

Now, $3x^2+13x-10$

$= 3x^2+15x-2x-10$

= 3x(x+5)-2(x-5)

= (x-5) (3x-2)

$$\frac{1}{3x^2+13x-10} \cong \frac{A}{x+5} + \frac{B}{3x-2}$$

1 ≅ A (3x-2) + B(x+5)

Equating ‘x’ coeff: -

0=3A+B

B=-3A

Equating constant:-

1=-2A+5B

1=-2A+5(-3A)

1=-2A-15A

1=-17A

$A = -\frac{1}{17}$

$B = -3(-\frac{1}{17})$

$B = \frac{3}{17}$

$\frac{1}{3x^2+13x-10} \cong -\frac{1}{17(x+5)} + \frac{3}{17(3x-2)}$

$\int \frac{1}{3x^2+13x-10} dx = \int -\frac{1}{17(x+5)} + \frac{3}{17(3x-2)} dx$

$= -\frac{1}{17}\int \frac{1}{x+5} dx + \frac{3}{17}\int \frac{1}{3x-2} dx$

$= -\frac{1}{17} log(x+5) + \frac{3}{17} log(3x-2) + c$

48. Question

Evaluate $\int \frac{\sin x}{\cos^2 x - 2\cos x - 3} dx$

Answer

Given, $\int \frac{sinx}{\cos^2 x - 2cosx - 3} dx$

Let cosx=t

-sinx dx=dt

$= \int \frac{dt}{t^2-2t-3}$

Now, t^2-2t-3

= t^2-3t+t-3

= t(t-3)+t-3

= (t-3) (t+1)

$\frac{1}{t^2-2t-3} \cong \frac{A}{t-3} + \frac{B}{t+1}$

1 ≅ A(t-1)+B(t-3)

Equating ’t’ coeff:-

0=A+B

A=-B

Equating constant:-

1=-A-3B

1=-(-B)-3B

1=-2B

$B = \frac{-1}{2}$

$A = -\left(\frac{-1}{2}\right)$

$A = \frac{1}{2}$

$\frac{1}{t^2 - 2t - 3} \cong \frac{1}{2(t-3)} + \frac{-1}{2(t+1)}$

$\int \frac{1}{t^2 - 2t - 3} dt = \frac{1}{2} \int \frac{1}{t-3} dt - \frac{1}{2} \int \frac{1}{t-1} dt$

$= \frac{1}{2} log(t-3) - \frac{1}{2} log(t-1) + c$

$= \frac{1}{2} [log(cos\, x - 3) - log(cos\, x - 1)] + c$

49. Question

Evaluate $\int \sqrt{\text{cosec x} - 1}\, \text{dx}$

Answer

Given, $\int \sqrt{cosec\ x - 1}$ dx

$= \int \sqrt{\frac{1}{sin\, x} - 1}\ dx$

$= \int \sqrt{\frac{1 - sin\, x}{sinx}}\ dx$

Rationalising the denominator:-

$= \int \sqrt{\frac{(1 - sin\, x)(1 + sin\, x)}{(sin\, x)(1 + sin\, x)}}\ dx$

$= \int \sqrt{\frac{(1 - sin^2 x)}{sin\, x(1 + sin\, x)}}\ dx$

$= \int \sqrt{\frac{cos^2 x}{sin\, x(1 + sinx)}}\ dx$

$= \int \frac{cos\, x}{\sqrt{sin\, x(1 + sin\, x)}}\ dx$

Let $\sin x = t$

$\cos x\ dx = dt$

$= \int \frac{dt}{\sqrt{t(t+1)}}$

$= \int \frac{dt}{\sqrt{t^2 + t}}$

$$= \int \frac{dt}{\sqrt{t^2 + t - \frac{1}{4} + \frac{1}{4}}}$$

$$= \int \frac{dt}{\sqrt{\left(t + \frac{1}{2}\right)^2 - \frac{1}{4}}}$$

$$= \int \frac{dt}{\sqrt{\left(t + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}$$

Clearly, it is of the form $\int \frac{1}{\sqrt{x^2-a^2}} dx = \cos^{-1}\left(\frac{x}{a}\right)$

Where $x = t + \frac{1}{2}$; $a = \frac{1}{2}$

$$= cos^{-1}\left(\frac{t + \frac{1}{2}}{\frac{1}{2}}\right) + c$$

$$= cos^{-1}[2(sinx + \frac{1}{2})] + c$$

50. Question

Evaluate $\int \frac{1}{\sqrt{3 - 2x - x^2}} dx$

Answer

Given, $\int \frac{1}{\sqrt{3-2x-x^2}} dx$

$$= \int \frac{1}{\sqrt{4 - 1 - 2x - x^2}} dx$$

$$= \int \frac{1}{\sqrt{4 - (x^2 + 2x + 1)}} dx$$

$$= \int \frac{1}{\sqrt{4 - (x + 1)^2}} dx$$

$$= \int \frac{1}{\sqrt{(2)^2 - (x + 1)^2}} dx$$

It is clearly of the form, $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

Where, a=2; x=x+1

$$= sin^{-1}\left(\frac{x + 1}{2}\right) + c$$

51. Question

Evaluate $\int \frac{x + 1}{x^2 + 4x + 5} dx$

Answer

Given, $\int \frac{x+1}{x^2+4x+5} dx$

Consider, x+1 $\cong$ A $\frac{dy}{dx}(x^2 + 4x + 5) + B$

x+1≅A(2x+4)+B

Equating ‘x’coeff:-

1=2A

$$A = \frac{1}{2}$$

equating constant:-

1=4A+B

$$1 = 4\left(\frac{1}{2}\right) + B$$

1=2+B

B=-1

x+1 ≅ 1/2 (2x+4)-1

Now, $\int \frac{x+1}{x^2+4x+5} dx$

$$= \int \frac{\frac{1}{2}(2x+4) - 1}{x^2 + 4x + 5} dx$$

$$= \frac{1}{2}\int \frac{2x+4}{x^2+4x+5} dx - \int \frac{1}{x^2+4x+5} dx$$

$$[Since, \int \frac{f^I(x)}{f(x)} dx = log[f(x)] + c]$$

$$= \frac{1}{2} log\,(x^2 + 4x + 5) - \int \frac{1}{x^2 + 4x + 4 + 1} dx$$

$$= \frac{1}{2} log\,(x^2 + 4x + 5) - \int \frac{1}{(x+2)^2 + (1)^2} dx$$

$$= \frac{1}{2} log\,(x^2 + 4x + 5) - \frac{1}{1} tan^{-1}(\frac{x+2}{1})\ dx$$

$$[Since, \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} tan^{-1}(\frac{x}{a}) + c]$$

$$= \frac{1}{2} log\,(x^2 + 4x + 5) - tan^{-1}(x+2) + c$$

52. Question

Evaluate $\int \frac{5x+7}{\sqrt{(x-5)(x-4)}} dx$

Answer

Given, $\int \frac{5x+7}{\sqrt{(x-5)(x-4)}} dx$

$$= \int \frac{5x+7}{\sqrt{x^2 - 9x + 20}} dx$$

Now, $5x + 7 \cong A \frac{dy}{dx}(x^2 - 9x + 20) + B$

5x+7 ≅ A (2x-9)+B

Equating’x’ coeff:-

5=2A

A=$\frac{5}{2}$

Equating constant:-

7=-9A+B $7 = -9\left(\frac{5}{2}\right) + B$

$B = 7 + \frac{45}{2}$

$B = \frac{59}{2}$

$5x + 7 \cong \frac{5}{2}(2x - 9) + \frac{59}{2}$

$= \int \frac{5x - 7}{\sqrt{x^2 - 9x + 20}} dx$

$= \int \frac{\frac{5}{2}(2x - 9) + \frac{59}{2}}{\sqrt{x^2 - 9x + 20}} dx$

$= \frac{5}{2} \int \frac{2x - 9}{\sqrt{x^2 - 9x + 20}} dx + \frac{59}{2} \int \frac{1}{\sqrt{x^2 - 9x + 20}} dx$

$[Since, \int \frac{f^I(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c]$

$= \frac{5}{2}.2(\sqrt{x^2 - 9x + 20}) + \frac{59}{2} \int \frac{1}{\sqrt{\left(x + \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx$

$= 5\sqrt{x^2 - 9x + 20} + \frac{59}{2} . \frac{1}{2(\frac{1}{2})} . cosh^{-1}[\frac{x + \frac{9}{2}}{\frac{1}{2}}] + c$ $[since, \int \frac{1}{\sqrt{x^2 - a^2}} dx = cosh^{-1}[\frac{x}{a}] + c]$

$= 5\sqrt{x^2 - 9x + 20} + \frac{59}{2} cosh^{-1} 2\left[x + \frac{9}{2}\right] + c$

53. Question

Evaluate $\int \sqrt{\frac{1+x}{x}}\, dx$

Answer

Given, $\int \sqrt{\frac{1+x}{x}}\, dx$

Let $\sqrt{x + 1} = u$

⇒ $u^2 = x+1$

⇒ $u^2 - 1 = x$

$\frac{1}{2\sqrt{x + 1}} dx = du$

2 du = dx

$$\int \sqrt{\frac{1+x}{x}}\ dx = \int \frac{u}{u^2-1} 2u\ du$$

$$= 2\int \frac{u^2}{u^2-1}\ du$$

$$= 2\int \frac{u^2-1+1}{u^2-1}\ du$$

$$= 2\left[\int \frac{u^2-1}{u^2-1}\ du + \int \frac{1}{u^2-1}\ du\right]$$

$$= 2\left[\int 1\ du + \int \frac{1}{u^2-1}\ du\right]$$

As we know,

$$\int \frac{1}{x^2-a^2}\ dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$$

$$= 2\left[u\ + \frac{1}{2}\log\left|\frac{u-1}{u+1}\right|\right] + c$$

Now substitute back the value of u.

$$= 2\sqrt{x+1} + \frac{1}{2}\log\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + c$$

54. Question

Evaluate $\int\sqrt{\frac{1-x}{x}}\ dx$

Answer

Given, $\sqrt{\frac{1-x}{x}}\ dx$

Let, $\sqrt{x} = t$

$$\frac{d}{dx}(\sqrt{x}) = dt$$

$$\frac{1}{2\sqrt{x}}dx = dt$$

dx =2t dt

Now, $\int \frac{\sqrt{1-t^2}}{t}\ 2t\ dt$

$$= 2\int \sqrt{1-t^2}\, dt$$

Consider, t=sin k

dt=cos k dk

$$= 2\int \sqrt{1-\sin^2 k}\ .cosk\ dk$$

$$= 2\int \sqrt{\cos^2 k}\, .\, cosk\ dk$$

$=2\int \cos^2 k\ dk$

$=\int 2\cos^2 k\ dk$

$=\int \cos 2k-1\ dk$ [since, $\cos 2x=2\cos^2 x-1$]

$$=\frac{\sin 2k}{2}-k+c$$

$$=\frac{2sink\ cosk}{2}-k+c$$

$=t\cos(\sin^{-1} t)-2\sin^{-1} t+2c$

$=\sqrt{x}\cos(\sin^{-1}\sqrt{x})-2\sin^{-1}\sqrt{x}+2c$

55. Question

Evaluate $\int \frac{\sqrt{a}-\sqrt{x}}{1-\sqrt{ax}}dx$

Answer

Given, $\int \frac{\sqrt{a}-\sqrt{x}}{1-\sqrt{ax}}dx$

Let $1-\sqrt{ax}=t$

$$-\frac{1}{2\sqrt{ax}}a\ dx=dt$$

$$dx=-\frac{2\sqrt{ax}}{a}dt$$

Now,

$\sqrt{ax}=1+t$

$ax=(1+t)^2$

$$x=\frac{(1+t)^2}{a}$$

$$=\int \frac{\sqrt{a}-\sqrt{\frac{(1+t)^2}{a}}}{t}\times\frac{-2\sqrt{a}(1+t)}{a}dt$$

$$=\int \frac{\sqrt{a}-\left(\frac{1+t}{\sqrt{a}}\right)}{t}\times\frac{-2\sqrt{a}(1+t)}{a}dt$$

$$=\int \frac{a-1-t}{t}\times\frac{-2\sqrt{a}(1+t)}{a\sqrt{a}}dt$$

$$=\int \frac{(a-1-t)}{t}\times\frac{-2(1+t)}{a}dt$$

$$=2\int \frac{(a-1-t)}{t}\times\frac{(-1-t)}{a}dt$$

$$=2\int \frac{(-a-at+1+t+t+t^2)}{at}dt$$

$$=2\int \frac{(-a-at+1+2t+t^2)}{at}dt$$

$$=2\int\left(-\frac{1}{t}-1+\frac{1}{at}+\frac{2}{a}+\frac{t}{a}\right)dt$$

$$= 2\left[-\log t - t + \frac{1}{a}\log t + \frac{2}{a}t + \frac{t^2}{2a}\right] + c$$

$$= \left[-2\log t - 2t + \frac{2}{a}\log t + \frac{4}{a}t + \frac{t^2}{a}\right] + c$$

Put back the value of t to get,

$$= \left[-2\log(1-\sqrt{ax}) - 2(1-\sqrt{ax}) + \frac{2}{a}\log(1-\sqrt{ax}) + \frac{4}{a}(1-\sqrt{ax}) + \frac{(1-\sqrt{ax})^2}{a}\right] + c$$

56. Question

Evaluate $\int \frac{1}{(\sin x - 2\cos x)(2\sin x + \cos x)} dx$

Answer

Given, $\int \frac{1}{(sinx-2cosx)(2sinx+cosx)} dx$

$$= \int \frac{1}{2\sin^2 x + sinxcosx - 4cosxsinx - 2\cos^2 x} dx$$

$$= \int \frac{1}{2\sin^2 x - 3\cos x \sin x - 2\cos^2 x} dx$$

$$= \int \frac{1}{\cos^2 x[2\tan^2 x - 3tanx - 2]} dx$$

$$= \int \frac{\sec^2 x}{2\tan^2 x - 3tanx - 2} dx$$

Let tanx=t

$$\frac{d}{dx}(tanx) = dt$$

Sec^2x dx=dt

Now, $\int \frac{dt}{2t^2-3t-2}$

$$= \int \frac{dt}{(2t+1)(t-2)}$$

Now, $\frac{1}{(2t+1)(t-2)} \cong \frac{A}{2t+1} + \frac{B}{t-2}$

1≅ A(t-2)+B(2t+1)

Equating ‘t’ coeff: -

0=A+2B

A=-2B

Equating constant: -

1=-2A+B

1=-2(-2B) +B

1=5B

$B=\frac{1}{5}$

$$A=\frac{-2}{5}$$

$$\frac{1}{(2t+1)(t-2)}=\frac{-2}{5(2t+1)}+\frac{1}{5(t-2)}$$

Now, $\int\frac{1}{(2t+1)(t-2)}dt=\frac{-2}{5}\int\frac{1}{2t+1}dt+\frac{1}{5}\int\frac{1}{t-2}dt$

$$=\frac{-2}{5}\log(2t+1)+\frac{1}{5}\log(t-2)+c$$

$$=\frac{-2}{5}\log(2tanx+1)+\frac{1}{5}\log(tanx-2)+c$$

57. Question

Evaluate $\int\frac{1}{4\sin^2 x+4\sin x\cos x+5\cos^2 x}dx$

Answer

Given, $\int\frac{1}{4\sin^2 x+4\sin x\cos x+5\cos^2 x}dx$

$$=\int\frac{1}{cos^2 x[4\,tan^2 x+4\,tan\,x+5]}dx$$

$$=\int\frac{sec^2 x}{4\,tan^2 x+4\,tan\,x+5}dx$$

Let tanx=t

$$\frac{d}{dx}(\tan x)=dt$$

$$\sec^2 x\;dx=dt$$

$$=\int\frac{dt}{4t^2+4t+5}$$

$$=\int\frac{dt}{4t^2+4t+1+4}$$

$$=\int\frac{dt}{(2t+1)^2+(2)^2}$$

$$=\frac{1}{2}tan^{-1}[\frac{2t+1}{2}]+c$$

$$=\frac{1}{2}tan^{-1}[\frac{2\,tan\,x+1}{2}]+c$$

58. Question

Evaluate $\int\frac{1}{a+b\tan x}dx$

Answer

Given, $\int\frac{1}{a+b\,tan\,x}dx$

Consider, a=b=1

$=\int\frac{1}{1+\tan x}dx$

$=\int \frac{1}{1+\frac{\sin x}{\cos x}} dx$

$=\int \frac{\cos x}{\cos x+\sin x} dx$

Now, cosx=A (cosx+sinx) +B $\frac{d}{dx}$(cosx+sinx)

=A (cosx+sinx) +B (-sinx+cosx)

Equating 'cosx' coeff:- Equating 'sinx' coeff:-

1=A+B 0=A-B

A=B

1=A+A

2A=1

A=1/2 B=1/2

$\cos x = \frac{1}{2}(\cos x + \sin x) + \frac{1}{2}(-\sin x + \cos x)$

$= \int \frac{\cos x}{\cos x + \sin x} dx$

$= \int \frac{\frac{1}{2}(\cos x + \sin x) + \frac{1}{2}(-\sin x + \cos x)}{\cos x + \sin x} dx$

$= \int \frac{\frac{1}{2}(\cos x + \sin x)}{\cos x + \sin x} dx + \int \frac{\frac{1}{2}(-\sin x + \cos x)}{\cos x + \sin x} dx$

$= \frac{1}{2}\int 1dx + \frac{1}{2}\int \frac{-\sin x + \cos x}{\cos x + \sin x} dx$

[since, $\int \frac{f^{I}(x)}{f(x)} dx = \log[f(x)] + c$]

$= \frac{1}{2}(x)+\frac{1}{2}\log(\cos x + \sin x) + c$

59. Question

Evaluate $\int \frac{1}{\sin^2 x + \sin 2x} dx$

Answer

Given, $\int \frac{1}{\sin^2 x+\sin 2x} dx$

$= \int \frac{1}{\sin^2 x + 2\sin x\ cos\ x} dx$

$= \int \frac{1}{\sin^2 x(1 + 2\cot x)} dx$

$= \int \frac{cosec^2 x}{1 + \cot x} dx$

Let $\cot x = t$

$\frac{d}{dx}(cotx) = dt$

$-$cosec^{2}x dx=dt

Now, $-\int \frac{dt}{1+t}$

$= -\log(1+t) + c$

$= -\log(1+\cot x) + c$

60. Question

Evaluate $\int \frac{\sin x + 2\cos x}{2\sin x + \cos x} dx$

Answer

Given, $\int \frac{\sin x + 2\cos x}{2\sin x + \cos x} dx$

$\sin x + 2\cos x = A(2\sin x + \cos x) + B\frac{d}{dx}(2\sin x - \cos x)$

$= A(2\sin x + \cos x) + B(2\cos x - \sin x)$

Equating 'sin x' coeff: -

1=2A-B

B=2A-1

Equating 'cos x' coeff:-

2=A+2B

2=A+2(2A-1)

2=A+4A-2

4=5A

$A = \frac{4}{5}$

$B = 2\left(\frac{4}{5}\right) - 1$

$B = \frac{8}{5} - 1$

$B = \frac{3}{5}$

Now, $\sin x + 2\cos x = \frac{4}{5}(2\sin x + \cos x) + \frac{3}{5}(2\cos x - \sin x)$

$= \int \frac{\frac{4}{5}(2\sin x + \cos x) + \frac{3}{5}(2\cos x - \sin x)}{2\sin x + \cos x} dx$

$= \frac{4}{5}\int 1dx + \frac{3}{5}\int \frac{2\cos x - \sin x}{2\sin x + \cos x} dx$

$= \frac{4}{5}(x) + \frac{3}{5}\log(2\sin x + \cos x) + c$

61. Question

Evaluate $\int \frac{x^3}{\sqrt{x^8+4}} dx$

Answer

Given, $\int \frac{x^3}{\sqrt{x^8+4}} dx$

Put, $x^4=t$

$4x^3dx=dt$

$x^3dx=\frac{1}{4}dt$

$$= \int \frac{x^3}{\sqrt{(x^4)^2+4}} dx$$

$$= \int \frac{\frac{1}{4}dt}{\sqrt{t^2+4}}$$

$$= \frac{1}{4}\int \frac{1}{\sqrt{t^2+2^2}} dx$$

$$= \frac{1}{4}sinh^{-1}[\frac{t}{2}] + c$$

$$= \frac{1}{4}sinh^{-1}\left[\frac{x^4}{2}\right] + c$$

62. Question

Evaluate $\int \frac{1}{2-3\cos 2x} dx$

Answer

Given, $\int \frac{1}{2-3\cos 2x} dx$

Put tanx=t

$$\frac{d}{dx}(\tan x) = dt$$

$sec^2x\ dx=dt$

$dx = \frac{dt}{1+t^2}$

and $\cos 2x = \frac{1-t^2}{1+t^2}$

Now, $\int \frac{1}{2-3[\frac{1-t^2}{1+t^2}]} \cdot \frac{dt}{1+t^2}$

$$= \int \frac{1+t^2}{2(1+t^2)-3(1-t^2)} \frac{dt}{1+t^2}$$

$$= \int \frac{1}{2+2t^2-3+3t^2} dt$$

$$= \int \frac{1}{5t^2-1} dt$$

$$= \frac{1}{5}\int \frac{1}{t^2-\frac{1}{5}} dt$$

$$= \frac{1}{5}\int \frac{1}{t^2-\left(\frac{1}{\sqrt{5}}\right)^2} dt \ [since, \int \frac{1}{x^2-a^2} dx = \frac{1}{2a}\ log\left|\frac{x-a}{x+a}\right| + c]$$

$$= \frac{1}{5} \cdot \frac{1}{2(\frac{1}{\sqrt{5}})} log \left| \frac{t - \frac{1}{\sqrt{5}}}{t + \frac{1}{\sqrt{5}}} \right| + c$$

$$= \frac{1}{2\sqrt{5}} log \left| \frac{tanx - \frac{1}{\sqrt{5}}}{tan x + \frac{1}{\sqrt{5}}} \right| + c$$

63. Question

Evaluate $\int \frac{\cos x}{\frac{1}{4} - \cos^2 x} dx$

Answer

Given, $\int \frac{\cos x}{\frac{1}{4} - \cos^2 x} dx$

$$= \int \frac{cos\, x}{\frac{1}{4} - (1 - sin^2\, x)} dx$$

Let $\sin x = t$

$\cos x dx = dt$

$$= \int \frac{dt}{\frac{1}{4} - (1 - t^2)}$$

$$= \int \frac{dt}{\frac{1 - 4 + 4t^2}{4}}$$

$$= \int \frac{4\, dt}{4t^2 - 3}$$

$$= 4 \int \frac{1}{(2t)^2 - (\sqrt{3})^2} dt$$

[since, $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$]

$$= 4. \frac{1}{2\sqrt{3}} log \left| \frac{2t - \sqrt{3}}{2t + \sqrt{3}} \right| + c$$

$$= \frac{2}{\sqrt{3}} log \left| \frac{2\, sin\, x - \sqrt{3}}{2\, sin\, x + \sqrt{3}} \right| + c$$

64. Question

Evaluate $\int \frac{1}{1 + 2\cos x} dx$

Answer

Given, $\int \frac{1}{1+2\cos x} dx$

Put $\tan \frac{x}{2} = t$

dx $= \frac{2}{1+t^2}$ dt and $\cos x = \frac{1-t^2}{1+t^2}$

$$= \int \frac{1}{1 + 2\left[\frac{1-t^2}{1+t^2}\right]} \cdot \frac{2}{1+t^2} dt$$

$$= \int \frac{1+t^2}{1+t^2+2-2t^2} \cdot \frac{2}{1+t^2} dt$$

$$= \int \frac{2}{3-t^2} dt$$

$$= \int \frac{2}{(\sqrt{3})^2 - (t)^2} dt \left[since, \int \frac{1}{a^2 - x^2} = \frac{1}{2a} log \left|\frac{a+x}{a-x}\right| + c\right]$$

$$= \frac{1}{2a} log \left|\frac{\sqrt{3}+t}{\sqrt{3}-t}\right| + c$$

$$= \frac{1}{2a} log \left|\frac{\sqrt{3} + tan\frac{x}{2}}{\sqrt{3} - tan\frac{x}{2}}\right| + c$$

65. Question

Evaluate $\int \frac{1}{1-2\sin x} dx$

Answer

Given, $\int \frac{1}{1-2\sin x} dx$

Let $\tan\frac{x}{2} = t$

dx $= \frac{2}{1+t^2} dt$ and $\sin x = \frac{2t}{1+t^2}$

$$= \int \frac{1}{1 - 2\left(\frac{2t}{1+t^2}\right)} \cdot \frac{2}{1+t^2} dt$$

$$= \int \frac{1+t^2}{1+t^2-4t} \cdot \frac{2}{1+t^2} dt$$

$$= \int \frac{2}{t^2-4t+1} dt$$

$$= \int \frac{2}{t^2-4t+4-3} dt$$

$$= \int \frac{2}{(t-2)^2 - (\sqrt{3})^2} dt$$

$$= \frac{2}{2\sqrt{3}} log \left|\frac{t-2-\sqrt{3}}{t-2+\sqrt{3}}\right| + c \left[since, \int \frac{1}{x^2-a^2} dx = \frac{1}{2a} log \left|\frac{x-a}{x+a}\right| + c\right]$$

$$= \frac{1}{\sqrt{3}} log \left|\frac{tan\frac{x}{2} - 2 - \sqrt{3}}{tan\frac{x}{2} - 2 + \sqrt{3}}\right| + c$$

31. Question

Evaluate $\int \cot^4 x \, dx$

Answer

In this question, first of all we expand $\cot^4 x$ as

$\cot^4 x = (\mathrm{cosec}^2 x - 1)^2$

$= \mathrm{cosec}^4 x - 2\mathrm{cosec}^2 x + 1$ ...(1)

Now, write $\mathrm{cosec}^4 x$ as

$\mathrm{cosec}^4 x = \mathrm{cosec}^2 x \mathrm{cosec}^2 x$

$= \mathrm{cosec}^2 x(1 + \cot^2 x)$

$= \mathrm{cosec}^2 x + \mathrm{cosec}^2 x \cot^2 x$

Putting the value of $\mathrm{cosec}^4 x$ in eq(1)

$\cot^4 x = \mathrm{cosec}^2 x + \mathrm{cosec}^2 x \cot^2 x - 2\mathrm{cosec}^2 x + 1$

$= \mathrm{cosec}^2 x \cot^2 x - \mathrm{cosec}^2 x + 1$

$y = \int \cot^4 x \, dx$

$= \int \mathrm{cosec}^2 x \cot^2 x \, dx + \int - \mathrm{cosec}^2 x + 1 \, dx$

$A = \int \mathrm{cosec}^2 x \cot^2 x \, dx$

Let, cot x = t

Differentiating both side with respect to x

$$\frac{dt}{dx} = -cosec^2 x$$

$\Rightarrow -dt = \mathrm{cosec}^2 x \, dx$

$A = \int -t^2 \, dt$

Using formula $\int t^n dt = \frac{t^{n+1}}{n+1}$

$$A = -\frac{t^3}{3} + c_1$$

Again, put t = cot x

$$A = -\frac{cot^3 x}{3} + c_1$$

Now, $B = \int -\mathrm{cosec}^2 x + 1 \, dx$

Using formula $\int \mathrm{cosec}^2 x \, dx = -\cot x$ and $\int c \, dx = cx$

$B = \cot x + x + c_2$

Now, the complete solution is

y = A + B

$$y = -\frac{cot^3 x}{3} + \cot x + x + c$$

32. Question

Evaluate $\int \cot^5 x \, dx$

Answer

$$y = \int \frac{cos^5 x}{sin^5 x} dx$$

$$y = \int \frac{\cos^4 x \cos x}{\sin^5 x} dx$$

$$y = \int \frac{(1 - \sin^2 x)^2 \cos x}{\sin^5 x} dx$$

Let, sin x = t

Differentiating both sides with respect to x

$\frac{dt}{dx} = \cos x \Rightarrow$ dt = cos x dx

$$y = \int \frac{(1 - t^2)^2}{t^5} dt$$

$$y = \int \frac{1 - 2t^2 + t^4}{t^5} dt$$

$$y = \int t^{-5} - 2t^{-3} + \frac{1}{t} dt$$

Using formula $\int t^n dt = \frac{t^{n+1}}{n+1}$ and $\int \frac{1}{t} dt = \ln t$

$$y = \frac{t^{-4}}{-4} - 2\frac{t^{-2}}{-2} + \ln t + c$$

Again, put t = sin x

$$y = -\frac{\sin^{-4} x}{4} + \sin^{-2} x + \ln t + c$$

33. Question

Evaluate $\int \frac{x^2}{(x-1)^3} dx$

Answer

$$y = \int \frac{(x-1+1)^2}{(x-1)^3} dx$$

$$y = \int \frac{(x-1)^2 + 2(x-1) + 1}{(x-1)^3} dx$$

$$y = \int \frac{1}{(x-1)} + 2\frac{1}{(x-1)^2} + \frac{1}{(x-1)^3} dx$$

Using formula $\int \frac{1}{x} dx = \ln x$ and $\int x^n dx = \frac{x^{n+1}}{n+1}$

$$y = \ln(x-1) + 2\frac{(x-1)^{-1}}{-1} + \frac{(x-1)^{-2}}{-2} + c$$

$$y = \ln(x-1) - 2(x-1)^{-1} - \frac{(x-1)^{-2}}{2} + c$$

34. Question

Evaluate $\int x\sqrt{2x+3}\, dx$

Answer

In this question we write $x\sqrt{2x+3}$ as

$$x\sqrt{2x+3} = \frac{2x\sqrt{2x+3}}{2}$$

$$= \frac{(2x+3-3)\sqrt{2x+3}}{2}$$

$$= \frac{(2x+3)\sqrt{2x+3}\ -\ 3\sqrt{2x+3}}{2}$$

$$= \frac{(2x+3)^{\frac{3}{2}}\ -\ 3\sqrt{2x+3}}{2}$$

$$y = \int x\sqrt{2x+3}\ dx$$

$$y = \int \frac{(2x+3)^{\frac{3}{2}}\ -\ 3\sqrt{2x+3}}{2}\ dx$$

Using formula $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)}$

$$y = \frac{(2x+3)^{\frac{5}{2}}}{2\times 2\times \frac{5}{2}} - \frac{3(2x+3)^{\frac{3}{2}}}{2\times 2\times \frac{3}{2}} + c$$

$$y = \frac{(2x+3)^{\frac{5}{2}}}{10} - \frac{(2x+3)^{\frac{3}{2}}}{2} + c$$

35. Question

Evaluate $\int \frac{x^3}{\left(1+x^2\right)^2} dx$

Answer

Let, x = tan t

Differentiating both side with respect to t

$\frac{dx}{dt} = sec^2 t \Rightarrow$ dx = sec^2t dt

$$y = \int \frac{tan^3 t}{sec^4 t} sec^2 t dt$$

$$y = \int \frac{sin^3 t}{\cos t} dt$$

$$y = \int \frac{(1-cos^2 t)\sin t}{\cos t}\ dt$$

Again, let cos t = z

Differentiating both side with respect to t

$\frac{dz}{dt} = -\sin t \Rightarrow$ -dz = sin t dt

$$y = -\int \frac{(1-z^2)}{z}\ dz$$

$$y = -\int \frac{1}{z} - z\ dz$$

Using formula $\int \frac{1}{z} dz = \ln z$ and $\int z^n dz = \frac{z^{n+1}}{n+1}$

$$y = -\ln z + \frac{z^2}{2} + c$$

Again, put z = cos t = cos($\tan^{-1}$x)

$$y = -\ln\cos(\tan^{-1} x) + \frac{\cos^2(\tan^{-1} x)}{2} + c$$

36. Question

Evaluate $\int x \sin^5 x^2 \cos x^2 \, dx$

Answer

Let, sin x^2 = t

Differentiating both sides with respect to x

$$\frac{dt}{dx} = \cos x^2 \times 2x \Rightarrow \frac{dt}{2} = x\cos x^2 \, dx$$

$$y = \int \frac{t^5}{2} dt$$

Using formula $\int t^n \, dt = \frac{t^{n+1}}{n+1}$

$$y = \frac{t^6}{2 \times 6} + c$$

Again, put t = sin x^2

$$y = \frac{\sin^6 x^2}{12} + c$$

37. Question

Evaluate $\int \sin^3 x \cos^4 x \, dx$

Answer

$$y = \int (1 - \cos^2 x)\cos^4 x \sin x \, dx$$

Let, cos x = t

Differentiating both side with respect to x

$\frac{dt}{dx} = -\sin x \Rightarrow$ -dt = sin x dx

$$y = \int -(1 - t^2)t^4 \, dt$$

$$y = -\int t^4 - t^6 \, dt$$

Using formula $\int t^n \, dt = \frac{t^{n+1}}{n+1}$

$$y = -\left(\frac{t^5}{5} - \frac{t^7}{7}\right) + c$$

Again, put t = cos x

$$y = \frac{cos^7 x}{7} - \frac{cos^5 x}{5} + c$$

38. Question

Evaluate $\int \sin^5 x \, dx$

Answer

$$y = \int (1 - cos^2 x)^2 \sin x \; dx$$

Let, cos x = t

Differentiating both side with respect to x

$\frac{dt}{dx} = -\sin x \Rightarrow$ -dt = sin x dx

$$y = -\int (1 - t^2)^2 \; dt$$

$$y = -\int 1 + t^4 - 2t^2 \; dt$$

Using formula $\int t^n \, dt = \frac{t^{n+1}}{n+1}$ and $\int c \, dt = ct$

$$y = -\left(t + \frac{t^5}{5} - 2\frac{t^3}{3}\right) + c$$

Again, put t = cos x

$$y = -\left(\cos x + \frac{cos^5 x}{5} - 2\frac{cos^3 x}{3}\right) + c$$

39. Question

Evaluate $\int \cos^5 x \, dx$.

Answer

$$y = \int (1 - sin^2 x)^2 \cos x \; dx$$

Let, sin x = t

Differentiating both side with respect to x

$\frac{dt}{dx} = \cos x \Rightarrow$ dt = cos x dx

$$y = \int (1 - t^2)^2 \; dt$$

$$y = \int 1 + t^4 - 2t^2 \; dt$$

Using formula $\int t^n \, dt = \frac{t^{n+1}}{n+1}$ and $\int c \, dt = ct$

$$y = \left(t + \frac{t^5}{5} - 2\frac{t^3}{3}\right) + c$$

Again, put t = sin x

$$y = \left(\sin x + \frac{sin^5 x}{5} - 2\frac{sin^3 x}{3}\right) + c$$

40. Question

Evaluate $\int \sqrt{\sin x} \cos^3 x \, dx$

Answer

$y = \int \sqrt{\sin x}\ (1 - sin^2 x) \cos x \ dx$

Let, sin x = t

Differentiating both side with respect to x

$\frac{dt}{dx} = \cos x$ ⇒ dt = cos x dx

$$y = \int \sqrt{t}(1 - t^2)\, dt$$

$$y = \int t^{\frac{1}{2}} - t^{\frac{5}{2}}\, dt$$

Using formula $\int t^n \, dt = \frac{t^{n+1}}{n+1}$

$$y = \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - \frac{t^{\frac{7}{2}}}{\frac{7}{2}} + c$$

Again, put t = sin x

$$y = \frac{\sin x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{\sin x^{\frac{7}{2}}}{\frac{7}{2}} + c$$

41. Question

Evaluate $\int \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx$

Answer

$y = \int \frac{sin\, 2x}{(sin^2 x)^2 + (1 - sin^2 x)^2} \ dx$

Let, sin^{2}x = t

Differentiating both side with respect to x

$\frac{dt}{dx} = 2 \sin x \cos x$ ⇒ dt = sin 2x dx

$$y = \int \frac{dt}{t^2 + (1 - t)^2}$$

$$y = \int \frac{dt}{2t^2 - 2t + 1}$$

Try to make perfect square in denominator

$$y = \int \frac{dt}{2t^2 - 2t + \frac{1}{2} + \frac{1}{2}}$$

$$y = \int \frac{dt}{(\sqrt{2}t)^2 - 2(\sqrt{2}t)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}}$$

$$y = \int \frac{dt}{\left(\sqrt{2}t - \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$$

Using formula $\int \frac{dt}{t^2+a^2} = \frac{1}{a}\tan^{-1}\frac{t}{a}$

$$y = \frac{1}{\sqrt{2}\times\frac{1}{\sqrt{2}}}\tan^{-1}\frac{\left(\sqrt{2}t - \frac{1}{\sqrt{2}}\right)}{\frac{1}{\sqrt{2}}} + c$$

$$y = \sqrt{2}\tan^{-1}\left(\sqrt{2}t - \frac{1}{\sqrt{2}}\right) + c$$

Again, put t = $\sin^2$x

$$y = \sqrt{2}\tan^{-1}\left(\sqrt{2}sin^2x - \frac{1}{\sqrt{2}}\right) + c$$

42. Question

Evaluate $\int \frac{1}{\sqrt{x^2 - a^2}}dx$

Answer

Let, x = a sec t

Differentiating both side with respect to t

$\frac{dx}{dt} = a\sec t\tan t \Rightarrow$ dx = a sec t tan t dt

$$y = \int \frac{a\sec t\tan t}{\sqrt{a^2sec^2t - a^2}}dt$$

$$y = \int \frac{\sec t\tan t}{\tan t}dt$$

$$y = \int \sec t\, dt$$

Using formula $\int \sec t\, dt = \ln(\tan t + \sec t)$

y = ln(tan t + sec t) + c_1

Again, put $t = \sec^{-1}\frac{x}{a}$

$$y = \ln\left(\tan\sec^{-1}\frac{x}{a} + sec\sec^{-1}\frac{x}{a}\right) + c_1$$

$$y = \ln\left(\sqrt{\left(\frac{x}{a}\right)^2 - 1} + \frac{x}{a}\right) + c_1$$

$$y = \ln(x + \sqrt{x^2 - a^2}) - \ln a + c_1$$

$$y = \ln(x + \sqrt{x^2 - a^2}) + c$$

43. Question

Evaluate $\int \frac{1}{\sqrt{x^2 + a^2}}dx$

Answer

Let, x = a tan t

Differentiating both side with respect to t

$\frac{dx}{dt} = a\ sec^2 t \Rightarrow dx = a\ sec^2 t\ dt$

$$y = \int \frac{a\ sec^2 t}{\sqrt{a^2 tan^2 t + a^2}} dt$$

$$y = \int \frac{sec^2 t}{\sec t} dt$$

$$y = \int \sec t\ dt$$

Tip: This is very important formula. It is use directly in the question. So, learn it by heart.

Using formula $\int \sec t\ dt = \ln(\tan t + \sec t)$

y = ln(tan t + sec t) + c_1

Again, put $t = \tan^{-1}\frac{x}{a}$

$$y = \ln\left(\tan\tan^{-1}\frac{x}{a} + sec\tan^{-1}\frac{x}{a}\right) + c_1$$

$$y = \ln\left(\sqrt{\left(\frac{x}{a}\right)^2 + 1} + \frac{x}{a}\right) + c_1$$

$$y = \ln(x + \sqrt{x^2 + a^2}) - \ln a + c_1$$

$$y = \ln(x + \sqrt{x^2 + a^2}) + c$$

44. Question

Evaluate $\int \frac{1}{4x^2 + 4x + 5} dx$

Answer

In this question we can try to make perfect square in

denominator

$$y = \int \frac{1}{(2x)^2 + 2(2x)(1) + 1 + 4} dx$$

$$y = \int \frac{1}{(2x+1)^2 + (2)^2} dx$$

Using formula $\int \frac{dt}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

$$y = \frac{1}{2 \times 2}\tan^{-1}\frac{(2x+1)}{2} + c$$

$$y = \frac{1}{4}\tan^{-1}\frac{(2x+1)}{2} + c$$

45. Question

Evaluate $\int \frac{1}{x^2 + 4x - 5} dx$

Answer

In this question we can try to make perfect square in

denominator

$$y=\int \frac{1}{x^2+2(x)(2)+4-(3)^2}\,dx$$

$$y=\int \frac{1}{(x+2)^2-(3)^2}\,dx$$

Using formula $\int \frac{dt}{x^2-a^2}=\frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)+c$

$$y=\frac{1}{2\times 3}\log\left(\frac{x+2-3}{x+2+3}\right)+c$$

$$y=\frac{1}{6}\log\left(\frac{x-1}{x+5}\right)+c$$

1. Question

Evaluate $\int \frac{1}{\sqrt{x}+\sqrt{x+1}}\,dx$

Answer

Rationalising denominator

We get, $\int \frac{\sqrt{x}-\sqrt{x+1}}{x-(x+1)}\,dx$

It becomes $\int \frac{\sqrt{x}-\sqrt{x+1}}{-1}\,dx$

$=-\int \sqrt{x}\,dx-\int \sqrt{x+1}$ dx

$$=-\frac{x^{\frac{3}{2}}}{\frac{3}{2}}-\frac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}}+c$$

1. Question

Evaluate $\int \frac{1}{\sqrt{x}+\sqrt{x+1}}\,dx$

Answer

Rationalising denominator

We get, $\int \frac{\sqrt{x}-\sqrt{x+1}}{x-(x+1)}\,dx$

It becomes $\int \frac{\sqrt{x}-\sqrt{x+1}}{-1}\,dx$

$=-\int \sqrt{x}\,dx-\int \sqrt{x+1}$ dx

$$=-\frac{x^{\frac{3}{2}}}{\frac{3}{2}}-\frac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}}+c$$

2. Question

Evaluate $\int \frac{1-x^4}{1-x}\,dx$

Answer

Factorising the equation

$$= \int \frac{(1-x^2)(1+x^2)}{1-x} dx$$

$$= \int \frac{(1-x)(1+x)(1+x^2)}{1-x} dx$$

On cancelling we get

$=\int(1+x)(1+x^2)dx$

$=\int(1+x+x^2+x^3)dx$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + c$$

2. Question

Evaluate $\int \frac{1-x^4}{1-x} dx$

Answer

Factorising the equation

$$= \int \frac{(1-x^2)(1+x^2)}{1-x} dx$$

$$= \int \frac{(1-x)(1+x)(1+x^2)}{1-x} dx$$

On cancelling we get

$=\int(1+x)(1+x^2)dx$

$=\int(1+x+x^2+x^3)dx$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + c$$

3. Question

Evaluate $\int \frac{x+2}{(x+1)^3} dx$

Answer

On simplifying we get,

$\int \frac{(x+1)+1}{(x+1)^3} dx$

$$= \int \frac{1}{(x+1)^2} dx + \int \frac{1}{(x+1)^3} dx$$

On solving we get

$$= -\frac{1}{x+1} - \frac{1}{2(x+1)^2} + c$$

3. Question

Evaluate $\int \frac{x+2}{(x+1)^3} dx$

Answer

On simplifying we get,

$\int \frac{(x+1)+1}{(x+1)^3} dx$

$$= \int \frac{1}{(x+1)^2} dx + \int \frac{1}{(x+1)^3} dx$$

On solving we get

$$= -\frac{1}{x+1} - \frac{1}{2(x+1)^2} + c$$

4. Question

Evaluate $\int \frac{8x+13}{\sqrt{4x+7}} dx$

Answer

On simplifying we get,

$$= \int \frac{4x+7}{\sqrt{4x+7}} dx + \int \frac{4x+7}{\sqrt{4x+7}} dx - \int \frac{1}{\sqrt{4x+7}} dx$$

$$= 2\int \frac{4x+7}{\sqrt{4x+7}} dx - \int \frac{1}{\sqrt{4x+7}} dx$$

$$= 2\int \sqrt{4x+7}\, dx - \int \frac{1}{\sqrt{4x+7}} dx$$

$$= 2\, x\, \frac{(4x+7)^{3/2}}{\frac{3}{2}}\, x \times \frac{1}{4} - \frac{(4x+7)^{\frac{1}{2}}}{\frac{1}{2}}\, x \times \frac{1}{4} + c$$

$$= \frac{(4x+7)^{3/2}}{3} - \frac{(4x+7)^{\frac{1}{2}}}{2} + c$$

4. Question

Evaluate $\int \frac{8x+13}{\sqrt{4x+7}} dx$

Answer

On simplifying we get,

$$= \int \frac{4x+7}{\sqrt{4x+7}} dx + \int \frac{4x+7}{\sqrt{4x+7}} dx - \int \frac{1}{\sqrt{4x+7}} dx$$

$$= 2\int \frac{4x+7}{\sqrt{4x+7}} dx - \int \frac{1}{\sqrt{4x+7}} dx$$

$$= 2\int \sqrt{4x+7}\, dx - \int \frac{1}{\sqrt{4x+7}} dx$$

$$= 2\, x\, \frac{(4x+7)^{3/2}}{\frac{3}{2}}\, x \times \frac{1}{4} - \frac{(4x+7)^{\frac{1}{2}}}{\frac{1}{2}}\, x \times \frac{1}{4} + c$$

$$= \frac{(4x+7)^{3/2}}{3} - \frac{(4x+7)^{\frac{1}{2}}}{2} + c$$

5. Question

Evaluate $\int \frac{1+x+x^2}{x^2(1+x)}dx$

Answer

On simplifying we get

$$\int \frac{1+x}{x^2(1+x)}dx + \int \frac{x^2}{x^2(1+x)}dx$$

$$= \int \frac{1}{x^2}dx + \int \frac{1}{1+x}dx$$

$= -x^{-1} + ln(1+x) + c$

5. Question

Evaluate $\int \frac{1+x+x^2}{x^2(1+x)}dx$

Answer

On simplifying we get

$$\int \frac{1+x}{x^2(1+x)}dx + \int \frac{x^2}{x^2(1+x)}dx$$

$$= \int \frac{1}{x^2}dx + \int \frac{1}{1+x}dx$$

$= -x^{-1} + ln(1+x) + c$

6. Question

Evaluate $\int \frac{\left(2^x+3^x\right)^2}{6^x}dx$

Answer

On squaring numerator we get

$$= \int \frac{2^{2x} + 2.2^x.3^x + 3^{2x}}{2^x.3^x}dx$$

$$= \int \left(\frac{2}{3}\right)^x + 2 + \left(\frac{3}{2}\right)^x dx$$

Formula for $\int a^x dx = \frac{a^x}{\ln(a)}$

Solving above equation we get,

$$= \frac{\left(\frac{2}{3}\right)^x}{\ln\left(\frac{2}{3}\right)} + 2x + \frac{\left(\frac{3}{2}\right)^x}{\ln\left(\frac{3}{2}\right)} + c$$

6. Question

Evaluate $\int \frac{\left(2^x+3^x\right)^2}{6^x}dx$

Answer

On squaring numerator we get

$$= \int \frac{2^{2x} + 2.2^x.3^x + 3^{2x}}{2^x.3^x} dx$$

$$= \int \left(\frac{2}{3}\right)^x + 2 + \left(\frac{3}{2}\right)^x dx$$

Formula for $\int a^x dx = \frac{a^x}{\ln(a)}$

Solving above equation we get,

$$= \frac{\left(\frac{2}{3}\right)^x}{\ln\left(\frac{2}{3}\right)} + 2x + \frac{\left(\frac{3}{2}\right)^x}{\ln\left(\frac{3}{2}\right)} + c$$

7. Question

Evaluate $\int \frac{\sin x}{1 + \sin x} dx$

Answer

Multiplying numerator and denominator with 1-sinx

We get $\int \frac{sinx(1-sinx)}{1-sin^2 x} dx$

$$= \int \frac{sinx(1 - sinx)}{cos^2 x} dx$$

$$= \int \frac{sinx - sin^2 x}{cos^2 x} dx$$

$$= \int \frac{sinx}{cos^2 x} dx - \int \frac{sin^2 x}{cos^2 x} dx$$

Taking $\int \frac{sinx}{cos^2 x} dx = A$ and $\int \frac{sin^2 x}{cos^2 x} dx = B$

Solving for A

Taking $cos\ x=t$

On differentiating both sides we get

$-sin\ x\ dx=dt$

Putting values in A we get our equation as

$$= \int \frac{-dt}{t^2}$$

$= t^{-1}+c$

Substituting value of t,

$=sec\ x + c$

Solving for B

$$\int \frac{1 - cos^2 x}{cos^2 x} dx$$

$= \int sec^2x - \int 1\ dx$

= tan x – x +c

Final answer is A+B

= sec x + tan x – x + c

7. Question

Evaluate $\int \frac{\sin x}{1+\sin x} dx$

Answer

Multiplying numerator and denominator with 1-sinx

We get $\int \frac{sinx(1-sinx)}{1-sin^2x} dx$

$$= \int \frac{sinx(1-sinx)}{cos^2x} dx$$

$$= \int \frac{sinx - sin^2x}{cos^2x} dx$$

$$= \int \frac{sinx}{cos^2x} dx - \int \frac{sin^2x}{cos^2x} dx$$

Taking $\int \frac{sinx}{cos^2x} dx = A$ and $\int \frac{sin^2x}{cos^2x} dx = B$

Solving for A

Taking *cos x=t*

On differentiating both sides we get

-*sin x dx=dt*

Putting values in A we get our equation as

$$= \int \frac{-dt}{t^2}$$

= t^{-1}+c

Substituting value of t,

=*sec x + c*

Solving for B

$$\int \frac{1-cos^2x}{cos^2x} dx$$

= ∫ sec^2x - ∫ 1 dx

= tan x – x +c

Final answer is A+B

= sec x + tan x – x + c

8. Question

Evaluate $\int \frac{x^4+x^2-1}{x^2+1} dx$

Answer

On simplifying we get

$$\int \frac{x^2(x^2+1)}{(x^2+1)} - \frac{1}{(x^2+1)}\, dx$$

$$= \int x^2 dx - \int \frac{1}{x^2+1} dx$$

$$= \frac{x^3}{3} - tan^{-1} x + c$$

8. Question

Evaluate $\int \frac{x^4 + x^2 - 1}{x^2 + 1} dx$

Answer

On simplifying we get

$$\int \frac{x^2(x^2+1)}{(x^2+1)} - \frac{1}{(x^2+1)}\, dx$$

$$= \int x^2 dx - \int \frac{1}{x^2+1} dx$$

$$= \frac{x^3}{3} - tan^{-1} x + c$$

9. Question

Evaluate $\int \sec^2 x \cos^2 2x\, dx$

Answer

$\int \sec^2 x\, (\cos^2 x - \sin^2 x\,)^2\, dx$

Opening the square

$$= \int \frac{cos^4 x - 2.cos^2 x.sin^2 x + sin^4 x}{cos^2 x} dx$$

$$= \int (cos^2 x - 2sin^2 x + \frac{sin^2 x.\, sin^2 x}{cos^2 x})\, dx$$

$$= \int (cos^2 x - 2sin^2 x + \frac{(1-cos^2 x).(1-cos^2 x)}{cos^2 x})\, dx$$

On multiplying $(1-cos^2 x).(1-cos^2 x)$ equation reduces to

$=\int(\cos^2 x-2\sin^2 x+\sec^2 x-2+\cos^2 x)dx$

$=\int(2\cos^2 x-2\sin^2 x+\sec^2 x-2)dx$

$=\int(2(\cos^2 x-\sin^2 x)+\sec^2 x-2)dx$

$=\int(2\cos 2x+\sec^2 x-2)dx$

On solving this we get our answer i.e

$$= \frac{2sin2x}{2} + tanx - 2x + c$$

$=\sin 2x+\tan x-2x+c$

9. Question

Evaluate $\int \sec^2 x \cos^2 2x\, dx$

Answer

$\int \sec^2 x\, (\cos^2 x - \sin^2 x)^2\, dx$

Opening the square

$$= \int \frac{cos^4 x - 2.cos^2 x.sin^2 x + sin^4 x}{cos^2 x} dx$$

$$= \int (cos^2 x - 2sin^2 x + \frac{sin^2 x.sin^2 x}{cos^2 x})\, dx$$

$$= \int (cos^2 x - 2sin^2 x + \frac{(1 - cos^2 x).(1 - cos^2 x)}{cos^2 x})\, dx$$

On multiplying $(1 - cos^2 x).(1 - cos^2 x)$ equation reduces to

$=\int(\cos^2 x-2\sin^2 x+\sec^2 x-2+\cos^2 x)dx$

$=\int(2\cos^2 x-2\sin^2 x+\sec^2 x-2)dx$

$=\int(2(\cos^2 x-\sin^2 x)+\sec^2 x-2)dx$

$=\int(2\cos 2x+\sec^2 x-2)dx$

On solving this we get our answer i.e

$$= \frac{2sin2x}{2} + tanx - 2x + c$$

$=\sin 2x+\tan x-2x+c$

10. Question

Evaluate $\int \operatorname{cosec}^2 x \cos^2 2x\, dx$

Answer

$\int \operatorname{cosec}^2 x(\cos^2 x-\sin^2 x)^2 dx$

Opening the square

$$= \int \frac{cos^4 x - 2.cos^2 x.sin^2 x + sin^4 x}{sin^2 x} dx$$

$$= \int (\frac{cos^2 x.cos^2 x}{sin^2 x} - 2cos^2 x + sin^2 x)\, dx$$

$$= \int (\frac{(1 - sin^2 x).(1 - sin^2 x)}{sin^2 x} - 2cos^2 x + sin^2 x)\, dx$$

On multiplying $(1-\sin^2 x)(1-\sin^2 x)$ equation reduces to

$=\int(\operatorname{cosec}^2 x-2+\sin^2 x-2\cos^2 x+ \sin^2 x)dx$

$=\int(\operatorname{cosec}^2 x-2+2\sin^2 x-2\cos^2 x)dx$

$=\int(-2(\cos^2 x-\sin^2 x)+\operatorname{cosec}^2 x-2)dx$

$=\int(-2\cos 2x+\operatorname{cosec}^2 x-2)dx$

On solving this we get our answer i.e

$$= \frac{-2sin2x}{2} - cotx - 2x + c$$

=-sin2x-cotx-2x+c

10. Question

Evaluate $\int \cos ec^2 x \cos^2 2x\, dx$

Answer

$\int cosec^2x(cos^2x-sin^2x)^2dx$

Opening the square

$$= \int \frac{cos^4 x - 2.cos^2 x.sin^2 x + sin^4 x}{sin^2 x} dx$$

$$= \int (\frac{cos^2 x.cos^2 x}{sin^2 x} - 2cos^2 x + sin^2 x)\, dx$$

$$= \int (\frac{(1 - sin^2 x).(1 - sin^2 x)}{sin^2 x} - 2cos^2 x + sin^2 x)\, dx$$

On multiplying $(1-sin^2x)(1-sin^2x)$ equation reduces to

$=\int(cosec^2x-2+sin^2x-2cos^2x+ sin^2x)dx$

$=\int(cosec^2x-2+2sin^2x-2cos^2x)dx$

$=\int(-2(cos^2x-sin^2x)+cosec^2x-2)dx$

$=\int(-2cos2x+cosec^2x-2)dx$

On solving this we get our answer i.e

$$= \frac{-2sin2x}{2} - cotx - 2x + c$$

=-sin2x-cotx-2x+c

11. Question

Evaluate $\int \sin^4 2x\, dx$

Answer

Replacing 2x by t

We get dx=dt/2 by differentiating both sides

Our equation has become

$$\frac{1}{2}\int sin^4 t\, dt$$

$$= \frac{1}{2}\int sin^2 t.sin^2 t\, dt = \frac{1}{2}\int sin^2 t.(1 - cos^2 t)\, dt$$

$$= \frac{1}{2}\int sin^2 tdt - \frac{1}{2}\int sin^2 t.cos^2 tdt$$

simplifying $sin^2t.cos^2t$

on multiplying and dividing by 4 we get $sin^2t.cos^2tdt$ as sin^22t

$$= \frac{1}{2}\int \frac{1 - cos2t}{2} dt - \frac{1}{2}\int \frac{sin^2 2t}{4}$$

$$= \frac{1}{2}\int \frac{1-cos2t}{2}dt - \frac{1}{2}\int \frac{1-cos4t}{4.2}$$

$$= \frac{1}{4}\int 1-cos2t\ dt - \frac{1}{16}\int 1-cos4t\ dt$$

$$= \frac{t}{4} - \frac{sin2t}{8} - \frac{t}{8} + \frac{sin4t}{64} + c$$

Hence our final answer is

$$= \frac{t}{8} - \frac{sin2t}{8} + \frac{sin4t}{64} + c$$

11. Question

Evaluate $\int \sin^4 2x\ dx$

Answer

Replacing 2x by t

We get dx=dt/2 by differentiating both sides

Our equation has become

$$\frac{1}{2}\int sin^4 t\ dt$$

$$= \frac{1}{2}\int sin^2 t . sin^2 t\ dt = \frac{1}{2}\int sin^2 t.(1-cos^2 t)\ dt$$

$$= \frac{1}{2}\int sin^2 tdt - \frac{1}{2}\int sin^2 t. cos^2 tdt$$

simplifying $sin^2t.cos^2t$

on multiplying and dividing by 4 we get $sin^2t.cos^2tdt$ as sin^22t

$$= \frac{1}{2}\int \frac{1-cos2t}{2}dt - \frac{1}{2}\int \frac{sin^2 2t}{4}$$

$$= \frac{1}{2}\int \frac{1-cos2t}{2}dt - \frac{1}{2}\int \frac{1-cos4t}{4.2}$$

$$= \frac{1}{4}\int 1-cos2t\ dt - \frac{1}{16}\int 1-cos4t\ dt$$

$$= \frac{t}{4} - \frac{sin2t}{8} - \frac{t}{8} + \frac{sin4t}{64} + c$$

Hence our final answer is

$$= \frac{t}{8} - \frac{sin2t}{8} + \frac{sin4t}{64} + c$$

12. Question

Evaluate $\int \cos^3 3x\ dx$

Answer

We can write $\int cos^3 3xdx$ as:

$\int cos3x(cos3x)^2dx$ $\int cos3x(cos^2 3x)dx$ and

further as:

$=\cos 3x(1-\sin^2 3x)dx$

$=\int \cos 3x dx-\int \cos 3x(\sin^2 3x)dx$

Taking $A=\int \cos 3x dx$

Solving for A

$A=\frac{\sin 3x}{3}$

Taking $B=\int \cos 3x(\sin^2 3x)dx$

In this taking $\sin 3x=t$

Differentiating on both sides we get

$3\cos 3x dx=dt$

Solving by putting these values we get

$B=\int \frac{t^2}{3}dt$

$=\frac{t^3}{9}+c$

Substituting values we get

$B=\frac{\sin^3 3x}{9}+c$

Our final answer is A+B i.e

$=\frac{\sin 3x}{3}+\frac{\sin 3x}{3}+c$

12. Question

Evaluate $\int \cos^3 3x\, dx$

Answer

We can write $\int \cos^3 3x dx$ as:

$\int \cos 3x(\cos 3x)^2 dx$ $\int \cos 3x(\cos^2 3x)dx$ and

further as:

$=\cos 3x(1-\sin^2 3x)dx$

$=\int \cos 3x dx-\int \cos 3x(\sin^2 3x)dx$

Taking $A=\int \cos 3x dx$

Solving for A

$A=\frac{\sin 3x}{3}$

Taking $B=\int \cos 3x(\sin^2 3x)dx$

In this taking $\sin 3x=t$

Differentiating on both sides we get

$3\cos 3x dx=dt$

Solving by putting these values we get

$B=\int \frac{t^2}{3}dt$

$= \frac{t^3}{9} + c$

Substituting values we get

B $= \frac{sin^3 3x}{9} + c$

Our final answer is A+B i.e

$= \frac{sin3x}{3} + \frac{sin3x}{3} + c$

13. Question

Evaluate $\int \frac{\sin 2x}{a^2 + b^2 \sin^2 x}$

Answer

Taking b^2 common, we get,

$\int \frac{sin2x}{b^2(\frac{a^2}{b^2} + sin^2 x)} dx$

taking $\frac{a^2}{b^2} + sin^2 x = t$

on differentiating both sides we get

2sinxcosxdx=dt

Means sin2xdx=dt

putting $\frac{a^2}{b^2} + sin^2 x = t$ and sin2xdx=dt in equation we get our equation as

$\int \frac{dt}{b^2(t)}$

On solving this we get

$= \frac{\ln(t)}{b^2} + c$

Substituting value of t we get our answer as

$= \frac{\ln(\frac{a^2}{b^2} + sin^2 x)}{b^2} + c$

13. Question

Evaluate $\int \frac{\sin 2x}{a^2 + b^2 \sin^2 x}$

Answer

Taking b^2 common, we get,

$\int \frac{sin2x}{b^2(\frac{a^2}{b^2} + sin^2 x)} dx$

taking $\frac{a^2}{b^2} + sin^2 x = t$

on differentiating both sides we get

2sinxcosxdx=dt

Means sin2xdx=dt

putting $\frac{a^2}{b^2} + sin^2 x = t$ and sin2xdx=dt in equation we get our equation as

$$\int \frac{dt}{b^2(t)}$$

On solving this we get

$$= \frac{\ln(t)}{b^2} + c$$

Substituting value of t we get our answer as

$$= \frac{\ln(\frac{a^2}{b^2} + sin^2 x\ )}{b^2} + c$$

14. Question

Evaluate $\int \frac{1}{\left(\sin^{-1} x\right)\sqrt{1 - x^2}} dx$

Answer

Taking $\sin^{-1}x=t$

Differentiating both sides,

We get $\frac{1}{\sqrt{1-x^2}} dx = dt$

Our new equation has become

$$\int \frac{dt}{t}$$

On solving this we get

$$\int \frac{dt}{t} = \ln(t) + c$$

Substituting value of t= $\sin^{-1}x$

We get our answer as

=ln($\sin^{-1}x$)+c

14. Question

Evaluate $\int \frac{1}{\left(\sin^{-1} x\right)\sqrt{1 - x^2}} dx$

Answer

Taking $\sin^{-1}x=t$

Differentiating both sides,

We get $\frac{1}{\sqrt{1-x^2}} dx = dt$

Our new equation has become

$$\int \frac{dt}{t}$$

On solving this we get

$$\int \frac{dt}{t} = \ln(t) + c$$

Substituting value of t= $\sin^{-1}x$

We get our answer as

$= \ln(\sin^{-1}x) + c$

15. Question

Evaluate $\int \frac{\left(\sin^{-1}x\right)^3}{\sqrt{1-x^2}} dx$

Answer

Taking $\sin^{-1}x = t$

Differentiating both sides,

We get $\frac{1}{\sqrt{1-x^2}} dx = dt$

Our new equation has become

$\int t^3 dt$

On solving this we get

$$\int t^3 dt = \frac{t^4}{4} + c$$

Substituting value of t= $\sin^{-1}x$

We get our answer as

$$= \frac{(\sin^{-1}x)^4}{4} + c$$

15. Question

Evaluate $\int \frac{\left(\sin^{-1}x\right)^3}{\sqrt{1-x^2}} dx$

Answer

Taking $\sin^{-1}x = t$

Differentiating both sides,

We get $\frac{1}{\sqrt{1-x^2}} dx = dt$

Our new equation has become

$\int t^3 dt$

On solving this we get

$$\int t^3 dt = \frac{t^4}{4} + c$$

Substituting value of t= $\sin^{-1}x$

We get our answer as

$$= \frac{(sin^{-1}x)^4}{4} + c$$